Mixture representations for the joint distribution of lifetimes of two coherent systems with shared components

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Outline

Coherent systems

- A **coherent system** is

$$\psi = \psi(x_1, \ldots, x_n) : \{0, 1\}^n \to \{0, 1\},$$

where $x_i \in \{0, 1\}$ (it represents the state of the ith component) and where ψ (which represents the state of the system) is increasing in $x_1, \ldots, x_n$ and strictly increasing in x_i for at least a point $(x_1, \ldots, x_n)$, for all $i = 1, \ldots, n$.

- If $X_1, \ldots, X_n$ are the component lifetimes, then there exists ϕ such that the system lifetime $T = \phi(X_1, \ldots, X_n)$.

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Order statistics (OS)

- $X_1, \dots, X_n$ IID$\sim F$ random variables.
- $X_1, \dots, X_n$ exchangeable (EXC), i.e., for any permutation σ

$$(X_1, \dots, X_n) =_{ST} (X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

- Let $X_{1:n}, \dots, X_{n:n}$ be the associated OS which represent the lifetimes of k-out-of-n systems.
- $X_{1:n}$ is the series system lifetime and $X_{n:n}$ is the parallel system lifetime.
- Let $F_{i:n}(t) = \Pr(X_{i:n} \leq t)$ be the DF.
- Let $\overline{F}_{i:n}(t) = \Pr(X_{i:n} > t)$ be the RF.

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Mixture representation

- Samaniego (IEEE TR, 1985), IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} p_i \overline{F}_{i:n}(t), \tag{1.1}$$

where $p_i = \Pr(T = X_{i:n})$ and $\overline{F}_{i:n}(t) = \Pr(X_{i:n} > t)$.

- $\mathbf{p} = (p_1, \ldots, p_n)$ is the signature of the system.
- IID case: p_i only depends on ϕ

$$p_i = \frac{\left|\{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)}\}\right|}{n!} \tag{1.2}$$

- Navarro, Samaniego, Balakrishnan and Bhathacharya (NRL, 2008), (1.1) holds for EXC r.v. when $\mathbf{p}$ is given by (1.2).

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Generalized mixture representation

- Navarro, Ruiz and Sandoval (CSTM, 2007), EXC case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t). \tag{1.3}$$

- $\mathbf{a} = (a_1, \ldots, a_n)$ is the minimal signature of T.
- a_i only depends on ϕ but can be negative and so (1.3) is called a generalized mixture.
- In the IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}^i(t) = \overline{q}_\phi(\overline{F}(t)), \tag{1.4}$$

$\overline{q}_\phi(x) = \sum_{i=1}^{n} a_i x^i$ is the domination (reliability) polynomial.

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Mixture representations order n

- Navarro et al.(NRL, 2008): If $T=\phi(X_1,\dots,X_m)$ and $X_1,\dots,X_n$ $(m<n)$ are IID, then

$$\overline{F}_T(t)=\sum_{i=1}^{n}p_i^{(n)}\overline{F}_{i:n}(t), \tag{1.5}$$

where $p_i^{(n)}=\Pr(T=X_{i:n})$.

- $\mathbf{p}^{(n)}=(p_1^{(n)},\dots,p_n^{(n)})$ is the signature of order n.
- $p_i^{(n)}$ only depends on ϕ

$$p_i^{(n)}=\frac{\left|\{\sigma:\phi(x_1,\dots,x_n)=x_{i:n},\ \text{when } x_{\sigma(1)}<\dots<x_{\sigma(n)}\}\right|}{n!} \tag{1.6}$$

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Example

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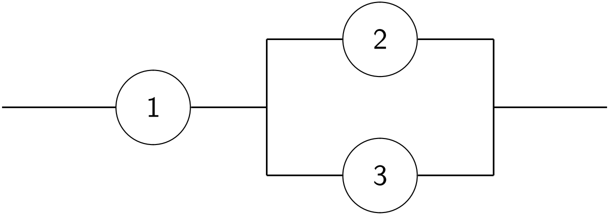

Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$.

