Stochastic comparisons of conditional distributions based on copula properties

Jorge Navarro[1] and Miguel A. Sordo
Universidad de Murcia and Universidad de Cádiz, Spain.

[1]Supported by Ministerio de Economía, Industria y Competitividad under Grant MTM2016-79943-P (AEI/FEDER).

Distortion functions

- The distorted distributions were introduced in Yaari's dual theory of choice under risk (Econometrica, 1987).

Distortion functions

- The distorted distributions were introduced in Yaari's dual theory of choice under risk (Econometrica, 1987).
- The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous **distortion function** $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \tag{1.1}$$

Distortion functions

- The distorted distributions were introduced in Yaari's dual theory of choice under risk (Econometrica, 1987).
- The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous **distortion function** $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \tag{1.1}$$

- For the reliability functions (RF) $\bar{F} = 1 - F$, $\bar{F}_q = 1 - F_q$, we have

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \tag{1.2}$$

where $\bar{q}(u) = 1 - q(1-u)$ is the **dual distortion function**; see Hürlimann (N Am Actuarial J, 2004).

Multivariate distortion functions

- The **generalized distorted distribution** (GDD) associated to n DF $F_1, \ldots, F_n$ and to an increasing continuous **multivariate distortion function** $Q : [0,1]^n \to [0,1]$ such that $Q(0,\ldots,0) = 0$ and $Q(1,\ldots,1) = 1$, is

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \tag{1.3}$$

Multivariate distortion functions

- The **generalized distorted distribution** (GDD) associated to n DF $F_1, \ldots, F_n$ and to an increasing continuous **multivariate distortion function** $Q : [0,1]^n \to [0,1]$ such that $Q(0, \ldots, 0) = 0$ and $Q(1, \ldots, 1) = 1$, is

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \tag{1.3}$$

- For the RF we have

$$\bar{F}_Q(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \tag{1.4}$$

where $\bar{F} = 1 - F$, $\bar{F}_Q = 1 - F_Q$ and $\bar{Q}(u_1, \ldots, u_n) = 1 - Q(1 - u_1, \ldots, 1 - u_n)$ is the **multivariate dual distortion function**; see Navarro et al. (ASMBI, 2014).

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1 - u)^\alpha$.

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.
- Proportional reversed hazard rate (PRHR) model $F_\alpha = F^\alpha$, $\alpha > 0$ with $q(u) = u^\alpha$ and $\bar{q}(u) = 1 - (1-u)^\alpha$.

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.
- Proportional reversed hazard rate (PRHR) model $F_\alpha = F^\alpha$, $\alpha > 0$ with $q(u) = u^\alpha$ and $\bar{q}(u) = 1 - (1-u)^\alpha$.
- Mixtures $F = p_1 F_1 + \cdots + p_n F_n$, $p_i \geq 0$ and $\sum_i p_i = 1$ with

$$Q(\boldsymbol{u}) = \bar{Q}(\boldsymbol{u}) = p_1 u_1 + \cdots + p_n u_n.$$

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.
- Proportional reversed hazard rate (PRHR) model $F_\alpha = F^\alpha$, $\alpha > 0$ with $q(u) = u^\alpha$ and $\bar{q}(u) = 1 - (1-u)^\alpha$.
- Mixtures $F = p_1 F_1 + \cdots + p_n F_n$, $p_i \geq 0$ and $\sum_i p_i = 1$ with

$$Q(\boldsymbol{u}) = \bar{Q}(\boldsymbol{u}) = p_1 u_1 + \cdots + p_n u_n.$$

- Order statistics $X_{i:n}$ from a sample of IID $X_1, \ldots, X_n$.

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.
- Proportional reversed hazard rate (PRHR) model $F_\alpha = F^\alpha$, $\alpha > 0$ with $q(u) = u^\alpha$ and $\bar{q}(u) = 1 - (1-u)^\alpha$.
- Mixtures $F = p_1 F_1 + \cdots + p_n F_n$, $p_i \geq 0$ and $\sum_i p_i = 1$ with

$$Q(\boldsymbol{u}) = \bar{Q}(\boldsymbol{u}) = p_1 u_1 + \cdots + p_n u_n.$$

- Order statistics $X_{i:n}$ from a sample of IID $X_1, \ldots, X_n$.
- Record values from IID $X_1, \ldots, X_n$ with

$$\bar{q}_n(u) = u \sum_{i=0}^{n} \frac{1}{i!} (-\ln u)^i.$$

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.
- Proportional reversed hazard rate (PRHR) model $F_\alpha = F^\alpha$, $\alpha > 0$ with $q(u) = u^\alpha$ and $\bar{q}(u) = 1 - (1-u)^\alpha$.
- Mixtures $F = p_1 F_1 + \cdots + p_n F_n$, $p_i \geq 0$ and $\sum_i p_i = 1$ with

$$Q(\boldsymbol{u}) = \bar{Q}(\boldsymbol{u}) = p_1 u_1 + \cdots + p_n u_n.$$

- Order statistics $X_{i:n}$ from a sample of IID $X_1, \dots, X_n$.
- Record values from IID $X_1, \dots, X_n$ with

$$\bar{q}_n(u) = u \sum_{i=0}^{n} \frac{1}{i!} (-\ln u)^i.$$

- Coherent systems $T = \phi(X_1, \dots, X_n)$.

Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1-u)^\alpha$.
- Proportional reversed hazard rate (PRHR) model $F_\alpha = F^\alpha$, $\alpha > 0$ with $q(u) = u^\alpha$ and $\bar{q}(u) = 1 - (1-u)^\alpha$.
- Mixtures $F = p_1 F_1 + \dots + p_n F_n$, $p_i \geq 0$ and $\sum_i p_i = 1$ with

$$Q(\boldsymbol{u}) = \bar{Q}(\boldsymbol{u}) = p_1 u_1 + \dots + p_n u_n.$$

- Order statistics $X_{i:n}$ from a sample of IID $X_1, \dots, X_n$.
- Record values from IID $X_1, \dots, X_n$ with

$$\bar{q}_n(u) = u \sum_{i=0}^{n} \frac{1}{i!} (-\ln u)^i.$$

- Coherent systems $T = \phi(X_1, \dots, X_n)$.
- Conditional distributions from $(X_1, \dots, X_n)$.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y/\bar{F}_X$ increases, hazard rate order.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y/\bar{F}_X$ increases, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y/\bar{F}_X$ increases, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all t.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y/\bar{F}_X$ increases, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X-t|X>t) \leq_{ST} (Y-t|Y>t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X-t|X>t) \leq E(Y-t|Y>t)$ for all t.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ increases, likelihood ratio order.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y/\bar{F}_X$ increases, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all t.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ increases, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow F_Y(t)/F_X(t)$ increases, reversed hazard rate order.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y/\bar{F}_X$ increases, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all t.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ increases, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow F_Y(t)/F_X(t)$ increases, reversed hazard rate order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all t.

