Applications of Generalized Distorted Distributions

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Outline

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Definition Distorted Distributions (DD)

- The distorted distributions were introduced in Yaari's dual theory of choice under risk (Econometrica 55 (1987):95–115).
- The *distorted distribution* (DD) associated to a distribution function (DF) F and to an increasing continuous *distortion function* $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \tag{1.1}$$

- If q is strictly increasing, then F and F_q have the same support.
- For the reliability functions (RF) $\overline{F} = 1 - F$, $\overline{F}_q = 1 - F_q$, we have

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \tag{1.2}$$

where $\overline{q}(u) = 1 - q(1-u)$ is the *dual distortion function*.

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Generalized Distorted Distributions (GDD)

- The *generalized distorted distribution* (GDD) associated to n DF $F_1, \ldots, F_n$ and to an increasing continuous *multivariate distortion function* $Q : [0,1]^n \to [0,1]$ such that $Q(0,\ldots,0) = 0$ and $Q(1,\ldots,1) = 1$, is

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \tag{1.3}$$

- If Q is strictly increasing and $F_1, \ldots, F_n$ have the same support, then F_Q also has the same support.
- For the RF we have

$$\overline{F}_Q(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \tag{1.4}$$

where $\overline{F} = 1 - F$, $\overline{F}_Q = 1 - F_Q$ and $\overline{Q}(u_1, \ldots, u_n) = 1 - Q(1-u_1, \ldots, 1-u_n)$ is the *multivariate dual distortion function*.

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Proportional hazard rate (PHR) model

- The PHR (Cox) model associated to a RF $\overline{F}$ is

$$\overline{F}_\alpha(t) = \left(\overline{F}(t)\right)^\alpha = \overline{q}\left(\overline{F}(t)\right)$$

for $\alpha > 0$. $\overline{F}_\alpha$ a DD with $\overline{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.

- The hazard (failure) rate function is defined by $h(t) = f(t)/\overline{F}(t)$ where f is the PDF.
- Under the PHR model, $h_\alpha(t) = \alpha h(t)$.
- The proportional reversed hazard rate (PRHR) model is

$$F_\alpha(t) = (F(t))^\alpha = q\left(F(t)\right)$$

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Order statistics (OS)

- $X_1, \ldots, X_n$ IID$\sim F$ random variables.
- $X_1, \ldots, X_n$ exchangeable (EXC), i.e., for any permutation σ

$$(X_1, \ldots, X_n) =_{ST} (X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

- $(X_1, \ldots, X_n)$ is an arbitrary random vector with

$$\mathbf{F}(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n).$$

- Let $X_{1:n}, \ldots, X_{n:n}$ be the associated OS.
- Let $F_{i:n}(t) = \Pr(X_{i:n} \leq t)$ be the DF.
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Distorted Distribution Representation-IID case

- In the IID case, we have

$$F_{i:n}(t) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} F_{j:j}(t) = q_{i:n}(F(t)), \quad (1.5)$$

(see David and Nagaraja 2003, p. 46) where

$$F_{j:j}(t) = \Pr(X_{j:j} \le t) = \Pr(\max(X_1, \ldots, X_j) \le t) = F^j(t)$$

and

$$q_{i:n}(u) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} u^j$$

is a strictly increasing polinomial in $[0,1]$.

- Both $F_{j:j}$ and $F_{i:n}$ are DD from F.

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Distorted Distribution Representation-IID case

- The upper OS $X_{j:j}$ (lifetime of the parallel system) satisfies the PRHR model with $\alpha = j$ since

$$F_{j:j}(t) = \Pr(X_{j:j} \leq t) = \Pr(\max(X_1, \ldots, X_j) \leq t) = (F(t))^j .$$

- The lower OS $X_{1:j}$ (lifetime of the series system) satisfies the PHR model

$$\overline{F}_{1:j}(t) = \Pr(X_{1:j} \leq t) = \Pr(\min(X_1, \ldots, X_j) > t) = \left(\overline{F}(t)\right)^j .$$

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Distorted Distribution Representation- EXC case

- In the EXC case the left hand side of (1.5) holds with

$$F_{j:j}(t) = \Pr(\max(X_1, \dots, X_j) \le t) = \mathbf{F}(\underbrace{t, \dots, t}_{j}, \underbrace{\infty, \dots, \infty}_{n-j}).$$

- The copula representation for $\mathbf{F}$ is

$$\mathbf{F}(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \qquad (1.6)$$

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- In the EXC case, $F_1 = \cdots = F_n = F$ and

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Coherent systems

- A **coherent system** is

$$\phi = \phi(x_1, \ldots, x_n) : \{0,1\}^n \to \{0,1\},$$

where $x_i \in \{0,1\}$ (it represents the state of the ith component) and where ϕ (which represents the state of the system) is increasing in $x_1, \ldots, x_n$ and strictly increasing in x_i for at least a point $(x_1, \ldots, x_n)$, for all $i = 1, \ldots, n$.

- If $X_1, \ldots, X_n$ are the component lifetimes, then there exists ψ such that the system lifetime $T = \psi(X_1, \ldots, X_n)$.
- $X_{1:n}, \ldots, X_{n:n}$ are the lifetimes of k-out-of-n systems.
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- $X_{1:n}$ is the series system lifetime and $X_{n:n}$ is the parallel system lifetime.

Coherent systems- IID and EXC case

- Samaniego (IEEE TR, 1985), IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} p_i \overline{F}_{i:n}(t), \tag{1.7}$$

where $p_i = \Pr(T = X_{i:n})$.

- $\mathbf{p} = (p_1, \ldots, p_n)$ is the signature of the system.
- IID case: p_i only depends on ϕ

$$p_i = \frac{\left|\{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)}\}\right|}{n!} \tag{1.8}$$

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Generalized mixture representations

- Navarro, Ruiz and Sandoval (CSTM, 2007), EXC case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t). \tag{1.9}$$

- $\mathbf{a} = (a_1, \ldots, a_n)$ is the minimal signature of T.
- a_i only depends on ϕ but can be negative and so (1.9) is called a generalized mixture.
- In the IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}^i(t) = \overline{q}_\phi(\overline{F}(t)), \tag{1.10}$$

$\overline{q}_\phi(x) = \sum_{i=1}^{n} a_i x^i$ is the domination (reliability) polynomial.

