Stochastic comparisons of conditional distributions based on copula properties

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Distorted Distributions
  Definitions
  Comparisons
  Bounds

Conditional distributions. Representations
  Case I: \((Y|X \leq x)\)
  Case II: \((Y|X > x)\)
  Case III: \((Y|X = x)\)

Comparisons, dependence and bounds
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  Dependence
  Bounds
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Distortion functions

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- The **distorted distribution** (DD) associated to a distribution function (DF) $F$ and to an increasing continuous distortion function $q : [0, 1] \to [0, 1]$ such that $q(0) = 0$ and $q(1) = 1$, is

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- For the reliability functions (RF) $\bar{F} = 1 - F$, $\bar{F}_q = 1 - F_q$, we have

  $$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \quad (1.2)$$

  where $\bar{q}(u) = 1 - q(1 - u)$ is the dual distortion function; see Hürlimann (N Am Actuarial J, 2004).
The **generalized distorted distribution** (GDD) associated to \( n \) DF \( F_1, \ldots, F_n \) and to an increasing continuous **multivariate** distortion function \( Q : [0, 1]^n \rightarrow [0, 1] \) such that \( Q(0, \ldots, 0) = 0 \) and \( Q(1, \ldots, 1) = 1 \), is

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F_Q(t) = Q(F_1(t), \ldots, F_n(t)).
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For the RF we have

\[
\bar{F}_Q(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)),
\] (1.4)

where \( \bar{F} = 1 - F \), \( \bar{F}_Q = 1 - F_Q \) and \( \bar{Q}(u_1, \ldots, u_n) = 1 - Q(1 - u_1, \ldots, 1 - u_n) \) is the multivariate dual distortion function; see Navarro et al. (ASMBI, 2014).
Examples

- Proportional hazard rate (PHR) Cox model $\bar{F}_\alpha = \bar{F}^\alpha$, $\alpha > 0$ with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1 - u)^\alpha$. 

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- Proportional reversed hazard rate (PRHR) model \( F_\alpha = F^\alpha, \alpha > 0 \) with \( q(u) = u^\alpha \) and \( \bar{q}(u) = 1 - (1 - u)^\alpha \).
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- Mixtures \( F = p_1 F_1 + \cdots + p_n F_n, p_i \geq 0 \) and \( \sum_i p_i = 1 \) with
  \[
  Q(u) = \bar{Q}(u) = p_1 u_1 + \cdots + p_n u_n.
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- Order statistics $X_{i:n}$ from a sample of IID $X_1, \ldots, X_n$. 
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  $$\bar{q}_n(u) = u \sum_{i=0}^n \frac{1}{i!} (-\ln u)^i.$$
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- Coherent systems \( T = \phi(X_1, \ldots, X_n) \).
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- Coherent systems $T = \phi(X_1, \ldots, X_n)$.
- Conditional distributions from $(X_1, \ldots, X_n)$. 
Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
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- \( X \leq_{ST} Y \iff \bar{F}_X(t) \leq \bar{F}_Y(t) \), stochastic order.
- \( X \leq_{HR} Y \iff \bar{F}_Y / \bar{F}_X \) increases, hazard rate order.
- \( X \leq_{HR} Y \iff (X - t | X > t) \leq_{ST} (Y - t | Y > t) \) for all \( t \).
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- \( X \leq_{MRL} Y \iff E(X - t | X > t) \leq E(Y - t | Y > t) \) for all \( t \).
Main stochastic orderings

- $X \leq_{ST} Y \iff \tilde{F}_X(t) \leq \tilde{F}_Y(t)$, stochastic order.
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- $X \leq_{MRL} Y \iff E(X - t | X > t) \leq E(Y - t | Y > t)$ for all $t$.
- $X \leq_{LR} Y \iff f_Y(t) / f_X(t)$ increases, likelihood ratio order.
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- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ increases, likelihood ratio order.
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- $X \leq_{RHR} Y \iff (t - X | X < t) \geq_{ST} (t - Y | Y < t)$ for all $t$. 
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- \( X \leq_{RHR} Y \iff (t - X | X < t) \geq_{ST} (t - Y | Y < t) \) for all \( t \).
- Then

\[
\begin{align*}
X \leq_{LR} Y & \quad \Rightarrow \quad X \leq_{HR} Y \quad \Rightarrow \quad X \leq_{MRL} Y \\
\downarrow & \quad \downarrow & \quad \downarrow \\
X \leq_{RHR} Y & \quad \Rightarrow \quad X \leq_{ST} Y \quad \Rightarrow \quad E(X) \leq E(Y)
\end{align*}
\]
Comparisons of distorted distributions

- If $T_i$ has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:

  ▶ If $T_1$ has the DF $F_1(t) = q_1(F(t))$, $i = 1, 2$, then:
Comparisons of distorted distributions

- If $T_i$ has the DF $F_i(t) = q_i(F(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all $F$ if and only if $\bar{q}_1 \leq \bar{q}_2$ (or $q_2 \leq q_1$) in $(0, 1)$.
If \( T_i \) has the DF \( F_i(t) = q_i(F(t)) \), \( i = 1, 2 \), then:

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- \( T_1 \leq_{HR} T_2 \) for all \( F \) if and only if \( \bar{q}_2 / \bar{q}_1 \) decreases in \((0, 1)\).
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  - $T_1 \leq_{LR} T_2$ for all $F$ if and only if $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.
  - $T_1 \leq_{MRL} T_2$ for all $F$ such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.

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Comparisons of GDD

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Navarro et al. (Methodology and Computing in Applied Probability, 2016).

Comparisons for ordered distributions were given in Navarro and del Águila (Metrika, 2017).
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- Comparisons for ordered distributions were given in Navarro and del Águila (Metrika, 2017).
If \( T \) has the RF \( \bar{q}(\bar{F}(t)) \), then:

\[
\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t) \leq \bar{F}(t) \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u}.
\]
Bounds for distorted distributions

- If $T$ has the RF $\bar{q}(\bar{F}(t))$, then:

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t) \leq \bar{F}(t) \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u}.$$ 

- Analogously,

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t) \geq \bar{F}(t) \inf_{u \in (0,1]} \frac{\bar{q}(u)}{u}.$$
If $T$ has the RF $\tilde{q}(\bar{F}(t))$, then:

$$\tilde{q}(\bar{F}(t)) = \frac{\tilde{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t) \leq \bar{F}(t) \sup_{u \in (0,1]} \frac{\tilde{q}(u)}{u}.$$ 

Analogously,

$$\tilde{q}(\bar{F}(t)) = \frac{\tilde{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t) \geq \bar{F}(t) \inf_{u \in (0,1]} \frac{\tilde{q}(u)}{u}.$$ 

These bounds are sharp.
Moreover, if $T \geq 0$, then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = \int_0^\infty \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t)dt$$

$$\leq \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u} \int_0^\infty \bar{F}(t)dt = E(X) \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u},$$

where $X$ has the RF $\bar{F}$. 
Moreover, if $T \geq 0$, then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t)) \, dt = \int_0^\infty \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t) \, dt$$

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where $X$ has the RF $\bar{F}$.