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow \bar{F}_Y / \bar{F}_X$ increases, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t | X > t) \leq_{ST} (Y - t | Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t | X > t) \leq E(Y - t | Y > t)$ for all t.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ increases, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow F_Y(t)/F_X(t)$ increases, reversed hazard rate order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X | X < t) \geq_{ST} (t - Y | Y < t)$ for all t.
- Then

$$
\begin{array}{ccccc}
X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\
\Downarrow & & \Downarrow & & \Downarrow \\
X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y)
\end{array}
$$

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $\bar{q}_1 \leq \bar{q}_2$ (or $q_2 \leq q_1$) in $(0, 1)$.

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $\bar{q}_1 \leq \bar{q}_2$ (or $q_2 \leq q_1$) in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all F if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $\bar{q}_1 \leq \bar{q}_2$ (or $q_2 \leq q_1$) in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all F if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all F if and only if q_2/q_1 increases in $(0, 1)$.

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $\bar{q}_1 \leq \bar{q}_2$ (or $q_2 \leq q_1$) in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all F if and only if $\bar{q}_2 / \bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all F if and only if q_2 / q_1 increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F if and only if $\bar{q}'_2 / \bar{q}'_1$ decreases in $(0, 1)$.

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $\bar{q}_1 \leq \bar{q}_2$ (or $q_2 \leq q_1$) in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all F if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all F if and only if q_2/q_1 increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F if and only if $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.
- $T_1 \leq_{MRL} T_2$ for all F such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.

Comparisons of distorted distributions

- If T_i has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \le_{ST} T_2$ for all F if and only if $\bar{q}_1 \le \bar{q}_2$ (or $q_2 \le q_1$) in $(0, 1)$.
- $T_1 \le_{HR} T_2$ for all F if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \le_{RHR} T_2$ for all F if and only if q_2/q_1 increases in $(0, 1)$.
- $T_1 \le_{LR} T_2$ for all F if and only if $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.
- $T_1 \le_{MRL} T_2$ for all F such that $E(T_1) \le E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.
- Navarro et al. ASMBI, 2013 and Navarro and Gomis ASMBI, 2016.

Comparisons of GDD

- If T_i has DF $Q_i(F_1,\dots,F_r)$, $i=1,2$, then:

Comparisons of GDD

- If T_i has DF $Q_i(F_1, \ldots, F_r)$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all $F_1, \ldots, F_r$ if and only if $\bar{Q}_2 - \bar{Q}_1 \geq 0$ in $(0, 1)^r$.

Comparisons of GDD

- If T_i has DF $Q_i(F_1, \ldots, F_r)$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all $F_1, \ldots, F_r$ if and only if $\bar{Q}_2 - \bar{Q}_1 \geq 0$ in $(0,1)^r$.
- $T_1 \leq_{HR} T_2$ for all $F_1, \ldots, F_r$ if and only if $\bar{Q}_2 / \bar{Q}_1$ is decreasing in $(0,1)^r$.

Comparisons of GDD

- If T_i has DF $Q_i(F_1,\dots,F_r)$, $i=1,2$, then:
- $T_1 \leq_{ST} T_2$ for all $F_1,\dots,F_r$ if and only if $\bar{Q}_2 - \bar{Q}_1 \geq 0$ in $(0,1)^r$.
- $T_1 \leq_{HR} T_2$ for all $F_1,\dots,F_r$ if and only if $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^r$.
- $T_1 \leq_{RHR} T_2$ for all $F_1,\dots,F_r$ if and only if Q_2/Q_1 is increasing in $(0,1)^r$.

Comparisons of GDD

- If T_i has DF $Q_i(F_1,\dots,F_r)$, $i=1,2$, then:
- $T_1 \leq_{ST} T_2$ for all $F_1,\dots,F_r$ if and only if $\bar{Q}_2 - \bar{Q}_1 \geq 0$ in $(0,1)^r$.
- $T_1 \leq_{HR} T_2$ for all $F_1,\dots,F_r$ if and only if $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^r$.
- $T_1 \leq_{RHR} T_2$ for all $F_1,\dots,F_r$ if and only if Q_2/Q_1 is increasing in $(0,1)^r$.
- Navarro et al. (Methodology and Computing in Applied Probability, 2016).

Comparisons of GDD

- If T_i has DF $Q_i(F_1,\dots,F_r)$, $i=1,2$, then:
- $T_1 \leq_{ST} T_2$ for all $F_1,\dots,F_r$ if and only if $\bar{Q}_2 - \bar{Q}_1 \geq 0$ in $(0,1)^r$.
- $T_1 \leq_{HR} T_2$ for all $F_1,\dots,F_r$ if and only if $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^r$.
- $T_1 \leq_{RHR} T_2$ for all $F_1,\dots,F_r$ if and only if Q_2/Q_1 is increasing in $(0,1)^r$.
- Navarro et al. (Methodology and Computing in Applied Probability, 2016).
- Comparisons for ordered distributions were given in Navarro and del Águila (Metrika, 2017).

Bounds for distorted distributions

- If T has the RF $\bar{q}(\bar{F}(t))$, then:

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t) \leq \bar{F}(t) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

Bounds for distorted distributions

- If T has the RF $\bar{q}(\bar{F}(t))$, then:

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t) \leq \bar{F}(t) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

- Analogously,

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t) \geq \bar{F}(t) \inf_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

Bounds for distorted distributions

- If T has the RF $\bar{q}(\bar{F}(t))$, then:

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t) \leq \bar{F}(t) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

- Analogously,

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t) \geq \bar{F}(t) \inf_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

- These bounds are sharp.

Bounds for the means of distorted distributions

- Moreover, if $T \geq 0$, then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = \int_0^\infty \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t)dt$$
$$\leq \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u} \int_0^\infty \bar{F}(t)dt = E(X) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u},$$

where X has the RF $\bar{F}$.

Bounds for the means of distorted distributions

- Moreover, if $T \geq 0$, then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = \int_0^\infty \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t)dt$$
$$\leq \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u} \int_0^\infty \bar{F}(t)dt = E(X) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u},$$

where X has the RF $\bar{F}$.

- Analogously

$$E(T) \geq E(X) \inf_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

Bounds for the means of distorted distributions

- Moreover, if $T \geq 0$, then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = \int_0^\infty \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t)dt$$
$$\leq \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u} \int_0^\infty \bar{F}(t)dt = E(X) \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u},$$

where X has the RF $\bar{F}$.

- Analogously

$$E(T) \geq E(X) \inf_{u \in (0,1]} \frac{\bar{q}(u)}{u}.$$

- These bounds are sharp.

Bounds for the means of distorted distributions

Proposition

If T has the RF $\bar{q}(\bar{F}(t))$, then:

$$\inf_{u\in(0,1)} \frac{\bar{q}(u)-u}{2u(1-u)} \le \frac{E(T)-E(X)}{\Delta_F} \le \sup_{u\in(0,1)} \frac{\bar{q}(u)-u}{2u(1-u)} \qquad (1.5)$$

when $0=\inf\{x : F(x)>0\}$ *and the Gini mean difference dispersion measure*

$$\Delta_F = 2\int_0^\infty F(x)(1-F(x))dx$$

is positive. The bounds are sharp.