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Coherent systems-GENERAL case

- A **path set** of T is a set $P \subseteq \{1, \ldots, n\}$ such that if all the components in P work, then the system works.
- A **minimal path set** of T is a path set which does not contains other path sets.
- If $P_1, \ldots, P_r$ are the minimal path sets of T, then $T = \max_{j=1,\ldots,r} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

$$\overline{F}_T(t) = \Pr\left(\max_{j=1,\ldots,r} X_{P_j} > t\right) = \Pr\left(\cup_{j=1}^r \{X_{P_j} > t\}\right)$$
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- The copula representation for the RF of $(X_1, \ldots, X_n)$ is

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)),$$

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- Then

$$\overline{F}_P(t) = Q_{P,K}(\overline{F}_1(t), \ldots, \overline{F}_n(t)),$$

where $Q_{P,K}(u_1, \ldots, u_n) = K(u_1^P, \ldots, u_n^P)$ and $u_i^P = u_i$ for $i \in P$ and $u_i^P = 1$ for $i \notin P$.

- Therefore, from the minimal path set repres., we get

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Example

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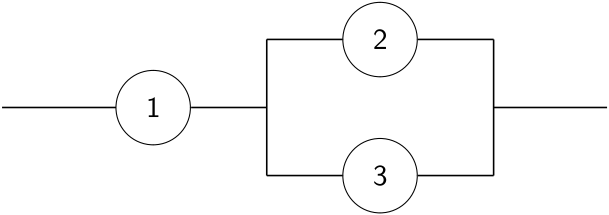
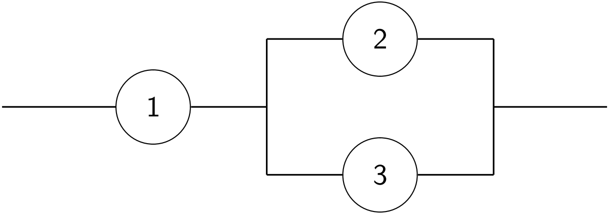


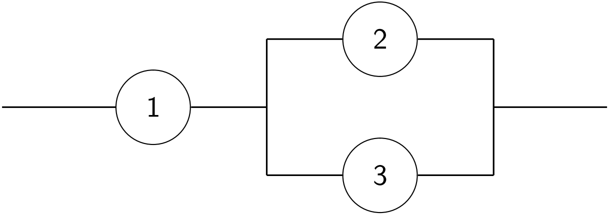
Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$.

Example



$3! = 6$ permutations.

Example



$$X_1 < X_2 < X_3 \Rightarrow T = X_1 = X_{1:3}$$

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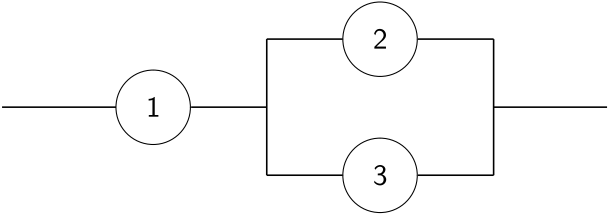
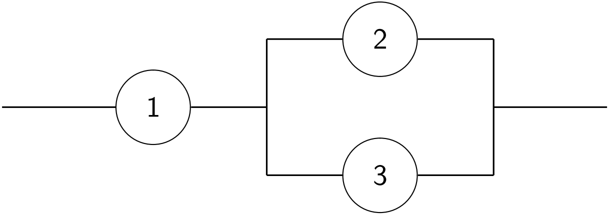


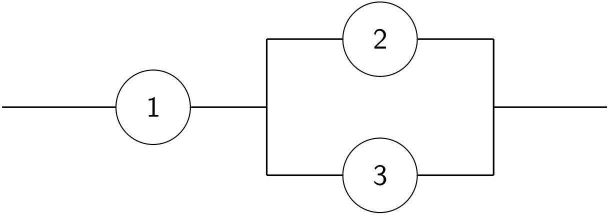
$$X_1 < X_3 < X_2 \Rightarrow T = X_1 = X_{1:3}$$

Example



$$X_2 < X_1 < X_3 \Rightarrow T = X_1 = X_{2:3}$$

Example



$$X_2 < X_3 < X_1 \Rightarrow T = X_3 = X_{2:3}$$

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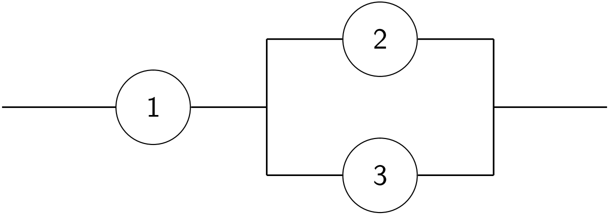

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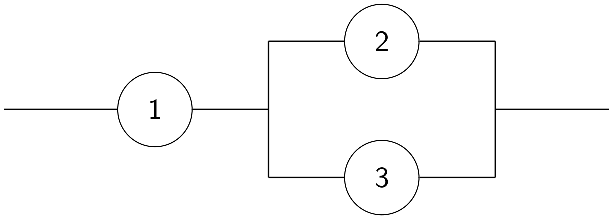
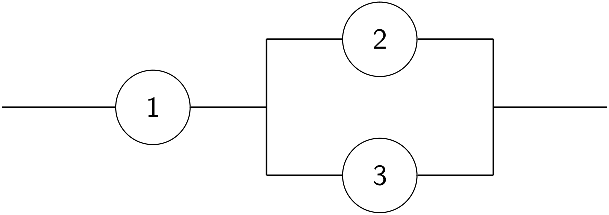