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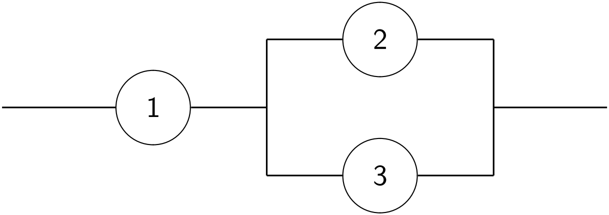
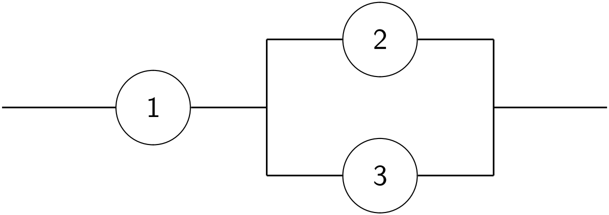


$3! = 6$ permutations.

Example



$$X_1 < X_2 < X_3 \Rightarrow T = X_1 = X_{1:3}$$

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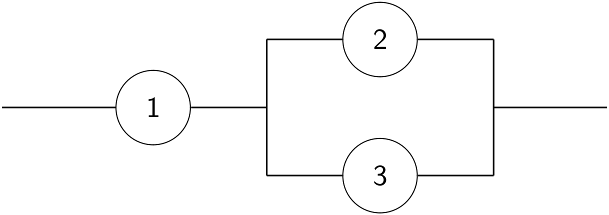
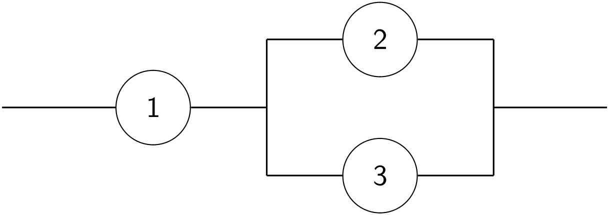


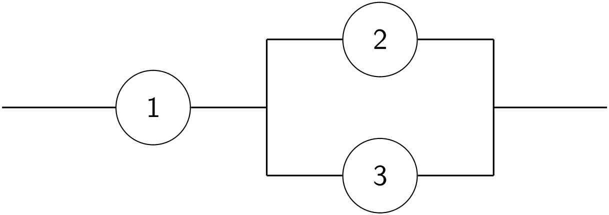
$$X_1 < X_3 < X_2 \Rightarrow T = X_1 = X_{1:3}$$

Example



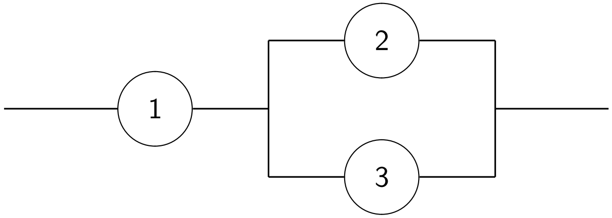
$$X_2 < X_1 < X_3 \Rightarrow T = X_1 = X_{2:3}$$

Example



$$X_2 < X_3 < X_1 \Rightarrow T = X_3 = X_{2:3}$$

Example



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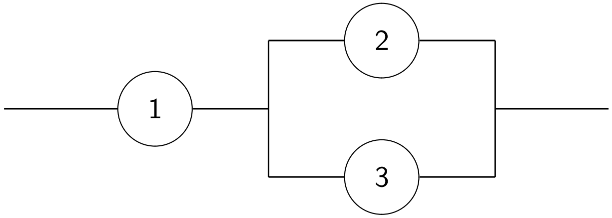