Analogously

$$E(T) \geq E(X) \inf_{u \in (0,1]} \frac{\bar{q}(u)}{u}.$$
Bounds for the means of distorted distributions

- Moreover, if $T \geq 0$, then
  
  \[ E(T) = \int_{0}^{\infty} \bar{q}(\bar{F}(t)) \, dt = \int_{0}^{\infty} \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \, \bar{F}(t) \, dt \]
  
  \[ \leq \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u} \int_{0}^{\infty} \bar{F}(t) \, dt = E(X) \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u}, \]
  
  where $X$ has the RF $\bar{F}$.

- Analogously
  
  \[ E(T) \geq E(X) \inf_{u \in (0,1]} \frac{\bar{q}(u)}{u}. \]

- These bounds are sharp.
Bounds for the means of distorted distributions

Proposition

If $T$ has the RF $\bar{q}(\bar{F}(t))$, then:

$$
\inf_{u \in (0,1)} \frac{\bar{q}(u) - u}{2u(1 - u)} \leq \frac{E(T) - E(X)}{\Delta_F} \leq \sup_{u \in (0,1)} \frac{\bar{q}(u) - u}{2u(1 - u)}
$$

(1.5)

when $0 = \inf\{x : F(x) > 0\}$ and the Gini mean difference dispersion measure

$$
\Delta_F = 2 \int_0^{\infty} F(x)(1 - F(x))dx
$$

is positive. The bounds are sharp.
Let \((X, Y)\) be a bivariate random vector. Then

\[
F(x, y) = \Pr(X \leq x, Y \leq y) = C(F(x), G(y)),
\]

where \(F(x) = \Pr(X \leq x)\) and \(G(y) = \Pr(Y \leq y)\) are the marginal DF and \(C\) is a copula (i.e., \(C\) a continuous distribution function with uniform marginals over \((0, 1))\).
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The joint reliability function can be represented as
\[
\bar{F}(x, y) = \Pr(X > x, Y > y) = \hat{C}(\bar{F}(x), \bar{G}(y)),
\]
where \(\bar{F} = 1 - F\) and \(\bar{G} = 1 - G\) and \(\hat{C}\) is also a copula, called *survival copula*. 

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Let \((X, Y)\) be a bivariate random vector. Then

\[
F(x, y) = \Pr(X \leq x, Y \leq y) = C(F(x), G(y)),
\]

where \(F(x) = \Pr(X \leq x)\) and \(G(y) = \Pr(Y \leq y)\) are the marginal DF and \(C\) is a copula (i.e., \(C\) a continuous distribution function with uniform marginals over \((0, 1)\)).

The joint reliability function can be represented as

\[
\bar{F}(x, y) = \Pr(X > x, Y > y) = \hat{C}(\bar{F}(x), \bar{G}(y)),
\]

where \(\bar{F} = 1 - F\) and \(\bar{G} = 1 - G\) and \(\hat{C}\) is also a copula, called survival copula.

\(\hat{C}\) is determined by \(C\) (and vice versa) by

\[
\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).
\]
We consider the following conditional distributions:
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- **Case I:** \((Y|X \leq x)\) when \(F(x) = \Pr(X \leq x) > 0\).
We consider the following conditional distributions:

- Case I: $(Y|X \leq x)$ when $F(x) = \Pr(X \leq x) > 0$.
- Case II: $(Y|X > x)$ when $\bar{F}(x) = \Pr(X > x) > 0$. 

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We consider the following conditional distributions:

- **Case I:** \((Y | X \leq x)\) when \(F(x) = \Pr(X \leq x) > 0\).
- **Case II:** \((Y | X > x)\) when \(\bar{F}(x) = \Pr(X > x) > 0\).
- **Case III:** \((Y | X = x)\) when \(f(x) = F'(x) > 0\).
Conditional distributions. Case I.

For \( (Y|X \leq x) \) we have:

\[
\Pr(Y \leq y|X \leq x) = \frac{\Pr(X \leq x, Y \leq y)}{\Pr(X \leq x)} = \frac{C(F(x), G(y))}{F(x)} = q_1(G(y)),
\]

where the distortion function is given by

\[
q_1(u) = \frac{C(F(x), u)}{F(x)}.
\]
Conditional distributions. Case I.

> For \((Y|X \leq x)\) we have:

\[
Pr(Y \leq y|X \leq x) = \frac{Pr(X \leq x, Y \leq y)}{Pr(X \leq x)} = \frac{C(F(x), G(y))}{F(x)} = q_1(G(y)),
\]

where the distortion function is given by

\[
q_1(u) = \frac{C(F(x), u)}{F(x)}.
\]

> Analogously, its reliability can be written as

\[
Pr(Y > y|X \leq x) = \frac{F(x) - C(F(x), G(y))}{F(x)} = \bar{q}_1(\bar{G}(y)),
\]

where the dual distortion function is given by

\[
\bar{q}_1(u) = 1 - q_1(1-u) = \frac{F(x) - C(F(x), 1-u)}{F(x)} = \frac{u - \hat{C}(\bar{F}(x), u)}{F(x)}.
\]
Conditional distributions. Case II.

Analogously, for \((Y|X > x)\) we have:

\[
\Pr(Y > y|X > x) = \frac{\Pr(X > x, Y > y)}{\Pr(X > x)} = \frac{\hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = \bar{q}_2(\bar{G}(y)),
\]

where

\[
\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}.
\]
Conditional distributions. Case II.