Copula represenattion

- Let (X, Y) be a bivariate random vector. Then

$$\boldsymbol{F}(x, y) = \Pr(X \leq x, Y \leq y) = C(F(x), G(y)),$$

where $F(x) = \Pr(X \leq x)$ and $G(y) = \Pr(Y \leq y)$ are the marginal DF and C is a copula (i.e., C a continuous distribution function with uniform marginals over $(0, 1)$).

Copula represenattion

- Let (X, Y) be a bivariate random vector. Then

$$\boldsymbol{F}(x, y) = \Pr(X \le x, Y \le y) = C(F(x), G(y)),$$

where $F(x) = \Pr(X \le x)$ and $G(y) = \Pr(Y \le y)$ are the marginal DF and C is a copula (i.e., C a continuous distribution function with uniform marginals over $(0, 1)$).

- The joint reliability function can be represented as

$$\bar{\boldsymbol{F}}(x, y) = \Pr(X > x, Y > y) = \hat{C}(\bar{F}(x), \bar{G}(y)),$$

where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ and $\hat{C}$ is also a copula, called *survival copula*.

Copula represenattion

- Let (X, Y) be a bivariate random vector. Then

$$\boldsymbol{F}(x, y) = \Pr(X \leq x, Y \leq y) = C(F(x), G(y)),$$

where $F(x) = \Pr(X \leq x)$ and $G(y) = \Pr(Y \leq y)$ are the marginal DF and C is a copula (i.e., C a continuous distribution function with uniform marginals over $(0, 1)$).

- The joint reliability function can be represented as

$$\bar{\boldsymbol{F}}(x, y) = \Pr(X > x, Y > y) = \hat{C}(\bar{F}(x), \bar{G}(y)),$$

where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ and $\hat{C}$ is also a copula, called *survival copula*.

- $\hat{C}$ is determined by C (and vice versa) by

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

Conditional distributions

- We consider the following conditional distributions:

Conditional distributions

- We consider the following conditional distributions:
- Case I: $(Y|X \leq x)$ when $F(x) = \Pr(X \leq x) > 0$.

Conditional distributions

- We consider the following conditional distributions:
- Case I: $(Y|X \leq x)$ when $F(x) = \Pr(X \leq x) > 0$.
- Case II: $(Y|X > x)$ when $\bar{F}(x) = \Pr(X > x) > 0$.

Conditional distributions

- We consider the following conditional distributions:
- Case I: $(Y|X \leq x)$ when $F(x) = \Pr(X \leq x) > 0$.
- Case II: $(Y|X > x)$ when $\bar{F}(x) = \Pr(X > x) > 0$.
- Case III: $(Y|X = x)$ when $f(x) = F'(x) > 0$.

Conditional distributions. Case I.

- For $(Y|X \leq x)$ we have:

$$\Pr(Y \leq y|X \leq x) = \frac{\Pr(X \leq x, Y \leq y)}{\Pr(X \leq x)} = \frac{C(F(x), G(y))}{F(x)} = q_1(G(y)),$$

where the distortion function is given by

$$q_1(u) = \frac{C(F(x), u)}{F(x)}.$$

Conditional distributions. Case I.

- For $(Y|X \leq x)$ we have:

$$\Pr(Y \leq y|X \leq x) = \frac{\Pr(X \leq x, Y \leq y)}{\Pr(X \leq x)} = \frac{C(F(x), G(y))}{F(x)} = q_1(G(y)),$$

where the distortion function is given by

$$q_1(u) = \frac{C(F(x), u)}{F(x)}.$$

- Analogously, its reliability can be written as

$$\Pr(Y > y|X \leq x) = \frac{F(x) - C(F(x), G(y))}{F(x)} = \bar{q}_1(\bar{G}(y)),$$

where the dual distortion function is given by

$$\bar{q}_1(u) = 1 - q_1(1-u) = \frac{F(x) - C(F(x), 1-u)}{F(x)} = \frac{u - \hat{C}(\bar{F}(x), u)}{F(x)}.$$

Conditional distributions. Case II.

- Analogously, for $(Y|X > x)$ we have:

$$\Pr(Y > y|X > x) = \frac{\Pr(X > x, Y > y)}{\Pr(X > x)} = \frac{\hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = \bar{q}_2(\bar{G}(y)),$$

where

$$\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}.$$

Conditional distributions. Case II.

- Analogously, for $(Y|X > x)$ we have:

$$\Pr(Y > y|X > x) = \frac{\Pr(X > x, Y > y)}{\Pr(X > x)} = \frac{\hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = \bar{q}_2(\bar{G}(y)),$$

where

$$\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}.$$

- Analogously, its distribution function can be written as

$$\Pr(Y \le y|X > x) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = q_2(G(y)),$$

where

$$q_2(u) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), 1-u)}{\bar{F}(x)} = \frac{u - C(F(x), u)}{\bar{F}(x)}.$$

Conditional distributions. Case III.

- Let us consider now $(Y|X = x)$ when $f(x) > 0$ and $\boldsymbol{F}$ is absolutely continuous. Then the joint density is

$$\boldsymbol{f}(x, y) = f(x)g(y)\partial_2\partial_1 C(F(x), G(y)).$$

Conditional distributions. Case III.

- Let us consider now $(Y|X = x)$ when $f(x) > 0$ and $\boldsymbol{F}$ is absolutely continuous. Then the joint density is

$$\boldsymbol{f}(x, y) = f(x)g(y)\partial_2\partial_1 C(F(x), G(y)).$$

- Hence $f_{Y|X=x}(y|x) = \boldsymbol{f}(x, y)/f(x)$ and

$$\begin{aligned}\Pr(Y \leq y|X = x) &= \int_{-\infty}^{y} \frac{\boldsymbol{f}(x, z)}{f(x)} dz \\ &= \int_{-\infty}^{y} g(z)\partial_2\partial_1 C(F(x), G(z))dz \\ &= \partial_1 C(F(x), G(y)) \\ &= q_3(G(y)) \qquad (2.1)\end{aligned}$$

when $\lim_{u\to 0^+} \partial_1 C(F(x), u) = 0$, where $q_3(u) = \partial_1 C(F(x), u)$.

Conditional distributions. Case III.

- Analogously, its reliability function can be written as

$$\Pr(Y > y|X = x) = \bar{q}_3(\bar{G}(y)), \tag{2.2}$$

where

$$\bar{q}_3(u) = 1 - q_3(1-u) = 1 - \partial_1 C(F(x), 1-u) = \partial_1 \hat{C}(\bar{F}(x), u).$$

Conditional distributions

- Similar representations can be obtained for

Conditional distributions

- Similar representations can be obtained for
- $(Y|X \leq x, Y \leq y)$,

Conditional distributions

- Similar representations can be obtained for
- $(Y|X \leq x, Y \leq y)$,
- $(Y|X > x, Y > y)$,

Conditional distributions

- Similar representations can be obtained for
- $(Y|X \leq x, Y \leq y)$,
- $(Y|X > x, Y > y)$,
- $(Y|X \leq x, Y > y)$,

Conditional distributions

- Similar representations can be obtained for
- $(Y|X \leq x, Y \leq y)$,
- $(Y|X > x, Y > y)$,
- $(Y|X \leq x, Y > y)$,
- $(Y|X > x, Y \leq y)$ and

Conditional distributions

- Similar representations can be obtained for
- $(Y|X \leq x, Y \leq y)$,
- $(Y|X > x, Y > y)$,
- $(Y|X \leq x, Y > y)$,
- $(Y|X > x, Y \leq y)$ and
- $(Y|X = x, Y \leq y)$.