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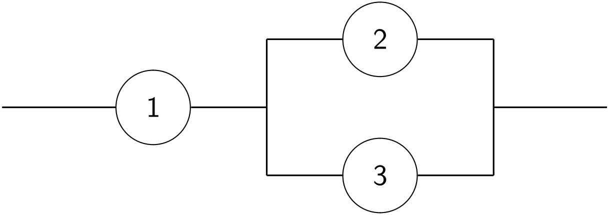
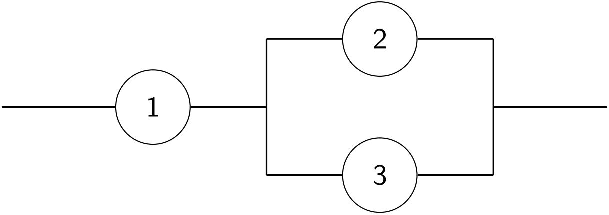


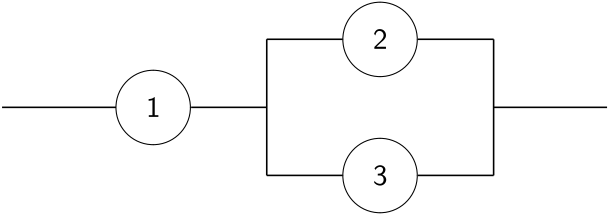
IID $\overline{F}$ cont.: $\mathbf{p} = (2/6, 4/6, 0) = (1/3, 2/3, 0)$.

Example



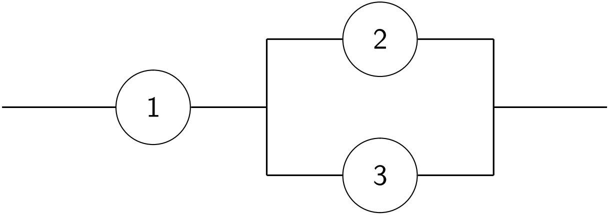
IID or EXC: $\overline{F}$ cont.: $\overline{F}_T(t) = \frac{1}{3}\overline{F}_{1:3}(t) + \frac{2}{3}\overline{F}_{2:3}(t)$.

Example



IID or EXC: $\overline{F}_T(t) = 2\overline{F}_{1:2}(t) - \overline{F}_{1:3}(t)$,
where $\mathbf{a} = (0, 2, -1)$ is the minimal signature.

Example



IID: $\overline{F}_T(t) = 2\overline{F}^2(t) - \overline{F}^3(t) = q_\phi(\overline{F}(t))$,
where $q_\phi(u) = 2u^2 - u^3$.

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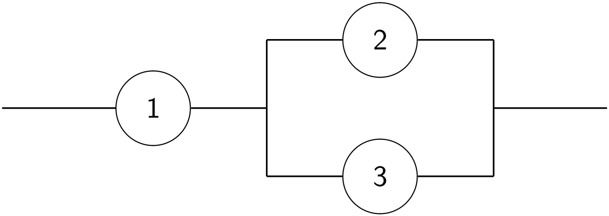
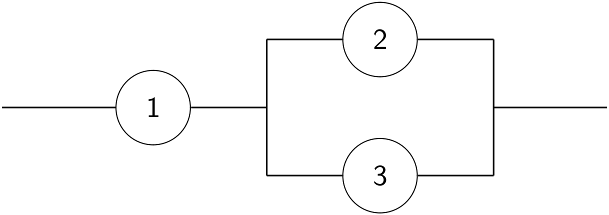


The minimal signatures for systems with $n \leq 5$ can be seen in: Navarro and Rubio (2010, Comm Stat Simul Comp 39, 68–84).

Signature of order n



Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$ from X_1, X_2, X_3, X_4.