Analogously, for \( (Y|X > x) \) we have:

\[
\Pr(Y > y|X > x) = \frac{\Pr(X > x, Y > y)}{\Pr(X > x)} = \frac{\hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = \bar{q}_2(\bar{G}(y)),
\]

where

\[
\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}.
\]

Analogously, its distribution function can be written as

\[
\Pr(Y \leq y|X > x) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = q_2(G(y)),
\]

where

\[
q_2(u) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), 1 - u)}{\bar{F}(x)} = \frac{u - C(F(x), u)}{\bar{F}(x)}.
\]
Conditional distributions. Case III.

Let us consider now $(Y|X = x)$ when $f(x) > 0$ and $F$ is absolutely continuous. Then the joint density is

$$f(x, y) = f(x)g(y)\partial_2 \partial_1 C(F(x), G(y)).$$
Conditional distributions. Case III.

Let us consider now \( (Y|X = x) \) when \( f(x) > 0 \) and \( F \) is absolutely continuous. Then the joint density is

\[
f(x, y) = f(x)g(y)\partial_2\partial_1 C(F(x), G(y)).
\]

Hence \( f_{Y|X=x}(y|x) = f(x, y)/f(x) \) and

\[
\Pr(Y \leq y|X = x) = \int_{-\infty}^{y} \frac{f(x, z)}{f(x)} dz
\]

\[
= \int_{-\infty}^{y} g(z)\partial_2\partial_1 C(F(x), G(z)) dz
\]

\[
= \partial_1 C(F(x), G(y))
\]

\[
= q_3(G(y))
\]

when \( \lim_{u \to 0^+} \partial_1 C(F(x), u) = 0 \), where \( q_3(u) = \partial_1 C(F(x), u) \).
Analogously, its reliability function can be written as

\[
Pr(Y > y | X = x) = \bar{q}_3(\bar{G}(y)),
\]

where

\[
\bar{q}_3(u) = 1 - q_3(1 - u) = 1 - \partial_1 C(F(x), 1 - u) = \partial_1 \hat{C}(\bar{F}(x), u).
\]
Conditional distributions

- Similar representations can be obtained for

> Case I: \((Y|X \leq x)\)
> Case II: \((Y|X > x)\)
> Case III: \((Y|X = x)\)
Conditional distributions

- Similar representations can be obtained for
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Conditional distributions

- Similar representations can be obtained for
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Conditional distributions

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- Similar representations can be obtained for
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  - \((Y|X > x, Y \leq y)\) and
  - \((Y|X = x, Y \leq y)\).
Conditional distributions

- Similar representations can be obtained for
  - \((Y|X \leq x, Y \leq y)\),
  - \((Y|X > x, Y > y)\),
  - \((Y|X \leq x, Y > y)\),
  - \((Y|X > x, Y \leq y)\) and
  - \((Y|X = x, Y \leq y)\).
- The same holds for \((X_1, \ldots, X_n)\) when \(n > 2\).
Comparisons between $Y$ and $(Y|X \leq x)$

Proposition

(i) $Y \geq_{ST} (Y|X \leq x)$ ($\leq_{ST}$) for all $F, G$ if and only if $C(F(x), u) \geq uF(x)$ ($\leq$) for all $u \in (0, 1)$.

(ii) $Y \geq_{HR} (Y|X \leq x)$ ($\leq_{HR}$) for all $F, G$ if and only if $\hat{C}(\bar{F}(x), u)/u$ is decreasing (increasing) in $u$ in $(0, 1)$.

(iii) $Y \geq_{RHR} (Y|X \leq x)$ ($\leq_{RHR}$) for all $F, G$ if and only if $\hat{C}(\bar{F}(x), u)/u$ is decreasing (increasing) in $u$ in $(0, 1)$.

(iv) $Y \geq_{LR} (Y|X \leq x)$ ($\leq_{LR}$) for all $F, G$ if and only if $C(F(x), u)$ is concave (convex) in $u$ in the interval $(0, 1)$.

(v) $Y \geq_{MRL} (Y|X \leq x)$ ($\leq_{MRL}$) for all $F, G$ if $\hat{C}(\bar{F}(x), u)/u$ is bathtub (upside-down bathtub) in $u$ in $(0, 1)$ and $E(Y) \geq E(Y|X \leq x)$ ($\leq$).
Comparisons between $Y$ and $(Y|X > x)$

Proposition

(i) $Y \leq_{ST} (Y|X > x) \ (\geq_{ST})$ for all $F, G$ if and only if $\hat{C}(\bar{F}(x), u) \geq u\bar{F}(x) \ (\leq)$ for all $u \in (0, 1)$.

(ii) $Y \leq_{HR} (Y|X > x) \ (\geq_{HR})$ for all $F, G$ if and only if $\hat{C}(\bar{F}(x), u)/u$ is decreasing (increasing) in $u$ in $(0, 1)$.

(iii) $Y \leq_{RHR} (Y|X > x) \ (\geq_{RHR})$ for all $F, G$ if and only if $C(F(x), u)/u$ is decreasing (increasing) in $u$ in $(0, 1)$.

(iv) $Y \leq_{LR} (Y|X > x) \ (\geq_{LR})$ for all $F, G$ if and only if $\hat{C}(\bar{F}(x), u)$ is concave (convex) in $u$ in $(0, 1)$.

(v) $Y \leq_{MRL} (Y|X > x) \ (\geq_{MRL})$ for all $F, G$ if $\hat{C}(\bar{F}(x), u)/u$ is bathtub (upside-down bathtub) in $u$ in $(0, 1)$ and $E(Y) \leq E(Y|X > x) \ (\geq)$. 
(\(X, Y\)) is positive (negative) quadrant dependent, PQD, (NQD) if \(C \geq \Pi (\leq)\), where \(\Pi(u, v) = uv\).
Dependence

- $(X, Y)$ is positive (negative) quadrant dependent, PQD, (NQD) if $C \geq \Pi$, where $\Pi(u, v) = uv$.
- PQD (NQD) implies Spearman $\rho_S \geq 0$ and Kendall $\tau_K \geq 0$ ($\leq 0$).
(X, Y) is \textit{positive (negative) quadrant dependent}, PQD, (NQD) if \( C \geq \Pi (\leq) \), where \( \Pi(u, v) = uv \).

PQD (NQD) implies Spearman \( \rho_S \geq 0 \) and Kendall \( \tau_K \geq 0 \) (\( \leq 0 \)).