Conditional distributions

- Similar representations can be obtained for
- $(Y|X \leq x, Y \leq y)$,
- $(Y|X > x, Y > y)$,
- $(Y|X \leq x, Y > y)$,
- $(Y|X > x, Y \leq y)$ and
- $(Y|X = x, Y \leq y)$.
- The same holds for $(X_1, \ldots, X_n)$ when $n > 2$.

Comparisons between Y and $(Y|X \le x)$

Proposition

(i) $Y \ge_{ST} (Y|X \le x)$ ($\le_{ST}$) *for all* F, G *if and only if* $C(F(x), u) \ge uF(x)$ ($\le$) *for all* $u \in (0, 1)$.

(ii) $Y \ge_{HR} (Y|X \le x)$ ($\le_{HR}$) *for all* F, G *if and only if* $\hat{C}(\bar{F}(x), u)/u$ *is decreasing (increasing) in* u *in* $(0, 1)$.

(iii) $Y \ge_{RHR} (Y|X \le x)$ ($\le_{RHR}$) *for all* F, G *if and only if* $C(F(x), u)/u$ *is decreasing (increasing) in* u *in* $(0, 1)$.

(iv) $Y \ge_{LR} (Y|X \le x)$ ($\le_{LR}$) *for all* F, G *if and only if* $C(F(x), u)$ *is concave (convex) in* u *in the interval* $(0, 1)$.

(v) $Y \ge_{MRL} (Y|X \le x)$ ($\le_{MRL}$) *for all* F, G *if* $\hat{C}(\bar{F}(x), u)/u$ *is bathtub (upside-down bathtub) in* u *in* $(0, 1)$ *and* $E(Y) \ge E(Y|X \le x)$ ($\le$).

Comparisons between Y and $(Y|X > x)$

Proposition

(i) $Y \leq_{ST} (Y|X > x)$ ($\geq_{ST}$) *for all* F, G *if and only if* $\hat{C}(\bar{F}(x), u) \geq u\bar{F}(x)$ ($\leq$) *for all* $u \in (0, 1)$.

(ii) $Y \leq_{HR} (Y|X > x)$ ($\geq_{HR}$) *for all* F, G *if and only if* $\hat{C}(\bar{F}(x), u)/u$ *is decreasing (increasing) in* u *in* $(0, 1)$.

(iii) $Y \leq_{RHR} (Y|X > x)$ ($\geq_{RHR}$) *for all* F, G *if and only if* $C(F(x), u)/u$ *is decreasing (increasing) in* u *in* $(0, 1)$.

(iv) $Y \leq_{LR} (Y|X > x)$ ($\geq_{LR}$) *for all* F, G *if and only if* $\hat{C}(\bar{F}(x), u)$ *is concave (convex) in* u *in* $(0, 1)$.

(v) $Y \leq_{MRL} (Y|X > x)$ ($\geq_{MRL}$) *for all* F, G *if* $\hat{C}(\bar{F}(x), u)/u$ *is bathtub (upside-down bathtub) in* u *in* $(0, 1)$ *and* $E(Y) \leq E(Y|X > x)$ ($\geq$).

Dependence

- (X, Y) is *positive (negative) quadrant dependent, PQD, (NQD)* if $C \geq \Pi$ $(\leq)$, where $\Pi(u, v) = uv$.

Dependence

- (X, Y) is *positive (negative) quadrant dependent, PQD, (NQD)* if $C \geq \Pi$ $(\leq)$, where $\Pi(u, v) = uv$.
- PQD (NQD) implies Spearman $\rho_S \geq 0$ and Kendall $\tau_K \geq 0$ (≤ 0).

Dependence

- (X, Y) is *positive (negative) quadrant dependent, PQD, (NQD)* if $C \geq \Pi$ $(\leq)$, where $\Pi(u, v) = uv$.
- PQD (NQD) implies Spearman $\rho_S \geq 0$ and Kendall $\tau_K \geq 0$ (≤ 0).
- The following conditions are equivalent:
  (i) (X, Y) is PQD, (NQD).
  (ii) $Y \geq_{ST} (Y|X \leq x)$ $(\leq_{ST})$ for all F, G.
  (iii) $Y \leq_{ST} (Y|X > x)$ $(\geq_{ST})$ for all F, G.

Dependence

- (X,Y) is *positive (negative) quadrant dependent, PQD, (NQD)* if $C \geq \Pi$ $(\leq)$, where $\Pi(u,v) = uv$.
- PQD (NQD) implies Spearman $\rho_S \geq 0$ and Kendall $\tau_K \geq 0$ (≤ 0).
- The following conditions are equivalent:
  (i) (X,Y) is PQD, (NQD).
  (ii) $Y \geq_{ST} (Y|X \leq x)$ $(\leq_{ST})$ for all F,G.
  (iii) $Y \leq_{ST} (Y|X > x)$ $(\geq_{ST})$ for all F,G.
- Note that

$$\Pr(Y \leq y) = F(x)\Pr(Y \leq y|X \leq x) + \bar{F}(x)\Pr(Y \leq y|X > x).$$

Comparisons between Y and $(Y|X=x)$

Proposition

(i) $Y \leq_{ST} (Y|X=x)$ $(\geq_{ST})$ *for all* F, G *if and only if* $\partial_1 C(F(x), u) \leq u$ $(\geq)$ *for all* $u \in (0,1)$.

(ii) $Y \leq_{HR} (Y|X=x)$ $(\geq_{HR})$ *for all* F, G *if and only if* $\partial_1 \hat{C}(\bar{F}(x), u)/u$ *is decreasing (increasing) in* u *in* $(0,1)$.

(iii) $Y \leq_{RHR} (Y|X=x)$ $(\geq_{RHR})$ *for all* F, G *if and only if* $\partial_1 C(F(x), u)/u$ *is increasing (decreasing) in* u *in* $(0,1)$.

(iv) $Y \leq_{LR} (Y|X=x)$ $(\geq_{LR})$ *for all* F, G *if and only if* $\partial_1 C(F(x), u)$ *is concave (convex) in* u *in* $(0,1)$.

(v) $Y \leq_{MRL} (Y|X=x)$ $(\geq_{MRL})$ *for all* F, G *if* $\partial_1 \hat{C}(\bar{F}(x), u)/u$ *is bathtub (upside-down bathtub) in* u *in* $(0,1)$ *and* $E(Y) \leq E(Y|X=x)$ $(\geq)$.

Comparisons between $(Y|X \le x_1)$ and $(Y|X \le x_2)$

- $(Y|X \le x_1) \le_{ST} (Y|X \le x_2)$ holds for all F, G if, and only if,

$$\frac{C(F(x_1), v)}{F(x_1)} \ge \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0,1). \tag{3.1}$$

Comparisons between $(Y|X \leq x_1)$ and $(Y|X \leq x_2)$

- $(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)$ holds for all F, G if, and only if,

$$\frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0,1). \tag{3.1}$$

- (X, Y) is *Left Tail Decreasing (Increasing) LTD*$(Y|X)$ *(LTI*$(Y|X)$*)* in Y if $(Y|X \leq x)$ is ST-increasing (decreasing) in x.