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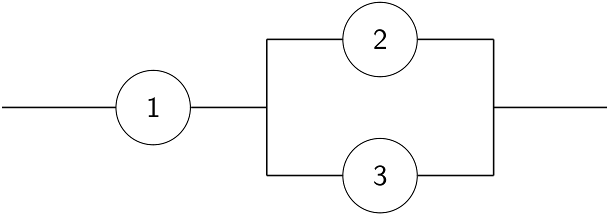
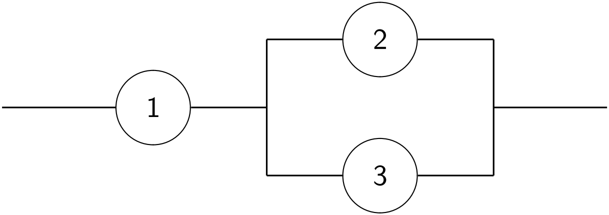


$4! = 24$ permutations.

Signature of order n



$$X_1 < X_2 < X_3 < X_4 \Rightarrow T = X_1 = X_{1:4}$$

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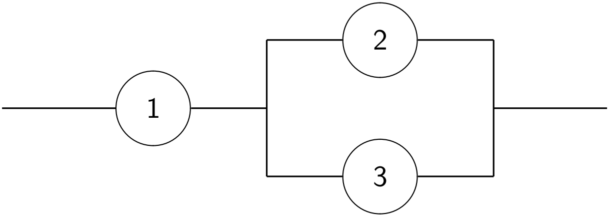
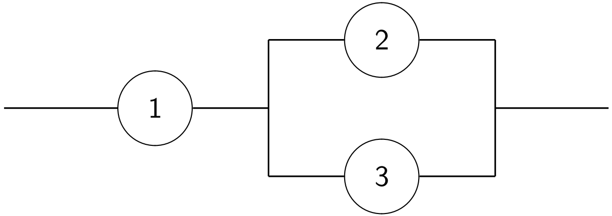


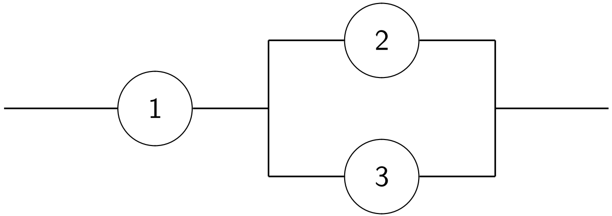
$3! = 6$ permutations lead to $T = X_1 = X_{1:4}$

Signature of order n



The signature of order 4 is
$(6/24, 10/24, 8/24, 0) = (1/4, 5/12, 1/3, 0)$.

Signature of order n



The signatures of order 5 and minimal signatures for systems with $n \leq 5$ can be seen in: Navarro and Rubio (2010, Comm Stat Simul Comp 39, 68–84).

Bivariate Signature Matrix (BSM)

- T_1 and T_2 are the lifetimes of two coherent systems based on components with IID lifetimes $X_1, \ldots, X_n$ with a continuous DF F.
- Then $\Pr(X_{1:n} < \ldots < X_{n:n}) = 1$.
- The two systems may share one or more components.
- The systems may be of order less than n.
- We define the random vector $\mathbf{I} = (I_1, I_2)$ by

$$\mathbf{I} = (i, j) \text{ whenever } T_1 = X_{i:n} \text{ and } T_2 = X_{j:n}. \qquad (1.7)$$

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Bivariate Signature Matrix (BSM)

- T_1 and T_2 are the lifetimes of two coherent systems based on components with IID lifetimes $X_1, \ldots, X_n$ with a continuous DF F.
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- The bivariate probability mass function of $\mathbf{I}$ is denoted by $p_{i,j} = \Pr(\mathbf{I} = (i,j))$, for $i,j = 1,\ldots,n$.
- Note that

$$p_{i,j} = |A_{i,j}|/n!, \tag{1.8}$$

where $|A_{i,j}|$ is the size of the set

$$A_{i,j} = \{\sigma \in \mathcal{P}_n : T_1 = X_{i:n} \text{ and } T_2 = X_{j:n} \text{ when } X_{\sigma(1)} < \cdots < X_{\sigma(n)}\}$$