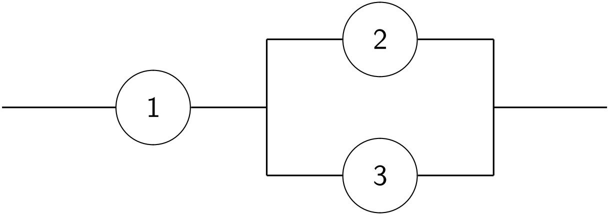
$$X_3 < X_2 < X_1 \Rightarrow T = X_2 = X_{2:3}$$

Example



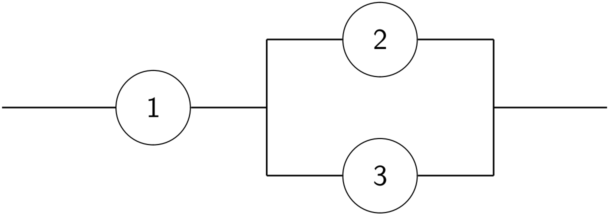
IID $\overline{F}$ cont.: $\mathbf{p} = (2/6, 4/6, 0) = (1/3, 2/3, 0)$.

Example



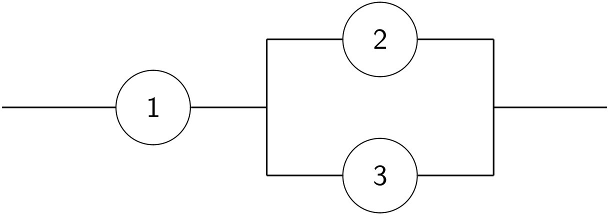
IID $\overline{F}$ cont.: $\overline{F}_T(t) = \frac{1}{3}\overline{F}_{1:3}(t) + \frac{2}{3}\overline{F}_{2:3}(t)$.

Example-general case



Coherent system lifetime $T = \max(\min(X_1, X_2), \min(X_1, X_3))$
Minimal path sets $P_1 = \{1, 2\}$ and $P_1 = \{1, 3\}$.

Example-general case



$$\overline{F}_T(t) = \Pr(\{X_{\{1,3\}} > t\} \cup \{X_{\{1,2\}} > t\})$$
$$= \overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) - \overline{F}_{\{1,2,3\}}(t).$$

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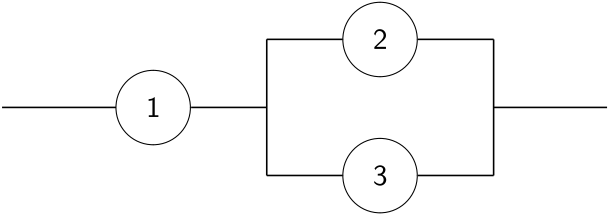
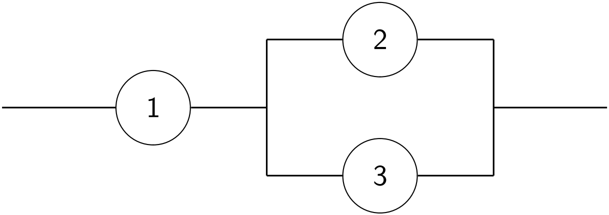


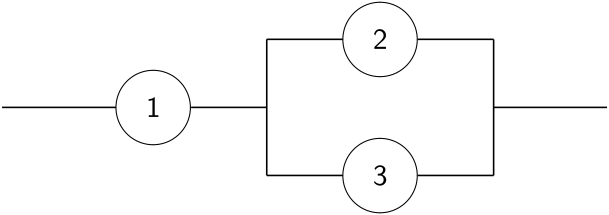
$\overline{F}_{\{1,2\}}(t) = \overline{\mathbf{F}}(t,t,0) = K(\overline{F}_1(t), \overline{F}_2(t), 1),\ldots$
$\overline{F}_T(t) = Q_{\phi,K}(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t))$ where
$Q_{\phi,K}(u_1,u_2,u_3) = K(u_1,u_2,1) + K(u_1,1,u_3) - K(u_1,u_2,u_3).$

Example-general case



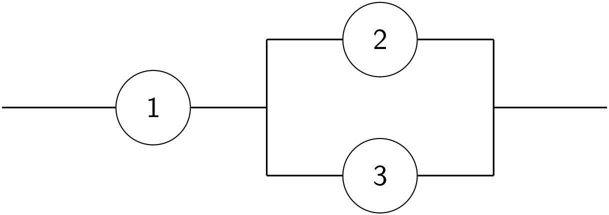
EXC: $\overline{F}_T(t) = 2\overline{F}_{1:2}(t) - \overline{F}_{1:3}(t) = q_{\phi,K}(\overline{F}(t))$,
where $q_{\phi,K}(u) = 2K(u,u,1) - K(u,u,u)$.
Minimal signature $\mathbf{a} = (0,2,-1)$.

Example-general case



IID: $\overline{F}_T(t) = 2\overline{F}^2(t) - \overline{F}^3(t) = q_\phi(\overline{F}(t))$,
where $q_\phi(u) = 2u^2 - u^3$.

Example-general case



The minimal signatures for $n \leq 5$ can be seen in:
Navarro and Rubio (2010, Comm Stat Simul Comp 39, 68–84).

Generalized Order Statistics (GOS)

- For an arbitrary DF F, GOS $X_{1:n}^{GOS}, \dots, X_{n:n}^{GOS}$ based on F can be obtained (Kamps, 1995, B. G. Teubner Stuttgart, p.49) via the quantile transformation

$$X_{r:n}^{GOS} = F^{-1}(U_{r:n}^{GOS}), \quad r = 1, \dots, n,$$

where $(U_{1:n}^{*}, \dots, U_{n:n}^{*})$ has the joint PDF

$$g^{GOS}(u_1, \dots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1 - u_i)^{m_i} \right) (1 - u_n)^{k-1}$$

for $0 \leq u_1 \leq \dots \leq u_n < 1$, $n \geq 2$, $k \geq 1$, $\gamma_1, \dots, \gamma_n > 0$ and $m_i = \gamma_i - \gamma_{i+1} - 1$.