The following conditions are equivalent:

(i) \((X, Y)\) is PQD, (NQD).
(ii) \( Y \geq_{ST} (Y|X \leq x) \) (\( \leq_{ST} \)) for all \( F, G \).
(iii) \( Y \leq_{ST} (Y|X > x) \) (\( \geq_{ST} \)) for all \( F, G \).
(X, Y) is positive (negative) quadrant dependent, PQD, (NQD) if C ≥ Π (≤), where Π(u, v) = uv.

PQD (NQD) implies Spearman ρ_S ≥ 0 and Kendall τ_K ≥ 0 (≤ 0).

The following conditions are equivalent:

(i) (X, Y) is PQD, (NQD).
(ii) Y ≥_{ST} (Y|X ≤ x) (≤_{ST}) for all F, G.
(iii) Y ≤_{ST} (Y|X > x) (≥_{ST}) for all F, G.

Note that

Pr(Y ≤ y) = F(x) Pr(Y ≤ y|X ≤ x) + \bar{F}(x) Pr(Y ≤ y|X > x).
Comparisons between $Y$ and $(Y|X = x)$

Proposition

(i) $Y \leq_{ST} (Y|X = x) \ (\geq_{ST})$ for all $F, G$ if and only if $\partial_1 C(F(x), u) \leq u \ (\geq)$ for all $u \in (0, 1)$.

(ii) $Y \leq_{HR} (Y|X = x) \ (\geq_{HR})$ for all $F, G$ if and only if $\partial_1 \hat{C}(\tilde{F}(x), u)/u$ is decreasing (increasing) in $u$ in $(0, 1)$.

(iii) $Y \leq_{RHR} (Y|X = x) \ (\geq_{RHR})$ for all $F, G$ if and only if $\partial_1 \hat{C}(\tilde{F}(x), u)/u$ is increasing (decreasing) in $u$ in $(0, 1)$.

(iv) $Y \leq_{LR} (Y|X = x) \ (\geq_{LR})$ for all $F, G$ if and only if $\partial_1 C(F(x), u)$ is concave (convex) in $u$ in $(0, 1)$.

(v) $Y \leq_{MRL} (Y|X = x) \ (\geq_{MRL})$ for all $F, G$ if $\partial_1 \hat{C}(\tilde{F}(x), u)/u$ is bathtub (upside-down bathtub) in $u$ in $(0, 1)$ and $E(Y) \leq E(Y|X = x) \ (\geq)$. 
Comparisons between \((Y|X \leq x_1)\) and \((Y|X \leq x_2)\)

\[\begin{align*}
(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2) & \text{ holds for all } F, G \text{ if, and only if, } \\
\frac{C(F(x_1), v)}{F(x_1)} & \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0, 1). \quad (3.1)
\end{align*}\]
Comparisons between \((Y|X \leq x_1)\) and \((Y|X \leq x_2)\)

\[ (Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2) \] holds for all \(F, G\) if, and only if,

\[
\frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0, 1).
\] (3.1)

\( (X, Y) \) is **Left Tail Decreasing (Increasing) LTD\((Y|X)\)** \((LTD(Y|X))\) in \(Y\) if \((Y|X \leq x)\) is ST-increasing (decreasing) in \(x\).
Comparisons between \((Y|X \leq x_1)\) and \((Y|X \leq x_2)\)

- \((Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)\) holds for all \(F, G\) if, and only if,
  \[
  \frac{C(F(x_1), \nu)}{F(x_1)} \geq \frac{C(F(x_2), \nu)}{F(x_2)}, \quad \forall \nu \in (0, 1). \quad (3.1)
  \]

- \((X, Y)\) is Left Tail Decreasing (Increasing) LTD\((Y|X)\) (LTI\((Y|X)\)) in \(Y\) if \((Y|X \leq x)\) is ST-increasing (decreasing) in \(x\).

- Hence, \((X, Y)\) is LTD\((Y|X)\) (LTI\((Y|X)\)) for all \(F, G\) if and only if \(C(u, \nu)/u\) is decreasing (increasing) in \(u\) for all \(\nu\) (see Nelsen p. 192, Th. 5.2.5).
Comparisons between \((Y|X \leq x_1)\) and \((Y|X \leq x_2)\)

- \((Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)\) holds for all \(F, G\) if, and only if,
  \[
  \frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)}, \quad \forall v \in (0, 1).
  \] (3.1)

- \((X, Y)\) is \textit{Left Tail Decreasing (Increasing) LTD}(\(Y|X\)) (\(LTI(Y|X)\)) in \(Y\) if \((Y|X \leq x)\) is ST-increasing (decreasing) in \(x\).

- Hence, \((X, Y)\) is LTD\((Y|X)\) (LTI\((Y|X)\)) for all \(F, G\) if and only if \(C(u, v)/u\) is decreasing (increasing) in \(u\) for all \(v\) (see Nelsen p. 192, Th. 5.2.5).

- It is a positive (negative) dependence property.
Comparisons between \((Y|X \leq x_1)\) and \((Y|X \leq x_2)\)

- \((Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)\) holds for all \(F, G\) if, and only if,
  \[
  \frac{C(F(x_1), \nu)}{F(x_1)} \geq \frac{C(F(x_2), \nu)}{F(x_2)}, \quad \forall \nu \in (0, 1). \tag{3.1}
  \]

- \((X, Y)\) is Left Tail Decreasing (Increasing) LTD\((Y|X)\) (LTI\((Y|X)\)) in \(Y\) if \((Y|X \leq x)\) is ST-increasing (decreasing) in \(x\).

- Hence, \((X, Y)\) is LTD\((Y|X)\) (LTI\((Y|X)\)) for all \(F, G\) if and only if \(C(u, \nu)/u\) is decreasing (increasing) in \(u\) for all \(\nu\) (see Nelsen p. 192, Th. 5.2.5).

- It is a positive (negative) dependence property.

- LTD\((Y|X)\) (LTI\((Y|X)\)) implies the positive (negative) quadrant dependent, PQD, (NQD) property of \((X, Y)\).
(i) \((Y|X \leq x)\) is ST-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(C(u, v)/u\) is decreasing (increasing) in \(u\) for all \(v\).