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Immediate properties

- The BSM $P = (p_{i,j})$ does not depend on F and can be computed using (1.8).
- Of course, $p_{i,j} \geq 0$ and $\sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j} = 1$.
- The univariate signature $(p_1, \ldots, p_n)$ of order n of T_1, can be computed from the BSM as $p_i = \sum_{j=1}^{n} p_{i,j}$. A similar result holds for T_2.
- If $T_2 = X_{k:n}$ then $p_{i,k} = p_i$ and $p_{i,j} = 0$ for $i = 1, \ldots, n$ and $j \neq k$. In this case, I_1 and I_2 are independent.

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Example

- Let X_1, X_2, X_3, X_4 be the IID lifetimes of four components.
- $T_1 = X_{2:3} = \min(\max(X_1, X_2), \max(X_1, X_3), \max(X_2, X_3))$.
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- There are $4! = 24$ permutations. Then:

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- From the above, the bivariate signature matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 0 \\ 1/3 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The marginal probability mass function of I_1 is $(0, 1/2, 1/2, 0)$ and that of I_2 is $(1/2, 1/3, 1/6, 0)$.
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Main results

Theorem (Navarro, Samaniego and Balakrishnan, Adv. Appl. Prob., 2013)

Let T_1 and T_2 be the lifetimes of two coherent systems based IID (or EXC) components with lifetimes $X_1, \dots, X_n$ with a common continuous DF F. Then, the joint distribution function $G(t_1, t_2) = \Pr(T_1 \leq t_1, T_2 \leq t_2)$ of (T_1, T_2) can be written as

$$G(t_1, t_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j} F_{i,j:n}(t_1, t_2), \tag{1.9}$$

where $P = (p_{i,j})$ is the bivariate signature matrix of (T_1, T_2) and $F_{i,j:n}(t_1, t_2) = \Pr(X_{i:n} \leq t_1, X_{j:n} \leq t_2)$.

Jorge Navarro, E-mail: jorgenav@um.es

Main results

Theorem (Navarro, Samaniego and Balakrishnan, J. Appl. Prob., 2010)

The joint distribution G of T_1 and T_2 based on IID components with lifetimes $X_1, \dots, X_n$ can be written as

$$G(t_1, t_2) = \sum_{i=1}^{n} \sum_{j=0}^{n} s_{i,j} F_{i:n}(t_1) F_{j:n}(t_2) \text{ for } t_1 \leq t_2 \tag{1.10}$$

$$G(t_1, t_2) = \sum_{i=0}^{n} \sum_{j=1}^{n} s^*_{i,j} F_{i:n}(t_1) F_{j:n}(t_2) \text{ for } t_1 > t_2, \tag{1.11}$$

where $F_{0:n} = 1$ (by convention) and $\{s_{i,j}\}$ and $\{s^*_{i,j}\}$ are collections of coefficients (which do not depend on F) such that $\sum_{i=1}^{n} \sum_{j=0}^{n} s_{i,j} = \sum_{i=0}^{n} \sum_{j=1}^{n} s^*_{i,j} = 1$.

Consequences

- (T_1, T_2) has a singular part whenever $\Pr(T_1 = T_2) > 0$.
- In the IID case, if F is absolutely continuous, then $F_{i:n}(t_1)F_{j:n}(t_2)$ and $F_{i,j:n}(t_1, t_2)$ are both absolutely continuous bivariate distributions when $i \neq j$.
- So, in the second theorem, we need two different linear combinations (one for $t_1 \leq t_2$ and another one for $t_1 > t_2$) based on $F_{i:n}(t_1)F_{j:n}(t_2)$.
- However, in the first theorem, note that

$$F_{i,i:n}(t_1, t_2) = \Pr(X_{i:n} \leq t_1, X_{i:n} \leq t_2) = F_{i:n}(\min(t_1, t_2))$$