Generalized Order statistics (GOS)

- If $\gamma_1, \ldots, \gamma_n$ are pairwise different, then

$$F_{r:n}^{GOS}(t) = 1 - c_{r-1} \sum_{i=1}^{r} \frac{a_{i,r}}{\gamma_i} (1 - F(t))^{\gamma_i} = q_{r:n}^{GOS}(F(t))$$

with the constants

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad a_{i,r} = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n$$

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Particular cases of GOS

- The GOS include:
- OS, IID case ($m_1 = \cdots = m_{n-1} = 0$ and $k = 1$).
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Preservation results

- If q_1 and q_2 are two DF,

$$q_1(F) \leq_{ord} q_2(F) \text{ for all } F?$$

- If q is a DF,

$$F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)?$$

- If Q_1 and Q_2 are two MDF,

$$Q_1(F_1, \dots, F_n) \leq_{ord} Q_2(F_1, \dots, F_n)?$$

- If Q is a MDF,

$$F_i \leq_{ord} G_i, i = 1, \dots, n, \Rightarrow Q(F_1, \dots, F_n) \leq_{ord} Q(G_1, \dots, G_n)?$$

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Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all t.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all t.
- Then

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- If T_i has the DD $q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $q_1(u)/q_2(u) \geq 1$ in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all F if and only if $\overline{q}_2/\overline{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ $(\geq_{LR})$ for all F if and only if $q_2(q_1^{-1}(u))$ is concave (convex) in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F if and only if $q_1'(u)/q_2'(u)$ decreases.
- **NEW** $T_1 \leq_{MRL} T_2$ for all F if $E(T_1) \leq E(T_2)$ and $\overline{q}_2/\overline{q}_1$ is bathtub in $(0, 1)$.

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- If $\alpha(u)$ is decreasing in $(0,1)$, then

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  where $\alpha(u) = uq'(1-u)/(1-q(1-u)) = u\overline{q}'(u)/\overline{q}(u)$.
- If $\beta_q(u)$ is decreasing and nonnegative in $(0,1)$, then

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$$\alpha_i^{\Phi}(u_1, \ldots, u_n) = \frac{u_i D_i \Phi(u_1, \ldots, u_n)}{\Phi(u_1, \ldots, u_n)} \tag{2.1}$$

  and $D_i \Phi(u_1, \ldots, u_n) = \frac{\partial}{\partial u_i} \Phi(u_1, \ldots, u_n)$.
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- $G_1 \leq_{RHR} G_2$ for all $F_1, \ldots, F_n$ if and only if Q_2/Q_1 is increasing in $(0,1)^n$.

Preservation of stochastic orders-GDD^{NEW}

- If $G_i = Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, then:
- $G_1 \leq_{ST} G_2$ for all $F_1, \ldots, F_n$ if and only if $Q_1/Q_2 \geq 1$ in $(0, 1)^n$.
- $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if and only if $\overline{Q}_2/\overline{Q}_1$ is decreasing in $(0, 1)^n$.
- $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if $\alpha_i^{\overline{Q}_1} \geq \alpha_i^{\overline{Q}_2}$ in $(0, 1)^n$ for $i = 1, \ldots, n$, where

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- If $F_i \leq_{HR} G_i$ for $i = 1, \dots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF Q such that $\alpha_i^{\overline{Q}}$ is decreasing in $(0,1)^n$ for $i = 1, \dots, n$.
- If $F_i \leq_{HR} G_i$ for $i = 1, \dots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF Q such that

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is decreasing in $u_1, \dots, u_n$ when $0 < u_i < 1$ and $0 < u_i v_i < 1$.

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- If $F_i \leq_{RHR} G_i$ for $i = 1, \ldots, n$, then $F_Q \leq_{RHR} G_Q$ for all MDF Q such that α_i^Q is decreasing in $(0,1)^n$ for $i = 1, \ldots, n$.
- If $F_i \leq_{LR} G_i$ and F_i is IHR (DHR) for $i = 1, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF Q such that

$$\gamma^{\overline{Q}} = \frac{w_1 z_1 u_1 D_1 \overline{Q}(u_1 v_1, \ldots, u_n v_n) + \cdots + w_n z_n u_n D_n \overline{Q}(u_1 v_1, \ldots, u_n v_n)}{z_1 u_1 D_1 \overline{Q}(u_1, \ldots, u_n) + \cdots + z_n u_n D_n \overline{Q}(u_1, \ldots, u_n)}$$

is decreasing in $u_1, \ldots, u_n$, increasing in $v_1, \ldots, v_n, w_1, \ldots, w_n$ and increasing (decreasing) in z_i in $(0,1)^n \times (1,\infty) \times (0,\infty)^{2n}$.

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is decreasing in u_1 and decreasing (increasing) in v_i, $i = 1, \dots, n$.
- If $F_1 \le_{LR} G_1$ and $F_1 \le_{LR} F_i$ $(\ge_{LR})$ for $i = 2, \dots, n$, then $F_Q \le_{LR} G_Q$ for all MDF Q such that

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is decreasing in u_1, increasing in v_1 and increasing (decreasing) in v_i and w_i for $i = 2, \dots, n$.