Comparisons between $(Y|X \leq x_1)$ and $(Y|X \leq x_2)$

- $(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)$ holds for all F, G if, and only if,

$$\frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0, 1). \tag{3.1}$$

- (X, Y) is *Left Tail Decreasing (Increasing)* $LTD(Y|X)$ *(*$LTI(Y|X)$*)* in Y if $(Y|X \leq x)$ is ST-increasing (decreasing) in x.
- Hence, (X, Y) is LTD$(Y|X)$ (LTI$(Y|X)$) for all F, G if and only if $C(u, v)/u$ is decreasing (increasing) in u for all v (see Nelsen p. 192, Th. 5.2.5).

Comparisons between $(Y|X \leq x_1)$ and $(Y|X \leq x_2)$

- $(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)$ holds for all F, G if, and only if,

$$\frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0, 1). \tag{3.1}$$

- (X, Y) is *Left Tail Decreasing (Increasing)* $LTD(Y|X)$ *(*$LTI(Y|X)$*)* in Y if $(Y|X \leq x)$ is ST-increasing (decreasing) in x.
- Hence, (X, Y) is LTD$(Y|X)$ (LTI$(Y|X)$) for all F, G if and only if $C(u, v)/u$ is decreasing (increasing) in u for all v (see Nelsen p. 192, Th. 5.2.5).
- It is a positive (negative) dependence property.

Comparisons between $(Y|X \leq x_1)$ and $(Y|X \leq x_2)$

- $(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)$ holds for all F, G if, and only if,

$$\frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0,1). \tag{3.1}$$

- (X, Y) is *Left Tail Decreasing (Increasing)* $LTD(Y|X)$ *(*$LTI(Y|X)$*)* in Y if $(Y|X \leq x)$ is ST-increasing (decreasing) in x.
- Hence, (X, Y) is LTD($Y|X$) (LTI($Y|X$)) for all F, G if and only if $C(u, v)/u$ is decreasing (increasing) in u for all v (see Nelsen p. 192, Th. 5.2.5).
- It is a positive (negative) dependence property.
- LTD($Y|X$) (LTI($Y|X$)) implies the positive (negative) quadrant dependent, PQD, (NQD) property of (X, Y).

(i) $(Y|X \leq x)$ is ST-increasing (decreasing) in x for all F, G if and only if $C(u,v)/u$ is decreasing (increasing) in u for all v.

(ii) $(Y|X \leq x)$ is HR-increasing (decreasing) in x for all F, G if and only if $(v - \hat{C}(u_2, v))/(v - \hat{C}(u_1, v))$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iii) $(Y|X \leq x)$ is RHR-increasing (decreasing) in x for all F, G if and only if $C(u_2, v)/C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iv) $(Y|X \leq x)$ is LR-increasing (decreasing) in x for all F, G if and only if $\partial_2 C(u_2, v)/\partial_2 C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(v) $(Y|X \leq x)$ is MRL-increasing (decreasing) in x for all F, G if and only if $(v - \hat{C}(u_2, v))/(v - \hat{C}(u_1, v))$ is bathtub (upside-down bathtub) in v for all $0 < u_1 \leq u_2 < 1$ and $E(Y|X \leq x)$ is increasing (decreasing) in x.

Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

- $(Y|X > x_1) \leq_{ST} (Y|X > x_2)$ holds for all F, G if, and only if,

$$\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0, 1). \tag{3.2}$$

Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

- $(Y|X > x_1) \leq_{ST} (Y|X > x_2)$ holds for all F, G if, and only if,

$$\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0,1). \tag{3.2}$$

- (X, Y) is *Right Tail Increasing (Decreasing)* $RTI(Y|X)$ *(*$RTD(Y|X)$*)* in Y if $(Y|X > x)$ is ST-increasing (decreasing) in x.

Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

- $(Y|X > x_1) \leq_{ST} (Y|X > x_2)$ holds for all F, G if, and only if,

$$\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0,1). \qquad (3.2)$$

- (X, Y) is *Right Tail Increasing (Decreasing) RTI*$(Y|X)$ *(RTD*$(Y|X)$*)* in Y if $(Y|X > x)$ is ST-increasing (decreasing) in x.
- (X, Y) is RTI$(Y|X)$ (RTD$(Y|X)$) for all F, G if and only if $\hat{C}(u, v)/u$ is decreasing (increasing) in u for all v.

Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

- $(Y|X > x_1) \leq_{ST} (Y|X > x_2)$ holds for all F, G if, and only if,

$$\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0,1). \tag{3.2}$$

- (X, Y) is *Right Tail Increasing (Decreasing)* $RTI(Y|X)$ *($RTD(Y|X)$)* in Y if $(Y|X > x)$ is ST-increasing (decreasing) in x.
- (X, Y) is $\text{RTI}(Y|X)$ ($\text{RTD}(Y|X)$) for all F, G if and only if $\hat{C}(u, v)/u$ is decreasing (increasing) in u for all v.
- It is a positive (negative) dependence property.

Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

- $(Y|X > x_1) \leq_{ST} (Y|X > x_2)$ holds for all F, G if, and only if,

$$\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0,1). \tag{3.2}$$

- (X, Y) is *Right Tail Increasing (Decreasing)* $RTI(Y|X)$ *(*$RTD(Y|X)$*)* in Y if $(Y|X > x)$ is ST-increasing (decreasing) in x.
- (X, Y) is RTI$(Y|X)$ (RTD$(Y|X)$) for all F, G if and only if $\hat{C}(u, v)/u$ is decreasing (increasing) in u for all v.
- It is a positive (negative) dependence property.
- RTI$(Y|X)$ (RTD$(Y|X)$) implies the positive (negative) quadrant dependent, PQD, (NQD) property of (X, Y).

(i) $(Y|X > x)$ is ST-increasing (decreasing) in x for all F, G if and only if $\hat{C}(u, v)/u$ is decreasing (increasing) in u for all v.

(ii) $(Y|X > x)$ is HR-increasing (decreasing) in x for all F, G if and only if $\hat{C}(u_2, v)/\hat{C}(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iii) $(Y|X > x)$ is RHR-increasing (decreasing) in x for all F, G if and only if $(v - C(u_2, v))/(v - C(u_1, v))$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iv) $(Y|X > x)$ is LR-increasing (decreasing) in x for all F, G if and only if $\partial_2\hat{C}(u_2, v)/\partial_2\hat{C}(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(v) $(Y|X > x)$ is MRL-increasing (decreasing) in x for all F, G if and only if $\hat{C}(u_2, v)/\hat{C}(u_1, v)$ is upside-down bathtub (bathtub) in v for all $0 < u_1 \leq u_2 < 1$ and $E(Y|X > x)$ is increasing (decreasing) in x.

