Extensions of signature representations for coherent systems

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References

The talk is based on the following references:

- Navarro J, Fernández-Sánchez J. (2020). On the extension of signature-based representations for coherent systems with dependent non-exchangeable components. Journal of Applied Probability 57, 429–440.
- Navarro J., Rychlik T., Spizzichino F. (2020). Conditions on marginals and copula of component lifetimes for signature representation of system lifetime. Fuzzy Sets and Systems. Available online November 12, 2020. https://doi.org/10.1016/j.fss.2020.11.006

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- Then the system state $\psi(x_1, \dots, x_n) \in \{0,1\}$ is completely determined by the structure function ψ and the component states $x_1, \dots, x_n \in \{0,1\}$.
- A system ψ is **semi-coherent** if it is increasing, $\psi(0, \dots, 0) = 0$ and $\psi(1, \dots, 1) = 1$.

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- Barlow and Proschan (1975). *Statistical Theory of Reliability and Life Testing.* International Series in Decision Processes. Holt, Rinehart and Winston, Inc., New York.

Minimal path sets

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- If $P_1,\dots,P_r$ are the minimal path sets of a semi-coherent system ψ, then

$$\psi(x_1,\dots,x_n)=\max_{i=1,\dots,r}\min_{j\in P_i} x_j. \tag{1.1}$$

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$$\psi(x_1, \dots, x_n) = \max_{i=1,\dots,r} \min_{j \in P_i} x_j. \tag{1.1}$$

- Here $\psi_P = \min_{j \in P} x_j$ represents the series system with components in P.

Lifetimes

- Let T be the system lifetime and let $X_1, \ldots, X_n$ be the component lifetimes. Then

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- Let $\bar{F}_T(t) = \Pr(T > t)$ be the system reliability (or survival) function and let $\bar{F}_i(t) = \Pr(X_i > t)$ for $i = 1, \dots, n$ be the component reliability functions.

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- The purpose is to write

$$\bar{F}_T = \bar{Q}(\bar{F}_1, \dots, \bar{F}_n). \tag{1.3}$$

Samaniego's representation

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- Theorem (Samaniego, 1985)

If T is the lifetime of a coherent system with IID component lifetimes having a common continuous reliability function $\bar{F}$, then

$$\bar{F}_T(t) = s_1\bar{F}_{1:n}(t) + \cdots + s_n\bar{F}_{n:n}(t), \tag{1.4}$$

where $\bar{F}_{1:n}, \ldots, \bar{F}_{n:n}$ are the reliability functions of the ordered component lifetimes $X_{1:n} \leq \cdots \leq X_{n:n}$ (order statistics) and $s_1 + \cdots + s_n = 1$.

Signature vector

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- Under these assumptions $\mathbf{s}$ only depends on the structure ψ.
- It can be computed as $s_i = \Pr(T = X_{i:n})$, as

$$s_i = \frac{|\{\sigma : \psi(x_1, \dots, x_n) = x_{i:n} \text{ when } x_{\sigma(1)} \le \dots \le x_{\sigma(n)}\}|}{n!}$$

or as

$$s_i = \frac{1}{\binom{n}{n-i+1}} \sum_{\sum_{j=1}^{n} x_j = n-i+1} \psi(x_1, \dots x_n) - \frac{1}{\binom{n}{n-i}} \sum_{\sum_{j=1}^{n} x_j = n-i} \psi(x_1, \dots x_n) \tag{1.5}$$

Order statistics

- If $X_1, \ldots, X_n$ are IID$\sim F$, then

$$\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t). \tag{1.6}$$

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- Hence from Samaniego's theorem

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t). \tag{1.7}$$

Stochastic comparisons

Theorem (Kochar, Mukerjee and Samaniego, 1999)

Let T_1 and T_2 be the lifetimes of two coherent systems based on n IID components with a common continuous distribution function F. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be their respective signatures.

(i) *If $\mathbf{s}_1 \leq_{ST} \mathbf{s}_2$, then $T_1 \leq_{ST} T_2$ for all F;*

(ii) *If $\mathbf{s}_1 \leq_{HR} \mathbf{s}_2$, then $T_1 \leq_{HR} T_2$ for all F;*

(iii) *If $\mathbf{s}_1 \leq_{LR} \mathbf{s}_2$, then $T_1 \leq_{LR} T_2$ for all abs. cont. F.*

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- $T = X_{1:2} = \min(X_1, X_2)$.
- $s_1 = \Pr(T = X_{1:2}) = 1$ and $s_2 = \Pr(T = X_{2:2}) = 1/2$.
- Samaniego's representation does not hold

$$\bar{F}_{1:2} \neq 1\bar{F}_{1:2} + \frac{1}{2}\bar{F}_{2:2}.$$

- However, if we use (1.5), then $s_1 = 1$, $s_2 = 0$ and Samaniego's representation holds.