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Preservation results of aging classes

- Let $\mathcal{C}$ be an aging class.
- If q is a distorted function,

$$F \in \mathcal{C} \Rightarrow q(F) \in \mathcal{C}?$$

- If Q is a multivariate distorted function,

$$F_i \in \mathcal{C}, i = 1, \dots, n, \Rightarrow Q(F_1, \dots, F_n) \in \mathcal{C}?$$

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Stochastic aging classes

- X is Increasing (Decreasing) Hazard Rate IHR (DHR) if h is increasing (decreasing).
- X is IHR $\Leftrightarrow (X - s|X > s) \geq_{ST} (X - t|X > t)$ for all $s < t$.
- X is New Better (Worse) than Used NBU (NWU) if $X \geq_{ST} (X - t|X > t)$ $(\leq_{ST})$ for all $t > 0$.
- X is Increasing (Decreasing) Likelihood Ratio ILR (DLR) if f is log-concave (log-convex).
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Preservation of Stochastic aging classes DD

- Let $F_q = q(F)$ and $\alpha(u) = u\overline{q}'(u)/\overline{q}(u)$. Then:
- The IHR class is preserved (i.e. F_q is IHR for all F IHR) if and only if α is decreasing in $(0,1)$.
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- The NBU (NWU) class is preserved if and only if

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Preservation of Stochastic aging classes

- In the IID case:
- The IHR class and the HR order are preserved for $X_{i:n}$ since $\alpha_{i:n}(u)$ is decreasing (Esary and Proschan 1963, Tech.).
- The DHR class is not necessarily preserved for $X_{i:n}$! It is only preserved for $X_{1:n}$ since $\alpha_{1:n}(u)$ is constant.
- The IHR and DHR classes are not necessarily preserved under the formation of coherent systems! It depends on the system structure.
- In the ID case the IHR class is not necessarily preserved for $X_{i:n}$! It depends on the copula (dependence).

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- Let $F_q = q(F)$ and let

$$\beta(u) = u\overline{q}''(u)/\overline{q}'(u),$$

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Then:

- If F is ILR and there exists $a \in [0,1]$ such that β is non-negative and decreasing in $(0,a)$ and $\overline{\beta}$ is non-positive and decreasing in $(a,1)$, then F_q is ILR.
- If F is DLR with support (l,∞) $(l \geq 0)$, β is non-negative and increasing in $(0,1)$, then F_q is DLR.

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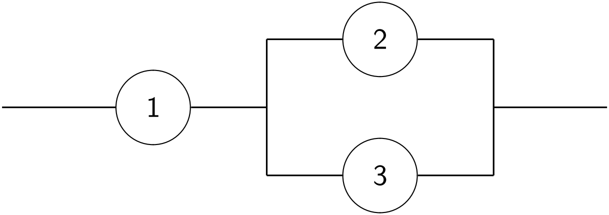

- Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$.
- In the IID case: $q(u) = u + u^2 - u^3$ and $\overline{q}(u) = 2u^2 - 3u^3$.
- Then $\alpha(u) = \frac{4-3u}{2-u}$ is strictly decreasing.
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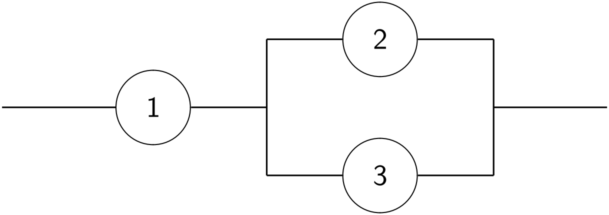

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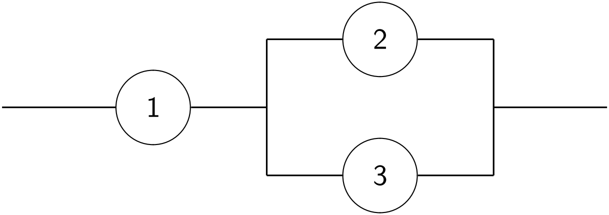

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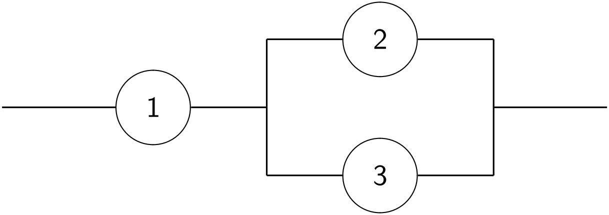

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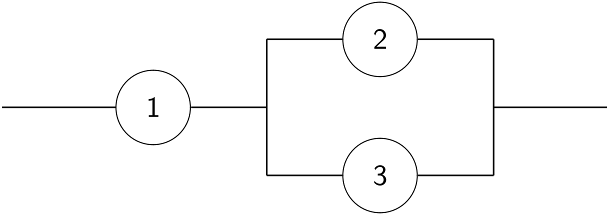

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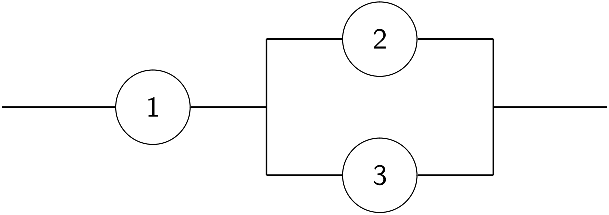

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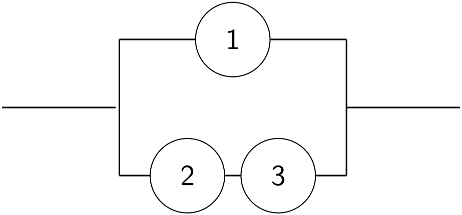

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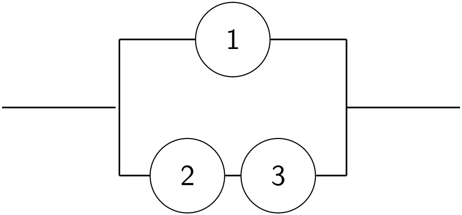

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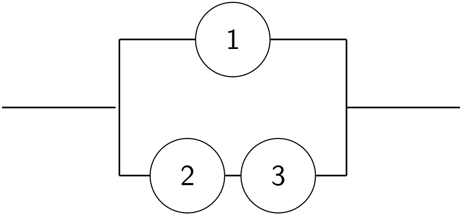