Comparisons between $(Y|X = x_1)$ and $(Y|X = x_2)$

- $(Y|X = x_1) \leq_{ST} (Y|X = x_2)$ holds for all F, G if, and only if,

$$\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1). \quad (3.3)$$

Comparisons between $(Y|X = x_1)$ and $(Y|X = x_2)$

- $(Y|X = x_1) \leq_{ST} (Y|X = x_2)$ holds for all F, G if, and only if,

$$\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1). \qquad (3.3)$$

- Y is *Stochastically Increasing (Decreasing) SI*$(Y|X)$ *(SD*$(Y|X)$*)* in X if $(Y|X = x)$ is ST-increasing (decreasing) in x.

Comparisons between $(Y|X = x_1)$ and $(Y|X = x_2)$

- $(Y|X = x_1) \leq_{ST} (Y|X = x_2)$ holds for all F, G if, and only if,

$$\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1). \tag{3.3}$$

- Y is *Stochastically Increasing (Decreasing) SI*$(Y|X)$ *(SD*$(Y|X)$*)* in X if $(Y|X = x)$ is ST-increasing (decreasing) in x.
- Y is SI$(Y|X)$ (SD$(Y|X)$) in X if and only if $\partial_1 \hat{C}(u, v)$ is decreasing (increasing) in u for all v (see Nelsen p. 196, Th. 5.2.10), that is, $\hat{C}(u, v)$ is concave (convex) in u.

Comparisons between $(Y|X = x_1)$ and $(Y|X = x_2)$

- $(Y|X = x_1) \leq_{ST} (Y|X = x_2)$ holds for all F, G if, and only if,

$$\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0,1). \tag{3.3}$$

- Y is *Stochastically Increasing (Decreasing)* $SI(Y|X)$ *(*$SD(Y|X)$*)* in X if $(Y|X = x)$ is ST-increasing (decreasing) in x.
- Y is SI$(Y|X)$ (SD$(Y|X)$) in X if and only if $\partial_1 \hat{C}(u, v)$ is decreasing (increasing) in u for all v (see Nelsen p. 196, Th. 5.2.10), that is, $\hat{C}(u, v)$ is concave (convex) in u.
- It is a positive (negative) dependence property.

Comparisons between $(Y|X = x_1)$ and $(Y|X = x_2)$

- $(Y|X = x_1) \leq_{ST} (Y|X = x_2)$ holds for all F, G if, and only if,

$$\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0,1). \qquad (3.3)$$

- Y is *Stochastically Increasing (Decreasing)* $SI(Y|X)$ *(*$SD(Y|X)$*)* in X if $(Y|X = x)$ is ST-increasing (decreasing) in x.
- Y is SI$(Y|X)$ (SD$(Y|X)$) in X if and only if $\partial_1 \hat{C}(u, v)$ is decreasing (increasing) in u for all v (see Nelsen p. 196, Th. 5.2.10), that is, $\hat{C}(u, v)$ is concave (convex) in u.
- It is a positive (negative) dependence property.
- SI$(Y|X)$ (SD$(Y|X)$) implies LTD$(Y|X)$ (LTI$(Y|X)$) and RTI$(Y|X)$ (RTD$(Y|X)$) and so PQD (NQD).

(i) $(Y|X=x)$ is ST-increasing (decreasing) in x for all F, G if and only if $C(u,v)$ (or $\hat{C}(u,v)$) is concave (convex) in u for all v.

(ii) $(Y|X=x)$ is HR-increasing (decreasing) in x for all F, G if and only if $\partial_1\hat{C}(u_2,v)/\partial_1\hat{C}(u_1,v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iii) $(Y|X=x)$ is RHR-increasing (decreasing) in x for all F, G if and only if $\partial_1 C(u_2,v)/\partial_1 C(u_1,v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iv) $(Y|X=x)$ is LR-increasing (decreasing) in x for all F, G if and only if $\partial_2\partial_1 C(u_2,v)/\partial_2\partial_1 C(u_1,v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(v) $(Y|X=x)$ is MRL-increasing (decreasing) in x for all F, G if and only if $\partial_1\hat{C}(u_2,v)/\partial_1\hat{C}(u_1,v)$ is upside-down bathtub (bathtub) in v for all $0 < u_1 \leq u_2 < 1$ and $E(Y|X=x)$ is increasing (decreasing) in x.

Dependence

- The following conditions are equivalent:
  - (i) (X, Y) is $LTD(X|Y)$.
  - (ii) $(X|Y \leq y)$ is ST-increasing.
  - (iii) $Y \geq_{RHR} (Y|X \leq x)$ for all x.
  - (iv) $Y \leq_{RHR} (Y|X > x)$ for all x.
  - (v) $(Y|X \leq x) \leq_{RHR} (Y|X > x)$ for all x.
  - (vi) $C(u, v)/v$ is decreasing in v for all u.

Dependence

- The following conditions are equivalent:
  - (i) (X, Y) is $LTD(X|Y)$.
  - (ii) $(X|Y \leq y)$ is ST-increasing.
  - (iii) $Y \geq_{RHR} (Y|X \leq x)$ for all x.
  - (iv) $Y \leq_{RHR} (Y|X > x)$ for all x.
  - (v) $(Y|X \leq x) \leq_{RHR} (Y|X > x)$ for all x.
  - (vi) $C(u, v)/v$ is decreasing in v for all u.
- Analogous equivalences can be stated for the respective negative notions.

Dependence

- The following conditions are equivalent:
  (i) (X, Y) is $RTI(X|Y)$.
  (ii) $(X|Y > y)$ is ST-increasing.
  (iii) $Y \geq_{HR} (Y|X \leq x)$ for all x.
  (iv) $Y \leq_{HR} (Y|X > x)$ for all x.
  (v) $(Y|X \leq x) \leq_{HR} (Y|X > x)$ for all x.
  (vi) $\hat{C}(u, v)/v$ is decreasing in v for all u.

Dependence

- The following conditions are equivalent:
  - (i) (X, Y) is $RTI(X|Y)$.
  - (ii) $(X|Y > y)$ is ST-increasing.
  - (iii) $Y \geq_{HR} (Y|X \leq x)$ for all x.
  - (iv) $Y \leq_{HR} (Y|X > x)$ for all x.
  - (v) $(Y|X \leq x) \leq_{HR} (Y|X > x)$ for all x.
  - (vi) $\hat{C}(u, v)/v$ is decreasing in v for all u.
- Analogous equivalences can be stated for the respective negative notions.

Dependence

- The following conditions are equivalent:
  - (i) (X,Y) is $SI(X|Y)$, i.e., $(X|Y=y)$ is ST-increasing in y.
  - (ii) $Y \geq_{LR} (Y|X \leq x)$ for all x.
  - (iii) $Y \leq_{LR} (Y|X > x)$ for all x.
  - (iv) $(Y|X \leq x) \leq_{LR} (Y|X > x)$ for all x.
  - (v) $C(u,v)$ is concave in v for all u.
  - (vi) $\hat{C}(u,v)$ is concave in v for all u.