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- Therefore, inthe IID case, G is absolutely continuous if and only if $p_{i,i} = 0$ for all $i = 1, \dots, n$.
- In this case, its PDF g can be written as

$$g(t_1, t_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j} f_{i,j:n}(t_1, t_2),$$

where $f_{i,j:n}$ is the PDF of $(X_{i:n}, X_{j:n})$ for $i \neq j$.
- A similar representation holds the joint reliability function of (T_1, T_2) with the same coefficients.
- The functions $F_{i:n}, F_{i,j:n}, \overline{F}_{i,j:n}$ and $f_{i,j:n}$ can all be computed from F using the expressions known in the theory of order statistics.
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Consequences

Theorem

If T_1 and T_2 have respective signatures $(p_1, ..., p_n)$ and $(p_1^*, ..., p_n^*)$ of order n and BSM $P = (p_{i,j})$, then

$$E(T_1 T_2) = \sum_{i=1}^{n} p_{i,i}\alpha_{i,i:n} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (p_{i,j} + p_{j,i})\alpha_{i,j:n}$$

$$Cov(T_1, T_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j}\sigma_{i,j:n} + \sum_{i=1}^{n} \sum_{j=1}^{n}(p_{i,j} - p_i p_j^*)\mu_{i:n}\mu_{j:n},$$

where $\mu_{i:n} = E(X_{i:n})$, $\alpha_{i,j:n} = E(X_{i:n}X_{j:n})$, $\sigma_{i,j:n} = Cov(X_{i:n}, X_{j:n})$ and $\sigma_{i,i:n} = \sigma^2_{i:n} = Var(X_{i:n})$ for $i, j = 1, \ldots, n$.

Consequences

- If $T_2 = X_{k:n}$, then

$$Cov(T_1, X_{k:n}) = \sum_{i=1}^{k-1} p_i \sigma_{i,k:n} + p_j \sigma^2_{k:n} + \sum_{i=k+1}^{n} p_i \sigma_{i,k:n}.$$

- If F is exponential and the signature of order n is $(0,\ldots,0,p_k,\ldots,p_n)$, then

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The multivariate stochastic order

- Let **X** and **Y** be two *n*-dimensional random vectors.
- We say that $\mathbf{X} \leq_{ST} \mathbf{Y}$ if $E(\phi(\mathbf{X})) \leq E(\phi(\mathbf{Y}))$ for all increasing real-valued functions ϕ for which that these expectations exist.
- $\mathbf{X} \leq_{ST} \mathbf{Y}$ implies

$$\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) \geq \Pr(X_1^* \leq x_1, \ldots, X_n^* \leq x_n) \tag{2.1}$$

(lower orthant ordering) and

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) \geq \Pr(X_1^* > x_1, \ldots, X_n^* > x_n) \tag{2.2}$$

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The south-east order

Definition

Let $A = (a_{i,j})$ and $A^* = (a^*_{i,j})$ be two $n \times m$ matrices with the same total mass, that is, with $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{m} a^*_{i,j}$. Then we say that A is less than A^* in the **south-east shift order** (shortly written as $A \leq_{S/E\rightarrow} A^*$) if A^* can be obtained from A through a finite sequence of transformations in which a positive mass $c > 0$ is moved from the term $a_{i,j}$ to the term $a_{r,s}$ with $r \geq i$ and $s \geq j$ (i.e., the new terms are $a_{i,j} - c$ and $a_{r,s} + c$, respectively).

Example

The following matrices are $S/E \rightarrow$ ordered:

$$\begin{pmatrix} 0 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1/6 & 1/3 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1/6 & 1/6 \\ 0 & 1/2 & 1/6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 1/6 & 1/6 \\ 0 & 1/6 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Main results

Theorem

Let T_1 and T_2 be the lifetimes of two coherent systems whose respective component lifetimes are subsets of $\{X_1, \ldots, X_n\}$ and $(X_1, \ldots, X_n)$ is an exchangeable random vector. Let T_1^* and T_2^* be the lifetimes of two coherent systems whose respective component lifetimes are subsets of $\{X_1^*, \ldots, X_n^*\}$ and $(X_1^*, \ldots, X_n^*)$ is an exchangeable random vector. If $P \leq_{S/E\rightarrow} P^*$ and

$$(X_1, \ldots, X_n) \leq_{ST} (X_1^*, \ldots, X_n^*),$$

then $(T_1, T_2) \leq_{ST} (T_1^*, T_2^*)$.