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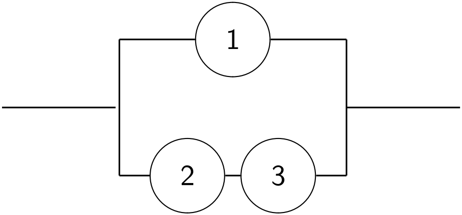

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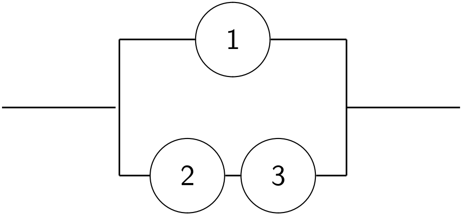

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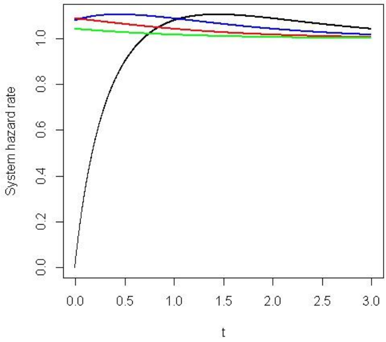

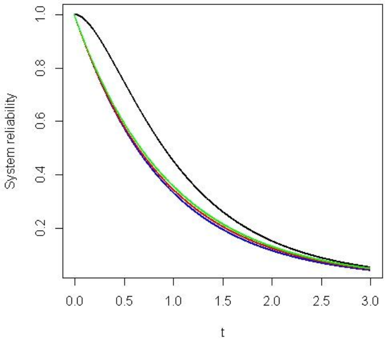

Figure: HR (left) and RF (left) of the residual lifetimes $(T - t | T > t)$ of the system $T = \max(X_1, \min(X_2, X_3))$ when X_i are IID$\sim$ $Exp(\mu = 1)$ with $t = 0, 1, 2, 3$ (black, blue, red, green).

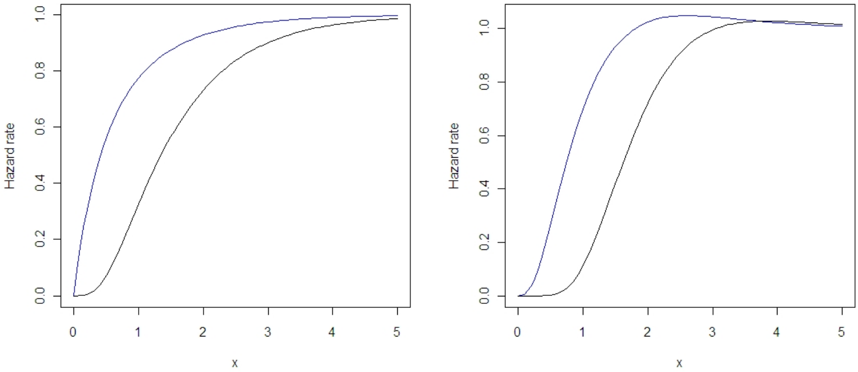

Figure: HR X_1 (left) and $T = \max(X_1, \min(X_2, X_3))$ (right) when X_i are IID with $\overline{F}(t) = 1 - (1 - e^{-t})^a$ for $t > 0$ and $a = 2, 5$ (blue, black).

Example-DID case

- Series system $X_{1:n} = \min(X_1, \dots, X_n)$ with ID components having a Clayton-Oakes survival copula

$$K(u_1, \dots, u_n) = \left(\sum_{i=1}^{n} u_i^{1-\theta} - (n-1) \right)^{1/(1-\theta)}, \quad \theta > 1.$$

- Then

$$\overline{q}(u) = K(u, \dots, u) = (nu^{1-\theta} - n + 1)^{1/(1-\theta)}.$$

- As $\alpha(u) = \frac{n}{n-(n-1)u^{\theta-1}}$ is a strictly increasing function for all $\theta > 1$, the DHR class is preserved for all n.
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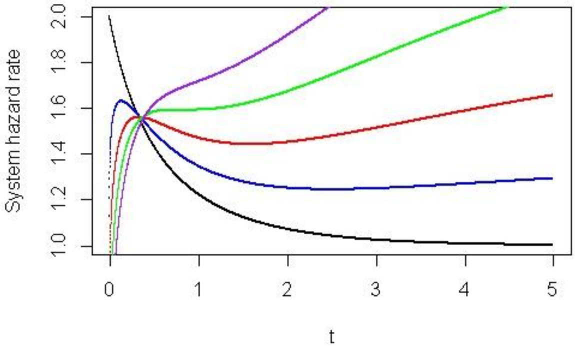

Figure: HR of $T = \min(X_1, X_2)$ when (X_1, X_2) has a C-O survival copula with $\theta = 2$ and $\overline{F}_i(t) = \exp(-t^a)$, $t > 0$, $i = 1, 2$ with $a = 1$ (black, Exponential), $a = 1.1, 1.2, 1.3, 1.4$ (blue, red, green, purple, IHR Weibull).

Example-Parallel system IND case

- Parallel system $X_{2:2} = \max(X_1, X_2)$ with IND components.
- Then $\overline{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$.
- As $\alpha_1^{\overline{Q}}(u_1, u_2) = (u_1 - u_1 u_2)/(u_1 + u_2 - u_1 u_2)$ is increasing in u_1 and decreasing in u_2, then the IHR and DHR classes are not necessarily preserved.
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$$\frac{\overline{Q}_{2:2}(u_1, u_2)}{u_1} = 1 + \frac{u_2}{u_1} - u_2$$

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- For the series system $X_{1:2} = \min(X_1, X_2)$, $\overline{Q}_{1:2}(u) = u_1 u_2$ and

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  is decreasing, then $X_{1:2} \leq_{HR} X_{2:2}$.
- X_1 and $X_{2:2}$ are not always HR-ordered since

$$\frac{\overline{Q}_{2:2}(u_1, u_2)}{u_1} = 1 + \frac{u_2}{u_1} - u_2$$

  is decreasing in u_1 but increasing in u_2.