(ii) \((Y|X \leq x)\) is HR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \((v - \hat{C}(u_2, v))/(v - \hat{C}(u_1, v))\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(iii) \((Y|X \leq x)\) is RHR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(C(u_2, v)/C(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(iv) \((Y|X \leq x)\) is LR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\partial_2 C(u_2, v)/\partial_2 C(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(v) \((Y|X \leq x)\) is MRL-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \((v - \hat{C}(u_2, v))/(v - \hat{C}(u_1, v))\) is bathtub (upside-down bathtub) in \(v\) for all \(0 < u_1 \leq u_2 < 1\) and \(E(Y|X \leq x)\) is increasing (decreasing) in \(x\).
Comparisons between \((Y \mid X > x_1)\) and \((Y \mid X > x_2)\)

\[\hat{C}(\bar{F}(x_1), v) \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0, 1). \quad (3.2)\]
Comparisons between \( (Y|X > x_1) \) and \( (Y|X > x_2) \)

\( (Y|X > x_1) \leq_{ST} (Y|X > x_2) \) holds for all \( F, G \) if, and only if,

\[
\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0, 1). \tag{3.2}
\]

\( (X, Y) \) is Right Tail Increasing (Decreasing) RTI\( (Y|X) \) (RTD\( (Y|X) \)) in \( Y \) if \( (Y|X > x) \) is ST-increasing (decreasing) in \( x \).
Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

- $(Y|X > x_1) \leq_{ST} (Y|X > x_2)$ holds for all $F, G$ if, and only if,

  $$
  \frac{\hat{C}(\bar{F}(x_1), \nu)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), \nu)}{\bar{F}(x_2)}, \quad \forall \nu \in (0, 1). \tag{3.2}
  $$

- $(X, Y)$ is Right Tail Increasing (Decreasing) RTI$(Y|X)$ (RTD$(Y|X)$) in $Y$ if $(Y|X > x)$ is ST-increasing (decreasing) in $x$.

- $(X, Y)$ is RTI$(Y|X)$ (RTD$(Y|X)$) for all $F, G$ if and only if $\hat{C}(u, \nu)/u$ is decreasing (increasing) in $u$ for all $\nu$. 

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Comparisons between \((Y|X > x_1)\) and \((Y|X > x_2)\)

\(\triangleright\) \((Y|X > x_1) \preceq_{ST} (Y|X > x_2)\) holds for all \(F, G\) if, and only if,

\[
\frac{\hat{C}(\bar{F}(x_1), \nu)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), \nu)}{\bar{F}(x_2)}, \quad \forall \nu \in (0, 1). \tag{3.2}
\]

\(\triangleright\) \((X, Y)\) is **Right Tail Increasing (Decreasing)** \(RTI(Y|X)\) (\(RTD(Y|X)\)) in \(Y\) if \((Y|X > x)\) is \(ST\)-increasing (decreasing) in \(x\).

\(\triangleright\) \((X, Y)\) is **RTI** \((Y|X)\) (\(RTD(Y|X)\)) for all \(F, G\) if and only if \(\hat{C}(u, \nu)/u\) is decreasing (increasing) in \(u\) for all \(\nu\).

\(\triangleright\) It is a positive (negative) dependence property.
Comparisons between \((Y|X > x_1)\) and \((Y|X > x_2)\)

- \((Y|X > x_1) \leq_{ST} (Y|X > x_2)\) holds for all \(F, G\) if, and only if,
  \[
  \frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \leq \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)}, \quad \forall v \in (0, 1).
  \]  
  (3.2)

- \((X, Y)\) is Right Tail Increasing (Decreasing) \(RTI(Y|X)\) \((RTD(Y|X))\) in \(Y\) if \((Y|X > x)\) is ST-increasing (decreasing) in \(x\).

- \((X, Y)\) is \(RTI(Y|X)\) \((RTD(Y|X))\) for all \(F, G\) if and only if \(\hat{C}(u, v)/u\) is decreasing (increasing) in \(u\) for all \(v\).

- It is a positive (negative) dependence property.

- \(RTI(Y|X)\) \((RTD(Y|X))\) implies the positive (negative) quadrant dependent, PQD, \((NQD)\) property of \((X, Y)\).
(i) \((Y|X > x)\) is ST-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\hat{C}(u, v)/u\) is decreasing (increasing) in \(u\) for all \(v\).

(ii) \((Y|X > x)\) is HR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\hat{C}(u_2, v)/\hat{C}(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(iii) \((Y|X > x)\) is RHR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \((v - C(u_2, v))/(v - C(u_1, v))\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(iv) \((Y|X > x)\) is LR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\partial^2 \hat{C}(u_2, v)/\partial^2 \hat{C}(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(v) \((Y|X > x)\) is MRL-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\hat{C}(u_2, v)/\hat{C}(u_1, v)\) is upside-down bathtub (bathtub) in \(v\) for all \(0 < u_1 \leq u_2 < 1\) and \(E(Y|X > x)\) is increasing (decreasing) in \(x\).
Comparisons between $(Y|X = x_1)$ and $(Y|X = x_2)$

- $(Y|X = x_1) \leq_{ST} (Y|X = x_2)$ holds for all $F, G$ if, and only if,
  \[
  \partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1).
  \] (3.3)
Comparisons between \((Y \mid X = x_1)\) and \((Y \mid X = x_2)\)

- \((Y \mid X = x_1) \leq_{ST} (Y \mid X = x_2)\) holds for all \(F, G\) if, and only if,
  \[ \partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1). \quad (3.3) \]

- \(Y\) is *Stochastically Increasing (Decreasing)* \(SI(Y \mid X)\) \((SD(Y \mid X))\) in \(X\) if \((Y \mid X = x)\) is ST-increasing (decreasing) in \(x\).
Comparisons between \((Y|X = x_1)\) and \((Y|X = x_2)\)

- \((Y|X = x_1) \leq_{ST} (Y|X = x_2)\) holds for all \(F, G\) if, and only if,
  \[
  \partial_1 \hat{C}(\tilde{F}(x_1), v) \leq \partial_1 \hat{C}(\tilde{F}(x_2), v), \forall v \in (0, 1).
  \]  
  \((3.3)\)

- \(Y\) is \textit{Stochastically Increasing (Decreasing) SI} \((Y|X)\) \((SD(Y|X))\) in \(X\) if \((Y|X = x)\) is ST-increasing (decreasing) in \(x\).

- \(Y\) is \textit{SI} \((Y|X)\) \((SD(Y|X))\) in \(X\) if and only if \(\partial_1 \hat{C}(u, v)\) is decreasing (increasing) in \(u\) for all \(v\) (see Nelsen p. 196, Th. 5.2.10), that is, \(\hat{C}(u, v)\) is concave (convex) in \(u\).
Comparisons between \((Y|X = x_1)\) and \((Y|X = x_2)\)

- \((Y|X = x_1) \leq_{ST} (Y|X = x_2)\) holds for all \(F, G\) if, and only if,
  \[\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1).\]  
  (3.3)

- \(Y\) is **Stochastically Increasing (Decreasing) SI\((Y|X)\)** \((SD(Y|X))\) in \(X\) if \((Y|X = x)\) is ST-increasing (decreasing) in \(x\).