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- The signature $\mathbf{s}$ only depends on ψ while $\mathbf{p}$ depends on both ψ and the joint distribution of $X_1, \ldots, X_n$.
- In the IID continuous case, they coincide.
- In the preceding example $\mathbf{p} = (1, 1/2)$ and $\mathbf{s} = (1, 0)$.

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$$(X_1, \ldots, X_n) =_{ST} (X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

- Theorem (Navarro and Rychlik, 2007)

If T is the lifetime of a coherent system with component lifetimes having an absolutely continuous joint EXC distribution, then $\mathbf{p} = \mathbf{s}$ *and*

$$\bar{F}_T(t) = p_1 \bar{F}_{1:n}(t) + \cdots + p_n \bar{F}_{n:n}(t). \tag{2.1}$$

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- Theorem (Navarro et al., 2008)

If T is the lifetime of a coherent system with component lifetimes having a common EXC distribution and structural signature s*, then*

$$\bar{F}_T(t) = s_1 \bar{F}_{1:n}(t) + \cdots + s_n \bar{F}_{n:n}(t). \tag{2.2}$$

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- It can be applied to the general IID case (as in the Bernoulli example above).

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- It will allow us to compare systems with different orders.
- It is based on the concept of signature of order n.

Theorem (Navarro et al., 2008)

If $T = \psi(X_1, \dots, X_k)$ is the lifetime of a semi-coherent system with component lifetimes $(X_1, \dots, X_n)$ ($k < n$) having a common EXC distribution, then

$$\bar{F}_T(t) = s_1^{(n)} \bar{F}_{1:n}(t) + \cdots + s_n^{(n)} \bar{F}_{n:n}(t) \qquad (2.3)$$

where $\mathbf{s}^{(n)} = (s_1^{(n)}, \dots, s_n^{(n)})$ is the structural signature of order n (i.e. the signature obtained from (1.5) in dimension n).

Theorem (Navarro et al., 2008)

Let T_1 and T_2 be the lifetimes of two semi-coherent systems with component lifetimes $(X_1, \ldots, X_n)$ having an EXC joint distribution $\boldsymbol{F}$, and signatures of order n, $\boldsymbol{s}_1^{(n)}$ and $\boldsymbol{s}_2^{(n)}$, respectively.

(i) *If $\boldsymbol{s}_1^{(n)} \leq_{ST} \boldsymbol{s}_2^{(n)}$, then $T_1 \leq_{ST} T_2$ for all $\boldsymbol{F}$;*

(ii) *If $\boldsymbol{s}_1^{(n)} \leq_{HR} \boldsymbol{s}_2^{(n)}$, then $T_1 \leq_{HR} T_2$ for all $\boldsymbol{F}$ such that*

$$X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}; \tag{2.4}$$

(iii) *If $\boldsymbol{s}_1^{(n)} \leq_{HR} \boldsymbol{s}_2^{(n)}$, then $T_1 \leq_{MRL} T_2$ for all $\boldsymbol{F}$ such that*

$$X_{1:n} \leq_{MRL} \cdots \leq_{MRL} X_{n:n}; \tag{2.5}$$

(iv) *If $\boldsymbol{s}_1^{(n)} \leq_{LR} \boldsymbol{s}_2^{(n)}$, then $T_1 \leq_{LR} T_2$ for all $\boldsymbol{F}$ such that*

$$X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}. \tag{2.6}$$

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- Therefore, the ID assumption is necessary for that representation.
- Let us consider the system $T = \min(X_1, \max(X_1, X_2))$:

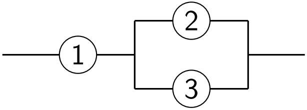

Figure: A coherent system of order 3.