Example-Parallel system IND case

- Parallel system $X_{2:2} = \max(X_1, X_2)$ with IND components.
- Then $\overline{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$.
- As $\alpha_1^{\overline{Q}}(u_1, u_2) = (u_1 - u_1 u_2)/(u_1 + u_2 - u_1 u_2)$ is increasing in u_1 and decreasing in u_2, then the IHR and DHR classes are not necessarily preserved.
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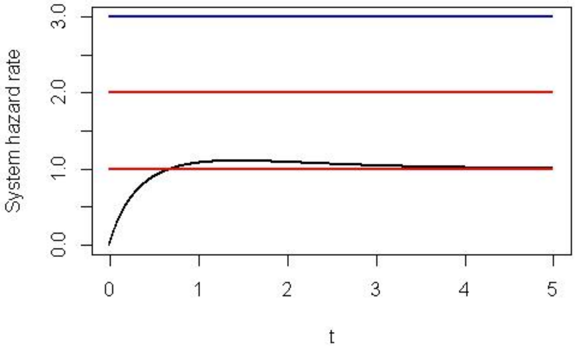

Figure: HR of X_i (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when $X_i \sim Exp(\mu = 1/i)$, $i = 1, 2$. X_i are IHR and DHR but $X_{2:2}$ is neither IHR nor DHR.

Example-Series system IND case

- $X_{1:n} = \min(X_1, \ldots, X_n)$ having DF $F_1, F_2, \ldots, F_n$.
- $Y_{1:n} = \min(Y_1, \ldots, Y_n)$ having DF $G_1, F_2, \ldots, F_n$,
- $F_1 \leq_{LR} G_1$
- If F_1 is DHR and F_i is IHR for $i = 2, \ldots, n$, then $X_{1:n} \leq_{LR} Y_{1:n}$.
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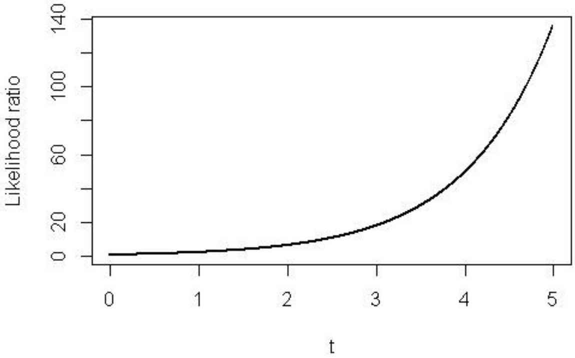

Figure: LR $g_{1:2}/f_{1:2}$ for $X_{1:2} = \min(X_1, X_2)$ and $Y_{1:2} = \min(Y_1, Y_2)$ when $X_1 \sim Exp(\mu = 1/2)$, $Y_1 \sim Exp(\mu = 1)$ and $\Pr(X_2 > t) = \Pr(Y_2 > t) = \exp(-t^2)$ (IHR Weibull).

Parrondo's paradox series systems-IID case

- Parrondo's paradox shows (Game Theory) how, in some games, a random strategy might be better than any deterministic strategy.
- The same paradox holds for coherent systems.
- Let us assume that we have two kind of units with reliability functions $\overline{F}_1 \geq \overline{F}_2$ (in a similar number) to build series systems with two independent units.
- Let $T = \min(X_1, X_2)$ be the system obtained when $\overline{F}_i(t) = \Pr(X_i > t)$, $i = 1, 2$.
- Let S be the system obtained when the units are chosen randomly.
- Then $T \leq_{ST} S$ since

$$\overline{F}_T(t) = \overline{F}_1(t)\overline{F}_2(t) \leq (0.5\overline{F}_1(t) + 0.5\overline{F}_1(t))^2 = \overline{F}_S(t).$$

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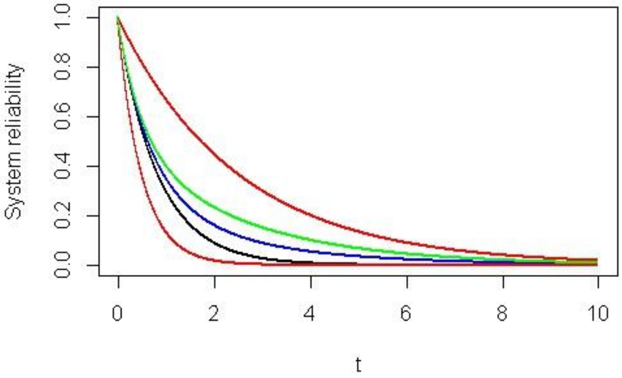

Figure: Reliability functions of systems T (black) and S (blue) when the units have exponential distributions with means 5 and 1.

Parrondo's paradox in other systems

- The same happen with series systems of size n with independent components.
- The orderings are reversed for parallel systems.
- In both cases, we compare the GDD $Q(F_1, \ldots, F_n)$ and $Q(G, \ldots, G)$, where $G = (F_1 + \cdots + F_n)/n$.
- A function $g : \mathbb{R}^n \to \mathbb{R}$ is weakly Schur-concave (convex) if

$$g(u_1, u_2, \ldots, u_n) \leq g(\overline{u}, \overline{u}, \ldots, \overline{u}) \quad (\geq)$$

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Parrondo's paradox

Theorem (Navarro and Spizzichino, ASMBI 2010)

If $(X_1, X_2, \ldots, X_n)$ *and* $(Y_1, Y_2, \ldots, Y_n)$ *have the same copula,* $\overline{F}_i(t) = \Pr(X_i > t)$ *and* $\overline{G}(t) = (\overline{F}_1(t) + \ldots + \overline{F}_n(t))/n = \Pr(Y_i > t)$ *for* $i = 1, \ldots, n$, *and* $\overline{Q}_{\phi,K}$ *is weakly Schur-concave (convex), then*

$$T = \phi(X_1, \ldots, X_n) \leq_{ST} S = \phi(Y_1, \ldots, Y_n) \quad (\geq_{ST}).$$