- \(Y\) is SI\((Y|X)\) \((SD(Y|X))\) in \(X\) if and only if \(\partial_1 \hat{C}(u, \nu)\) is decreasing (increasing) in \(u\) for all \(\nu\) (see Nelsen p. 196, Th. 5.2.10), that is, \(\hat{C}(u, \nu)\) is concave (convex) in \(u\).

- It is a positive (negative) dependence property.
Comparisons between \((Y|X = x_1)\) and \((Y|X = x_2)\)

- \((Y|X = x_1) \leq_{ST} (Y|X = x_2)\) holds for all \(F, G\) if, and only if,
\[
\partial_1 \hat{C}(\bar{F}(x_1), v) \leq \partial_1 \hat{C}(\bar{F}(x_2), v), \forall v \in (0, 1).
\]

\[(3.3)\]

- \(Y\) is *Stochastically Increasing (Decreasing)* SI(SD) \((Y|X)\) in \(X\) if \((Y|X = x)\) is ST-increasing (decreasing) in \(x\).

- \(Y\) is SI(SD) \((Y|X)\) in \(X\) if and only if \(\partial_1 \hat{C}(u, v)\) is decreasing (increasing) in \(u\) for all \(v\) (see Nelsen p. 196, Th. 5.2.10), that is, \(\hat{C}(u, v)\) is concave (convex) in \(u\).

- It is a positive (negative) dependence property.

- SI(SD) \((Y|X)\) implies LTD(LTI) \((Y|X)\) and RTI(RTD) \((Y|X)\) and so PQD (NQD).
(i) \((Y|X = x)\) is ST-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(C(u, v)\) (or \(\hat{C}(u, v)\)) is concave (convex) in \(u\) for all \(v\).

(ii) \((Y|X = x)\) is HR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\partial_1 \hat{C}(u_2, v)/\partial_1 \hat{C}(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(iii) \((Y|X = x)\) is RHR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\partial_1 C(u_2, v)/\partial_1 C(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(iv) \((Y|X = x)\) is LR-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\partial_2 \partial_1 C(u_2, v)/\partial_2 \partial_1 C(u_1, v)\) is increasing (decreasing) in \(v\) for all \(0 < u_1 \leq u_2 < 1\).

(v) \((Y|X = x)\) is MRL-increasing (decreasing) in \(x\) for all \(F, G\) if and only if \(\partial_1 \hat{C}(u_2, v)/\partial_1 \hat{C}(u_1, v)\) is upside-down bathtub (bathtub) in \(v\) for all \(0 < u_1 \leq u_2 < 1\) and \(E(Y|X = x)\) is increasing (decreasing) in \(x\).
The following conditions are equivalent:

(i) \((X, Y)\) is \(LTD(X|Y)\).
(ii) \((X|Y \leq y)\) is ST-increasing.
(iii) \(Y \geq_{RHR} (Y|X \leq x)\) for all \(x\).
(iv) \(Y \leq_{RHR} (Y|X > x)\) for all \(x\).
(v) \((Y|X \leq x) \leq_{RHR} (Y|X > x)\) for all \(x\).
(vi) \(C(u, v)/v\) is decreasing in \(v\) for all \(u\).
The following conditions are equivalent:

(i) \((X, Y)\) is LTD\((X|Y)\).
(ii) \((X|Y \leq y)\) is ST-increasing.
(iii) \(Y \geq_{RHR} (Y|X \leq x)\) for all \(x\).
(iv) \(Y \leq_{RHR} (Y|X > x)\) for all \(x\).
(v) \((Y|X \leq x) \leq_{RHR} (Y|X > x)\) for all \(x\).
(vi) \(C(u, v)/v\) is decreasing in \(v\) for all \(u\).

Analogous equivalences can be stated for the respective negative notions.
The following conditions are equivalent:

(i) \((X, Y)\) is \(RTI(X|Y)\).
(ii) \((X|Y > y)\) is \(ST\)-increasing.
(iii) \(Y \geq_{HR} (Y|X \leq x)\) for all \(x\).
(iv) \(Y \leq_{HR} (Y|X > x)\) for all \(x\).
(v) \((Y|X \leq x) \leq_{HR} (Y|X > x)\) for all \(x\).
(vi) \(\hat{C}(u, v)/v\) is decreasing in \(v\) for all \(u\).
The following conditions are equivalent:

(i) $(X, Y)$ is $RTI(X|Y)$.
(ii) $(X|Y > y)$ is ST-increasing.
(iii) $Y \geq_{HR} (Y|X \leq x)$ for all $x$.
(iv) $Y \leq_{HR} (Y|X > x)$ for all $x$.
(v) $(Y|X \leq x) \leq_{HR} (Y|X > x)$ for all $x$.
(vi) $\hat{C}(u, v)/v$ is decreasing in $v$ for all $u$.

Analogous equivalences can be stated for the respective negative notions.
The following conditions are equivalent:

(i) \((X, Y)\) is \(SI(X|Y)\), i.e., \((X|Y = y)\) is ST-increasing in \(y\).

(ii) \(Y \geq_{LR} (Y|X \leq x)\) for all \(x\).

(iii) \(Y \leq_{LR} (Y|X > x)\) for all \(x\).

(iv) \((Y|X \leq x) \leq_{LR} (Y|X > x)\) for all \(x\).

(v) \(C(u, v)\) is concave in \(v\) for all \(u\).

(vi) \(\hat{C}(u, v)\) is concave in \(v\) for all \(u\).
The following conditions are equivalent:

(i) $(X, Y)$ is $SI(X|Y)$, i.e., $(X|Y = y)$ is ST-increasing in $y$.
(ii) $Y \geq_{LR} (Y|X \leq x)$ for all $x$.
(iii) $Y \leq_{LR} (Y|X > x)$ for all $x$.
(iv) $(Y|X \leq x) \leq_{LR} (Y|X > x)$ for all $x$.
(v) $C(u, v)$ is concave in $v$ for all $u$.
(vi) $\hat{C}(u, v)$ is concave in $v$ for all $u$.