Dependence

- The following conditions are equivalent:
  (i) (X, Y) is $SI(X|Y)$, i.e., $(X|Y = y)$ is ST-increasing in y.
  (ii) $Y \geq_{LR} (Y|X \leq x)$ for all x.
  (iii) $Y \leq_{LR} (Y|X > x)$ for all x.
  (iv) $(Y|X \leq x) \leq_{LR} (Y|X > x)$ for all x.
  (v) $C(u, v)$ is concave in v for all u.
  (vi) $\hat{C}(u, v)$ is concave in v for all u.
- Analogous results can be stated for $(Y|X)$ and for the respective negative notions.

Dependence

- (X, Y) is *Left Corner Set Decreasing* $(LCSD)$ if $\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2.

Dependence

- (X, Y) is *Left Corner Set Decreasing* $(LCSD)$ if $\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2.
- (X, Y) is *Right Corner Set Increasing* $(RCSI)$ if $\Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)$ is increasing in x_2 and y_2.

Dependence

- (X, Y) is *Left Corner Set Decreasing* $(LCSD)$ if $\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2.
- (X, Y) is *Right Corner Set Increasing* $(RCSI)$ if $\Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)$ is increasing in x_2 and y_2.
- The following conditions are equivalent:
  - (i) *LCSD*.
  - (ii) $(Y|X \leq x)$ is RHR-increasing in x.
  - (iii) C is TP_2.

Dependence

- (X, Y) is *Left Corner Set Decreasing* (*LCSD*) if $\Pr(X \le x_1, Y \le y_1 | X \le x_2, Y \le y_2)$ is decreasing in x_2 and y_2.
- (X, Y) is *Right Corner Set Increasing* (*RCSI*) if $\Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)$ is increasing in x_2 and y_2.
- The following conditions are equivalent:
  - (i) *LCSD*.
  - (ii) $(Y|X \le x)$ is RHR-increasing in x.
  - (iii) C is TP_2.
- The following conditions are equivalent:
  - (i) *RCSI*.
  - (ii) $(Y|X > x)$ is HR-increasing in x.
  - (iii) $\hat{C}$ is TP_2.

Dependence

- (X, Y) is *Left Corner Set Decreasing* (*LCSD*) if $\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2.
- (X, Y) is *Right Corner Set Increasing* (*RCSI*) if $\Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)$ is increasing in x_2 and y_2.
- The following conditions are equivalent:
  (i) *LCSD*.
  (ii) $(Y|X \leq x)$ is RHR-increasing in x.
  (iii) C is TP_2.
- The following conditions are equivalent:
  (i) *RCSI*.
  (ii) $(Y|X > x)$ is HR-increasing in x.
  (iii) $\hat{C}$ is TP_2.
- Similar results can be obtained for *Left Corner Set Increasing* (*LCSI*) and *Right Corner Set Decreasing* (*RCSD*).

Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X = x)$ is ORD-increasing in x.

Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X=x)$ is ORD-increasing in x.
- $SI_{ORD}(X|Y)$ if $(X|Y=y)$ is ORD-increasing in y.

Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X = x)$ is ORD-increasing in x.
- $SI_{ORD}(X|Y)$ if $(X|Y = y)$ is ORD-increasing in y.
- The negative dependence properties $SD_{ORD}(Y|X)$ and $SD_{ORD}(X|Y)$ are defined in a similar way.

Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X=x)$ is ORD-increasing in x.
- $SI_{ORD}(X|Y)$ if $(X|Y=y)$ is ORD-increasing in y.
- The negative dependence properties $SD_{ORD}(Y|X)$ and $SD_{ORD}(X|Y)$ are defined in a similar way.
- The following conditions are equivalent:
  - (i) $SI_{RHR}(X|Y)$.
  - (ii) $(Y|X \leq x)$ is LR-increasing.
  - (iii) $\partial_2 C$ is TP_2.

Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X=x)$ is ORD-increasing in x.
- $SI_{ORD}(X|Y)$ if $(X|Y=y)$ is ORD-increasing in y.
- The negative dependence properties $SD_{ORD}(Y|X)$ and $SD_{ORD}(X|Y)$ are defined in a similar way.
- The following conditions are equivalent:
  (i) $SI_{RHR}(X|Y)$.
  (ii) $(Y|X\leq x)$ is LR-increasing.
  (iii) $\partial_2 C$ is TP_2.
- The following conditions are equivalent:
  (i) $SI_{HR}(X|Y)$.
  (ii) $(Y|X>x)$ is LR-increasing.
  (iii) $\partial_2 \hat{C}$ is TP_2.

Dependence relationships

$$
\begin{array}{ccccc}
SI_{LR}(X|Y) & \Rightarrow & SI_{RHR}(X|Y) & \Rightarrow & LCSD \\
\Downarrow & & \Downarrow & & \Downarrow \\
SI_{HR}(X|Y) & \Rightarrow & SI_{ST}(X|Y) & \Rightarrow & LTD(X|Y) \\
\Downarrow & & \Downarrow & & \Downarrow \\
RCSI & \Rightarrow & RTI(X|Y) & \Rightarrow & PQD
\end{array}
$$

Bounds for $(Y|X \le x)$

Proposition

If (X,Y) *has the copulas* C *and* $\hat{C}$ *and* $Y \ge 0$*, then:*

$$\inf_{u\in(0,1]} \frac{C(F(x),u)}{uF(x)} \le \frac{\Pr(Y\le y|X\le x)}{G(y)} \le \sup_{u\in(0,1]} \frac{C(F(x),u)}{uF(x)}$$

$$\inf_{u\in(0,1]} \frac{u-\hat{C}(\bar{F}(x),u)}{uF(x)} \le \frac{\Pr(Y> y|X\le x)}{\bar{G}(y)} \le \sup_{u\in(0,1]} \frac{u-\hat{C}(\bar{F}(x),u)}{uF(x)}$$

and

$$\inf_{u\in(0,1]} \frac{u-\hat{C}(\bar{F}(x),u)}{uF(x)} \le \frac{E(Y|X\le x)}{E(Y)} \le \sup_{u\in(0,1]} \frac{u-\hat{C}(\bar{F}(x),u)}{uF(x)}.$$

Bounds for $(Y|X > x)$

Proposition

If (X, Y) has the copulas C and $\hat{C}$ and $Y \geq 0$, then:

$$\inf_{u\in(0,1]} \frac{u - C(F(x),u)}{u\bar{F}(x)} \leq \frac{\Pr(Y \leq y|X > x)}{G(y)} \leq \sup_{u\in(0,1]} \frac{u - C(F(x),u)}{u\bar{F}(x)},$$

$$\inf_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x),u)}{u\bar{F}(x)} \leq \frac{\Pr(Y > y|X > x)}{\bar{G}(y)} \leq \sup_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x),u)}{u\bar{F}(x)}$$

and

$$\inf_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x),u)}{u\bar{F}(x)} \leq \frac{E(Y|X > x)}{E(Y)} \leq \sup_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x),u)}{u\bar{F}(x)}.$$

Bounds for $(Y|X=x)$

Proposition

If (X,Y) has the copulas C and $\hat{C}$ and $Y \geq 0$, then:

$$\inf_{u\in(0,1]} \frac{\partial_1 C(F(x),u)}{u} \leq \frac{\Pr(Y\leq y|X=x)}{G(y)} \leq \sup_{u\in(0,1]} \frac{\partial_1 C(F(x),u)}{u},$$

$$\inf_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u} \leq \frac{\Pr(Y>y|X=x)}{\bar{G}(y)} \leq \sup_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u}$$

and

$$\inf_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u} \leq \frac{E(Y|X=x)}{E(Y)} \leq \sup_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u}.$$