Parrondo's paradox in other systems

- This theorem can be applied to GDD.
- For $X_{1:n}$ with independent components $\overline{Q}_{1:n}(u_1, \dots, u_n) = u_1 \dots u_n$ which is Schur-concave and so Parrondo's paradox holds.
- For $X_{1:n}$ with dependent components $\overline{Q}_{1:n,K}(u_1, \dots, u_n) = K(u_1, \dots, u_n)$.
- Many copulas are Schur-concave (e.g. Archimedean copulas) and so Parrondo's paradox holds in many series systems.
- However there are copulas which are weakly Schur-convex and hence the ordering can be reversed for series systems (see Navarro and Spizzichino, ASMBI 2010).
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Randomized GDD

- If $\overline{Q}$ is a GDF, we consider the GDD with RF

$$\overline{F}_k(t) = \overline{Q}\big(\underbrace{\overline{F}_X(t), \dots, \overline{F}_X(t)}_{k-\text{times}}, \underbrace{\overline{F}_Y(t), \dots, \overline{F}_Y(t)}_{(n-k)-\text{times}}\big), k = 0, \dots, n \tag{3.1}$$

- Here, e.g., we can assume $X \geq_{ST} Y$.
- The randomized GDD is obtained when the number k of "good components" is chosen randomly according to a discrete random variable K with support included in $\{0, \dots, n\}$.
- It is represented by the random variable T_K.

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Proposition (Navarro, Pellerey and Di Crecenzo, 2014)

If k is chosen randomly according to K_1 or K_2 and

$$\varphi(k)=\overline{Q}(\underbrace{u,\dots,u}_{k-times},\underbrace{v,\dots,v}_{(n-k)-times})$$

is convex (concave) in $\{0,1,\dots,m\}$ for all $u,v\in(0,1)$, then:
(i) $K_1\le_{CX}K_2$ implies $T_{K_1}\le_{ST}T_{K_2}$ ($\ge_{ST}$).
(ii) $X\ge_{ST}Y$ and $K_1\le_{ICX}K_2$ ($\le_{ICV}$) imply $T_{K_1}\le_{ST}T_{K_2}$.

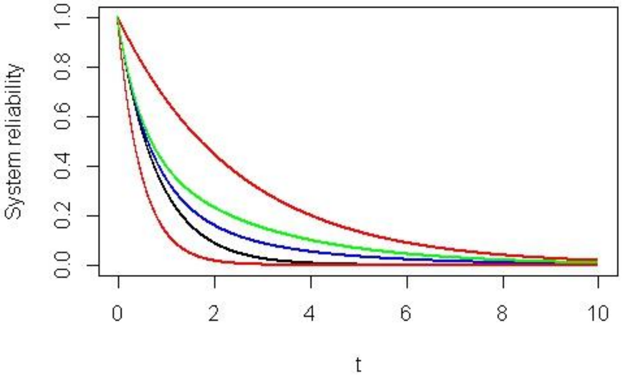

Figure: Reliability functions of systems T (black) and S (blue) when the units have exponential distributions with means 1 and 5.

Parrondo paradox example

- $T = \min(X_1, X_2)$ with $\overline{Q}(u, v) = uv$.
- It is obtained with K_1 such that $\Pr(K_1 = 1) = 1$.
- S is obtained with K_2 such that $\Pr(K_2 = 1) = 1/2$ and $\Pr(K_2 = 0) = \Pr(K_2 = 2) = 1/4$.
- Another reasonable option is obtained with K_3 such that $\Pr(K_3 = i) = 1/3$ for $i = 0, 1, 2$.
- Another option K_4 such that $\Pr(K_4 = 0) = \Pr(K_4 = 2) = 1/2$.
- Note that $E(K_i) = 1$ for $i = 1, 2, 3, 4$.
- As $\varphi(k) = u^k v^{1-k}$ is convex and $K_1 \leq_{CX} K_2 \leq_{CX} K_3 \leq_{CX} K_4$, then
$$\overline{F}_{K_1} \leq_{ST} \overline{F}_{K_2} \leq_{ST} \overline{F}_{K_3} \leq_{ST} \overline{F}_{K_4}.$$
- Actually, K_4 is the best option (the most convex) whenever $E(K) = 1$.

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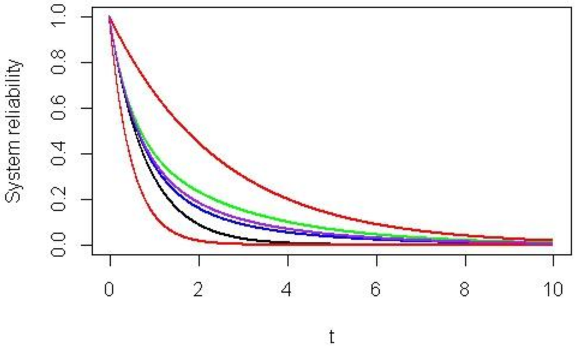
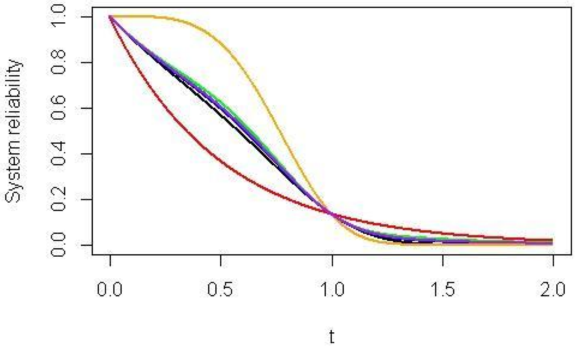


Figure: Reliability functions of systems $T = T_{K_1}$ (black), $S = T_{K_2}$ (blue), T_{K_3} (purple) and T_{K_4} (green) when the units have exponential distributions with means 5 and 1.



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References

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