Analogous results can be stated for $(Y|X)$ and for the respective negative notions.
(X, Y) is *Left Corner Set Decreasing* (LCSD) if \( \Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2) \) is decreasing in \( x_2 \) and \( y_2 \).
Dependence

- \((X, Y)\) is *Left Corner Set Decreasing (LCSD)* if
  \[\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)\]
  is decreasing in \(x_2\) and \(y_2\).

- \((X, Y)\) is *Right Corner Set Increasing (RCSI)* if
  \[\Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)\]
  is increasing in \(x_2\) and \(y_2\).
Dependence

- $(X, Y)$ is **Left Corner Set Decreasing (LCSD)** if
  \[
  \Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)
  \]
  is decreasing in $x_2$ and $y_2$.

- $(X, Y)$ is **Right Corner Set Increasing (RCSI)** if
  \[
  \Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)
  \]
  is increasing in $x_2$ and $y_2$.

- The following conditions are equivalent:
  1. **LCSD**.
  2. $(Y | X \leq x)$ is RHR-increasing in $x$.
  3. $C$ is $TP_2$. 

Similar results can be obtained for **Left Corner Set Increasing (LCSI)** and **Right Corner Set Decreasing (RCSD)**.
Dependence

- \((X, Y)\) is **Left Corner Set Decreasing (LCSD)** if
  \(\Pr(X \leq x_1, Y \leq y_1|X \leq x_2, Y \leq y_2)\) is decreasing in \(x_2\) and \(y_2\).

- \((X, Y)\) is **Right Corner Set Increasing (RCSI)** if
  \(\Pr(X > x_1, Y > y_1|X > x_2, Y > y_2)\) is increasing in \(x_2\) and \(y_2\).

- The following conditions are equivalent:
  1. **LCSD**.
  2. \((Y|X \leq x)\) is RHR-increasing in \(x\).
  3. \(C\) is \(TP_2\).

- The following conditions are equivalent:
  1. **RCSI**.
  2. \((Y|X > x)\) is HR-increasing in \(x\).
  3. \(\hat{C}\) is \(TP_2\).

Similar results can be obtained for **Left Corner Set Increasing (LCSI)** and **Right Corner Set Decreasing (RCSD)**.
Dependence

- $(X, Y)$ is **Left Corner Set Decreasing (LCSD)** if
  \[ \Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2) \]
  is decreasing in $x_2$ and $y_2$.

- $(X, Y)$ is **Right Corner Set Increasing (RCSI)** if
  \[ \Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2) \]
  is increasing in $x_2$ and $y_2$.

- The following conditions are equivalent:
  - (i) **LCSD**.
  - (ii) $(Y|X \leq x)$ is RHR-increasing in $x$.
  - (iii) $C$ is **TP$_2$**.

- The following conditions are equivalent:
  - (i) **RCSI**.
  - (ii) $(Y|X > x)$ is HR-increasing in $x$.
  - (iii) $\hat{C}$ is **TP$_2$**.

- Similar results can be obtained for **Left Corner Set Increasing (LCSI)** and **Right Corner Set Decreasing (RCSD)**.

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Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X = x)$ is ORD-increasing in $x$. 
Dependence

- $SI_{ORD}(Y|X)$ if $(Y|X = x)$ is ORD-increasing in $x$.
- $SI_{ORD}(X|Y)$ if $(X|Y = y)$ is ORD-increasing in $y$. 
Distorted Distributions

Dependence

Dependence

▶ $SI_{ORD}(Y|X)$ if $(Y|X = x)$ is ORD-increasing in $x$.
▶ $SI_{ORD}(X|Y)$ if $(X|Y = y)$ is ORD-increasing in $y$.
▶ The negative dependence properties $SD_{ORD}(Y|X)$ and $SD_{ORD}(X|Y)$ are defined in a similar way.
Dependence

- $S_{ORD}(Y|X)$ if $(Y|X = x)$ is ORD-increasing in $x$.
- $S_{ORD}(X|Y)$ if $(X|Y = y)$ is ORD-increasing in $y$.
- The negative dependence properties $S_{ORD}(Y|X)$ and $S_{ORD}(X|Y)$ are defined in a similar way.
- The following conditions are equivalent:
  
  (i) $S_{RHR}(X|Y)$.
  (ii) $(Y|X \leq x)$ is LR-increasing.
  (iii) $\partial_2 C$ is $TP_2$. 
Distorted Distributions
Conditional distributions. Representations
Comparisons, dependence and bounds

Dependence

- \( SI_{ORD}(Y|X) \) if \( (Y|X = x) \) is ORD-increasing in \( x \).
- \( SI_{ORD}(X|Y) \) if \( (X|Y = y) \) is ORD-increasing in \( y \).
- The negative dependence properties \( SD_{ORD}(Y|X) \) and \( SD_{ORD}(X|Y) \) are defined in a similar way.
- The following conditions are equivalent:
  - (i) \( SI_{RHR}(X|Y) \).
  - (ii) \( (Y|X \leq x) \) is LR-increasing.
  - (iii) \( \partial_2 C \) is \( TP_2 \).
- The following conditions are equivalent:
  - (i) \( SI_{HR}(X|Y) \).
  - (ii) \( (Y|X > x) \) is LR-increasing.
  - (iii) \( \partial_2 \hat{C} \) is \( TP_2 \).
Dependence relationships

\[ SI_{LR}(X|Y) \Rightarrow SI_{RHR}(X|Y) \Rightarrow LCSD \]
\[ SI_{HR}(X|Y) \Rightarrow SI_{ST}(X|Y) \Rightarrow LTD(X|Y) \]
\[ RCSI \Rightarrow RTI(X|Y) \Rightarrow PQD \]
Bounds for \((Y|X \leq x)\)

**Proposition**

If \((X, Y)\) has the copulas \(C\) and \(\hat{C}\) and \(Y \geq 0\), then:

\[
\inf_{u \in (0,1]} \frac{C(F(x), u)}{uF(x)} \leq \frac{\Pr(Y \leq y|X \leq x)}{G(y)} \leq \frac{C(F(x), u)}{uF(x)} \sup_{u \in (0,1]} \frac{C(F(x), u)}{uF(x)}
\]

\[
\inf_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \leq \frac{\Pr(Y > y|X \leq x)}{\bar{G}(y)} \leq \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \sup_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)}
\]

and

\[
\inf_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \leq \frac{E(Y|X \leq x)}{E(Y)} \leq \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \sup_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)}.
\]
Bounds for \( (Y|X > x) \)