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- If $X_{P_1} = \min(X_1, X_2)$ and $X_{P_2} = \min(X_1, X_3)$, then

$$\begin{aligned}
\bar{F}_T(t) &= \Pr(\{X_{P_1} > t\} \cup \{X_{P_2} > t\}) \\
&= \Pr(X_{P_1} > t) + \Pr(X_{P_2} > t) - \Pr(X_{P_1 \cup P_2} > t) \\
&= \Pr(X_1 > t, X_2 > t) + \Pr(X_1 > t, X_3 > t) \\
&\quad - \Pr(X_1 > t, X_2 > t, X_3 > t) \\
&=_{IND} \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)
\end{aligned}$$

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- If $X_{P_1} = \min(X_1, X_2)$ and $X_{P_2} = \min(X_1, X_3)$, then

$$\begin{aligned}\bar{F}_T(t) &= \Pr(\{X_{P_1} > t\} \cup \{X_{P_2} > t\})\\ &= \Pr(X_{P_1} > t) + \Pr(X_{P_2} > t) - \Pr(X_{P_1 \cup P_2} > t)\\ &= \Pr(X_1 > t, X_2 > t) + \Pr(X_1 > t, X_3 > t)\\ &\quad - \Pr(X_1 > t, X_2 > t, X_3 > t)\\ &=_{IND} \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)\end{aligned}$$

- If $\bar{F}_1(t) = e^{-2t}$ and $\bar{F}_2(t) = \bar{F}_3(t) = e^{-t}$, then

$$\bar{F}_T(t) = 2e^{-3t} - e^{-4t}, \text{ for } t \geq 0.$$

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- Analogously, for the order statistics we get

$$\bar{F}_{1:3}(t) = e^{-4t},$$
$$\bar{F}_{2:3}(t) = e^{-2t} + 2e^{-3t} - 2e^{-4t},$$
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- Therefore $\bar{F}_T = c_1\bar{F}_{1:3} + c_2\bar{F}_{2:3} + c_3\bar{F}_{3:3}$, that is,

$$2e^{-3t} - e^{-4t} = c_1 e^{-4t} + c_2(e^{-2t} + 2e^{-3t} - 2e^{-4t}) + c_3(2e^{-t} - 2e^{-3t} + e^{-4t})$$

does not hold for $c_1, c_2, c_3 \in \mathbb{R}$.

Example 2

- Hence $\bar{F}_T$ is not equal to the mixture obtained neither with the structural signature $\mathbf{s} = (1/3, 2/3, 0)$ given by

$$\bar{F}_s := \frac{1}{3}\bar{F}_{1:3} + \frac{2}{3}\bar{F}_{2:3}$$

nor with that obtained with the probabilistic signature

$$\bar{F}_p := p_1\bar{F}_{1:3} + p_2\bar{F}_{2:3},$$

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where X_1 and $Y = \min(X_2, X_3)$ are IID.

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- In this example

$$p_1 = \Pr(X_1 < \min(X_2, X_3)),$$

where X_1 and $Y = \min(X_2, X_3)$ are IID.

- Therefore, $p_1 = p_2 = 1/2$.

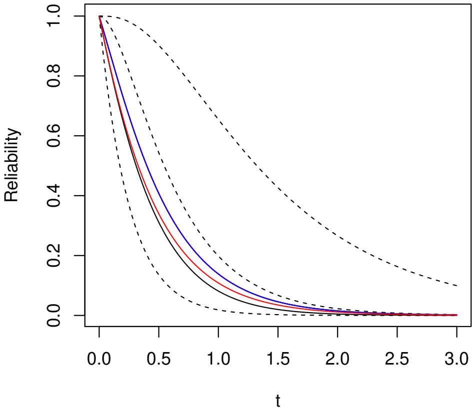

Figure: Reliability functions $\bar{F}_T$ (black), $\bar{F}_s$ (blue), $\bar{F}_p$ (red) and $\bar{F}_{k:3}$ (dashed lines) for $k = 1, 2, 3$.

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- It is based on the vector of the component states at time t, $(Z_1(t), \ldots, Z_n(t))$, where $Z_i(t) = 1$ (0) iff $X_i > t$ ($\leq$).
- It can be stated as follows:

Theorem (Marichal, Mathonet and Waldhauser, 2011)

If $n > 2$, the following conditons are equivalent:

(i) *Samaniego's representation holds with the structural signature for all the coherent systems of order n;*

(ii) *$(Z_1(t), \ldots, Z_n(t))$ is EXC for all $t \geq 0$.*

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- It is based on the copula representation for $(X_1, \ldots, X_n)$

$$\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(F_1(x_1), \ldots, F_n(x_n)),$$

where C is a copula function (i.e. a distribution function with uniform marginals on $(0,1)$).