Bounds based on Δ_G

Proposition

If (X,Y) has the copulas C and $\hat{C}$, $0=\inf\{y : G(y)>0\}$, $p=\bar{F}(x)$ and $\Delta_G := 2\int_0^\infty G(y)(1-G(y))dy>0$, then:

$$\inf_{u\in(0,1)} \frac{up-\hat{C}(p,u)}{2u(1-u)(1-p)} \le \frac{E(Y|X\le x)-E(Y)}{\Delta_G} \le \sup_{u\in(0,1)} \frac{up-\hat{C}(p,u)}{2u(1-u)(1-p)},$$

$$\inf_{u\in(0,1)} \frac{\hat{C}(p,u)-up}{2u(1-u)p} \le \frac{E(Y|X> x)-E(Y)}{\Delta_G} \le \sup_{u\in(0,1)} \frac{\hat{C}(p,u)-up}{2u(1-u)p}$$

and

$$\inf_{u\in(0,1)} \frac{\partial_1\hat{C}(p,u)-u}{2u(1-u)} \le \frac{E(Y|X= x)-E(Y)}{\Delta_G} \le \sup_{u\in(0,1)} \frac{\partial_1\hat{C}(p,u)-u}{2u(1-u)}.$$

Example 1

- If (X, Y) has the Clayton-Oakes copula:

$$C(u, v) = \frac{uv}{u + v - uv}$$

for $0 \leq u, v \leq 1$, then

$$q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.$$

Example 1

- If (X, Y) has the Clayton-Oakes copula:

$$C(u, v) = \frac{uv}{u + v - uv}$$

for $0 \leq u, v \leq 1$, then

$$q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.$$

- Then $Y \geq_{ST} (Y|X \leq x)$ for any G since $C(F(x), u) \geq uF(x)$.

Example 1

- If (X, Y) has the Clayton-Oakes copula:

$$C(u, v) = \frac{uv}{u + v - uv}$$

for $0 \leq u, v \leq 1$, then

$$q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.$$

- Then $Y \geq_{ST} (Y|X \leq x)$ for any G since $C(F(x), u) \geq uF(x)$.
- Even more, $Y \geq_{LR} (Y|X \leq x)$ for any G since q_1 is concave.

Example 1

- If (X, Y) has the Clayton-Oakes copula:

$$C(u, v) = \frac{uv}{u + v - uv}$$

for $0 \leq u, v \leq 1$, then

$$q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.$$

- Then $Y \geq_{ST} (Y|X \leq x)$ for any G since $C(F(x), u) \geq uF(x)$.
- Even more, $Y \geq_{LR} (Y|X \leq x)$ for any G since q_1 is concave.
- $(Y|X \leq x)$, $(Y|X > x)$ and $(Y|X = x)$ are LR-increasing in x.

Example 1

- If (X, Y) has the Clayton-Oakes copula:

$$C(u, v) = \frac{uv}{u + v - uv}$$

for $0 \leq u, v \leq 1$, then

$$q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.$$

- Then $Y \geq_{ST} (Y|X \leq x)$ for any G since $C(F(x), u) \geq uF(x)$.
- Even more, $Y \geq_{LR} (Y|X \leq x)$ for any G since q_1 is concave.
- $(Y|X \leq x)$, $(Y|X > x)$ and $(Y|X = x)$ are LR-increasing in x.
- $\rho_S = -39 + 4\pi^2 \cong 0.478417$ and $\tau_K = 1/3$.

An example

- We can obtain the following bounds:

$$G(y) \leq \Pr(Y \leq y | X \leq x) \leq \frac{1}{F(x)} G(y)$$

and

$$0 \leq \Pr(Y > y | X \leq x) \leq \bar{G}(y).$$

An example

- We can obtain the following bounds:

$$G(y) \leq \Pr(Y \leq y | X \leq x) \leq \frac{1}{F(x)} G(y)$$

and

$$0 \leq \Pr(Y > y | X \leq x) \leq \bar{G}(y).$$

- Then $E(Y|X \leq x) \leq E(Y)$ when $Y \geq 0$.

An example

- We can obtain the following bounds:

$$G(y) \leq \Pr(Y \leq y|X \leq x) \leq \frac{1}{F(x)} G(y)$$

and

$$0 \leq \Pr(Y > y|X \leq x) \leq \bar{G}(y).$$

- Then $E(Y|X \leq x) \leq E(Y)$ when $Y \geq 0$.
- Analogously, for $f(x) > 0$, we have

$$0 \leq \Pr(Y \leq y|X = x) \leq G(y)$$

when $F(x) \geq 1/2$ and

$$0 \leq \Pr(Y \leq y|X = x) \leq \frac{1}{4F(x)\bar{F}(x)} G(y)$$

when $F(x) < 1/2$.

References: Distorted distributions

- Navarro, del Aguila, Sordo and Suarez-Llorens (2013). Stochastic ordering properties for systems with dependent identically distributed components. Appl Stoch Mod Bus Ind 29, 264–278.
- Navarro, del Águila, Sordo and Suárez-Llorens (2014). Preservation of reliability classes under the formation of coherent systems. Appl Stoch Mod Bus Ind 30, 444–454.
- Navarro, del Águila, Sordo, Suárez-Llorens (2016). Preservation of stochastic orders under the formation of generalized distorted distributions. Applications to coherent systems. Meth Comp Appl Probab 18, 529–545.
- Navarro and Gomis (2015). Comparisons in the mean residual life order of coherent systems with identically distributed components. Appl Stoch Mod Bus Ind 32, 33–47.

References: Distorted distributions

- Navarro and del Águila (2017). Stochastic comparisons of distorted distributions, coherent systems and mixtures. Metrika 80, 627–648.
- Navarro and Rychlik (2010). Comparisons and bounds for expected lifetimes of reliability systems. European J Oper Res 207, 309–317.
- Miziula and Navarro (2017). Sharp bounds for the reliability of systems and mixtures with ordered components. Naval Res Logist 64, 108–116.
- Miziula and Navarro (2018). Bounds for the reliability functions of coherent systems with heterogeneous components. Appl Stoch Mod Bus Ind 34, 158–174.

Main References: Conditional distributions and systems

- Navarro and Sordo (2017). Stochastic comparisons and bounds for conditional distributions by using copula properties. Submitted.
- Navarro (2018). Distribution-free comparisons of residual lifetimes of coherent systems based on copula properties. Stat Papers.
- Navarro and Durante (2017). Copula-based representations for the reliability of the residual lifetimes of coherent systems with dependent components. J Mult Anal 158, 87-102.
- Navarro, Pellerey and Longobardi (2017). Comparison results for inactivity times of k-out-of-n and general coherent systems with dependent components. Test 26, 822–846.

References

- For the more references, please visit my personal web page:

$$https://webs.um.es/jorgenav/$$

References

- For the more references, please visit my personal web page:

$$https://webs.um.es/jorgenav/$$

- Thank you for your attention!!