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Example 1

Let $T_1 = \min(X_1, \max(X_2, X_3))$ and $T_2 = \max(X_1, \min(X_2, X_3))$. Then:

Equiprobable Orderings	T_1	T_2	$\mathbf{I}$
$X_1 < X_2 < X_3$	$X_1 = X_{1:3}$	$X_2 = X_{2:3}$	$(1, 2)$
$X_1 < X_3 < X_2$	$X_1 = X_{1:3}$	$X_3 = X_{2:3}$	$(1, 2)$
$X_2 < X_1 < X_3$	$X_1 = X_{2:3}$	$X_1 = X_{2:3}$	$(2, 2)$
$X_2 < X_3 < X_1$	$X_3 = X_{2:3}$	$X_1 = X_{3:3}$	$(2, 3)$
$X_3 < X_1 < X_2$	$X_1 = X_{2:3}$	$X_1 = X_{2:3}$	$(2, 2)$
$X_3 < X_2 < X_1$	$X_2 = X_{2:3}$	$X_1 = X_{3:3}$	$(2, 3)$

Example 1

- Hence, the bivariate signature of (T_1, T_2) is

$$P = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

- The joint distribution is

$$G(t_1,t_2) = \frac{1}{3}F_{1,2:3}(t_1,t_2) + \frac{1}{3}F_{2,3:3}(t_1,t_2) + \frac{1}{3}F_{2:3}(\min(t_1,t_2)).$$

- G is not absolutely continuous since $\Pr(T_1 = T_2) = p_{2,2} = 1/3$.
- The usual signatures are $(1/3, 2/3, 0)$ and $(0, 2/3, 1/3)$.

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Example 2

- Let $T_1 = X_{1:3}$ and $T_2 = \max(X_1, \min(X_2, X_3))$, then

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- The joint distribution is

$$G(t_1, t_2) = \frac{2}{3} F_{1,2:3}(t_1, t_2) + \frac{1}{3} F_{1,3:3}(t_1, t_2).$$

- If X_1, X_2, X_3 are IID and F is abs. cont., then G is abs. cont. since $\Pr(T_1 = T_2) = 0$ and

$$Cov(X_{1:3}, T_2) = \frac{2}{3}\sigma_{1,2:3} + \frac{1}{3}\sigma_{1,3:3}.$$

- If F is exponential, then

$$Cov(X_{1:3}, T_2) = \sigma_{1,1:3} = Var(X_{1:3}) = \frac{1}{9}\mu^2.$$

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Example 3

- Let $T_1 = X_{1:3}$ and $T_2 = \max(X_1, \min(X_2, X_3))$, then the BSM is

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Let $T_1^* = \min(X_1^*, \max(X_2^*, X_3^*))$ and $T_2^* = \max(X_1^*, \min(X_2^*, X_3^*))$, then the BSM is

$$P^* = \begin{pmatrix} 0 & 1/6 & 1/6 \\ 0 & 1/2 & 1/6 \\ 0 & 0 & 0 \end{pmatrix}.$$

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Example 3

- As seen in (2.3), we have $P \leq_{S/E\rightarrow} P^*$.
- If X_1, X_2, X_3 are IID and X_1^*, X_2^*, X_3^* are IID with $X_1 \leq_{ST} X_1^*$, then $(T_1, T_2) \leq_{ST} (T_1^*, T_2^*)$.
- If the components are dependent and EXC and

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Our Main References

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References

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