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- It is based on the copula representation for $(X_1, \ldots, X_n)$

$$\Pr(X_1 \le x_1, \ldots, X_n \le x_n) = C(F_1(x_1), \ldots, F_n(x_n)),$$

  where C is a copula function (i.e. a distribution function with uniform marginals on $(0, 1)$).
- The random vector $(X_1, \ldots, X_n)$ is EXC iff
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- So let us to relax (ii).

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- We say that a copula C es **diagonal dependent (DD)** if

$$C(u_1, \ldots, u_n) = C(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \quad (3.1)$$

for all permutations σ and all $1 < k < n$, where $u_i = u \in [0, 1]$ for all $i = 1, \ldots, k$ and $u_i = 1$ for $i = k + 1, \ldots, n$.

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- It means that all the copulas of the k-dimensional marginals have the same diagonal sections.
- For example, if $n=3$, then it is equivalent to

$$C(u,u,1)=C(u,1,u)=C(1,u,u),\quad \text{for all } u\in[0,1].$$

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- Now we can state the following theorem:

Theorem (Navarro and Fernández-Sánchez, 2020)

If T is the lifetime of a coherent system and the following conditions hold:

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- A similar property holds for semi-coherent systems with the structural signature of order n.

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- Therefore, for any copula C we can find a "close" DD copula C^*.

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- If $S = [0,1]$, then it is DD.
- Now we can state the following theorem.

Theorem (Navarro, Rychlik and Spizzichino, 2020)

If $n > 2$, the following conditions are equivalent:

(i) *Samaniego's representation holds with the structural signature for all the coherent systems of order n;*

(ii) *If $A_i = \{X_i \leq t\}$ and $\bar{A}_i = \{X_i > t\}$, then*

$$\Pr(A_1 \cap \cdots \cap A_k \cap \bar{A}_{k+1} \cap \cdots \cap \bar{A}_n) = \Pr(A_{\sigma(1)} \cap \cdots \cap A_{\sigma(k)} \cap \bar{A}_{\sigma(k+1)} \cap \cdots \cap \bar{A}_{\sigma(n)})$$

for all permutation σ, all $1 < k < n$ and all $t > 0$;

(iii) *The vector with the component states at time t is EXC for all $t \geq 0$;*

(iv) *The component lifetimes are ID $F_1 = \cdots = F_n = F$ and its copula is S-DD, where $S = ImF = \{u : F(t) = u \text{ for } t > 0\}$.*

Example 3

- Let us consider again $T = \min(X_1, \max(X_2, X_3))$ with

$$\bar{F}(t) = \Pr(X_1 > t, X_2 > t) + \Pr(X_1 > t, X_3 > t) - \Pr(X_1 > t, X_2 > t, X_3 > t).$$

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$$\Pr(X_1 > x_1, X_2 > x_2, X_3 > x_3) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \bar{F}_3(x_3)),$$

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  where $\hat{C}$ is the survival copula. C is DD iff $\hat{C}$ is DD.
- If we assume $\bar{F}_1 = \bar{F}_2 = \bar{F}_3 = \bar{F}$ (ID), then

$$\Pr(X_1 > t, X_2 > t) = \hat{C}(\bar{F}(t), \bar{F}(t), 1)$$
$$\Pr(X_1 > t, X_3 > t) = \hat{C}(\bar{F}(t), 1, \bar{F}(t))$$
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- Therefore, $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ with

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- As the signature is $s = (1/3, 2/3, 0)$ we do not need $\bar{F}_{3:3}$.

Example 3: IID components

- If the components are IID, $\hat{C}(u_1, u_2, u_3) = u_1 u_2 u_3$ and

$$\bar{q}(u) = 2u^2 - u^3$$
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holds since

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- If $\hat{C}$ is DD, then

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does hold for $\theta \neq 0$ since

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- For discrete distributions F, this assumption can be relaxed to S-DD copulas.
- Moreover, the signature comparisons do not detect all the orderings (see Rychlik, Navarro and Rubio JAP 2018, 55 (4), 1261–1271).

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- That's all,

  Thank you for your atention!!!
- The complete references can be seen in my webpage:

  $https://webs.um.es/jorgenav/miwiki/doku.php$