**Proposition**

If \((X, Y)\) has the copulas \(C\) and \(\hat{C}\) and \(Y \geq 0\), then:

\[
\inf_{u \in (0,1]} \frac{u - C(F(x), u)}{u \bar{F}(x)} \leq \frac{\Pr(Y \leq y|X > x)}{G(y)} \leq \sup_{u \in (0,1]} \frac{u - C(F(x), u)}{u \bar{F}(x)},
\]

\[
\inf_{u \in (0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u \bar{F}(x)} \leq \frac{\Pr(Y > y|X > x)}{\bar{G}(y)} \leq \sup_{u \in (0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u \bar{F}(x)}.
\]

and

\[
\inf_{u \in (0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u \bar{F}(x)} \leq \frac{E(Y|X > x)}{E(Y)} \leq \sup_{u \in (0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u \bar{F}(x)}.
\]
Bounds for \((Y|X = x)\)

**Proposition**

If \((X, Y)\) has the copulas \(C\) and \(\hat{C}\) and \(Y \geq 0\), then:

\[
\inf_{u \in (0,1]} \frac{\partial_1 C(F(x), u)}{u} \leq \frac{\Pr(Y \leq y|X = x)}{G(y)} \leq \sup_{u \in (0,1]} \frac{\partial_1 C(F(x), u)}{u},
\]

\[
\inf_{u \in (0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u} \leq \frac{\Pr(Y > y|X = x)}{\bar{G}(y)} \leq \sup_{u \in (0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u},
\]

and

\[
\inf_{u \in (0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u} \leq \frac{E(Y|X = x)}{E(Y)} \leq \sup_{u \in (0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u}.
\]
Bounds based on $\Delta_G$

**Proposition**

If $(X, Y)$ has the copulas $C$ and $\hat{C}$, $0 = \inf\{y : G(y) > 0\}$, $p = \bar{F}(x) \text{ and } \Delta_G := 2 \int_0^\infty G(y)(1 - G(y))dy > 0$, then:

$$\inf_{u \in (0, 1)} \frac{up - \hat{C}(p, u)}{2u(1 - u)(1 - p)} \leq \frac{E(Y|X \leq x) - E(Y)}{\Delta_G} \leq \sup_{u \in (0, 1)} \frac{up - \hat{C}(p, u)}{2u(1 - u)(1 - p)},$$

$$\inf_{u \in (0, 1)} \frac{\hat{C}(p, u) - up}{2u(1 - u)p} \leq \frac{E(Y|X > x) - E(Y)}{\Delta_G} \leq \sup_{u \in (0, 1)} \frac{\hat{C}(p, u) - up}{2u(1 - u)p}$$

and

$$\inf_{u \in (0, 1)} \frac{\partial_1 \hat{C}(p, u) - u}{2u(1 - u)} \leq \frac{E(Y|X = x) - E(Y)}{\Delta_G} \leq \sup_{u \in (0, 1)} \frac{\partial_1 \hat{C}(p, u) - u}{2u(1 - u)}.$$
Example 1

If \((X, Y)\) has the Clayton-Oakes copula:

\[
C(u, v) = \frac{uv}{u + v - uv}
\]

for \(0 \leq u, v \leq 1\), then

\[
q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.
\]
Example 1

- If \((X, Y)\) has the Clayton-Oakes copula:

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\[
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\]

- Then \(Y \geq_{ST} (Y\mid X \leq x)\) for any \(G\) since \(C(F(x), u) \geq uF(x)\).
Example 1

- If \((X, Y)\) has the Clayton-Oakes copula:

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- Even more, \(Y \geq_{LR} (Y|X \leq x)\) for any \(G\) since \(q_1\) is concave.
Example 1

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  q_1(u) = \frac{C(F(x), u)}{F(x)} = \frac{u}{F(x) + u - uF(x)}.
  \]
  
- Then \(Y \geq_{ST} (Y|X \leq x)\) for any \(G\) since \(C(F(x), u) \geq uF(x)\).
- Even more, \(Y \geq_{LR} (Y|X \leq x)\) for any \(G\) since \(q_1\) is concave.
- \((Y|X \leq x), (Y|X > x)\) and \((Y|X = x)\) are LR-increasing in \(x\).
Example 1

If \((X, Y)\) has the Clayton-Oakes copula:

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Then \(Y \geq_{ST} (Y|X \leq x)\) for any \(G\) since \(C(F(x), u) \geq uF(x)\).

Even more, \(Y \geq_{LR} (Y|X \leq x)\) for any \(G\) since \(q_1\) is concave.

\((Y|X \leq x), (Y|X > x)\) and \((Y|X = x)\) are LR-increasing in \(x\).

\[\rho_S = -39 + 4\pi^2 \approx 0.478417 \text{ and } \tau_K = 1/3.\]
An example

We can obtain the following bounds:

\[ G(y) \leq \Pr(Y \leq y|X \leq x) \leq \frac{1}{F(x)} G(y) \]

and

\[ 0 \leq \Pr(Y > y|X \leq x) \leq \bar{G}(y). \]
An example

- We can obtain the following bounds:

\[ G(y) \leq \Pr(Y \leq y \mid X \leq x) \leq \frac{1}{F(x)} G(y) \]

and

\[ 0 \leq \Pr(Y > y \mid X \leq x) \leq \tilde{G}(y). \]

- Then \( E(Y \mid X \leq x) \leq E(Y) \) when \( Y \geq 0 \).
An example

We can obtain the following bounds:

\[ G(y) \leq \Pr(Y \leq y | X \leq x) \leq \frac{1}{F(x)} G(y) \]

and

\[ 0 \leq \Pr(Y > y | X \leq x) \leq \tilde{G}(y). \]

Then \( E(Y | X \leq x) \leq E(Y) \) when \( Y \geq 0 \).

Analogously, for \( f(x) > 0 \), we have

\[ 0 \leq \Pr(Y \leq y | X = x) \leq G(y) \]

when \( F(x) \geq 1/2 \) and

\[ 0 \leq \Pr(Y \leq y | X = x) \leq \frac{1}{4F(x)\bar{F}(x)} G(y) \]

when \( F(x) < 1/2 \).
References: Distorted distributions

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Main References: Conditional distributions and systems

References

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- Thank you for your attention!!