Recent advances in system reliability theory using signatures

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[1]Supported by Ministerio de Ciencia y Tecnologia, MTM2006-12834
[2]Based on the paper Navarro, Samaniego, Balakrishnan, and Bhattacharya (2008, Naval Res. Logist. **55**, 313-327)

Coherent systems and order statistics

- $X_1, X_2, \ldots, X_n$ (positive) random variables.
- $X_1, X_2, \ldots, X_n$ IID
- $X_1, X_2, \ldots, X_n$ exchangeable (EXC), i.e., for any permutation σ

$$(X_1, X_2, \ldots, X_n) =_{ST} (X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)})$$

- $\overline{F}(t) = \Pr(X_i > t)$ reliability (survival) function.
- $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ the associated OS.
- $X_{k:n}$ represents the lifetime of the k-out-of-n:F system.
- $T = \phi(X_1, X_2, \ldots, X_n)$ lifetime of a coherent system.
- $T = \max_{1 \le j \le r} X_{P_j}$; P_j minimal path sets, $X_P = \min_{i \in P} X_i$.
- $T = X_{i:n}$ with probability $s_i = \Pr(T = X_{i:n})$.

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Mixture representations

- Samaniego (1985), IID and F continuous, then

$$\bar{F}_T = \sum_{i=1}^{n} s_i \bar{F}_{i:n}. \tag{1.1}$$

- $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ is the signature of T, $s_i = \Pr(T = X_{i:n})$.
- s_i does not depend on F and

$$s_i = \frac{1}{n!} \sum_{\sigma} 1(\sigma \in A_i)$$

$A_i = \{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)}\}$.
- Navarro and Rychlik (2007), (1.1) holds for EXC r.v. with absolutely continuous joint distribution.
- (1.1) does not necessarily hold if F is not a continuous function (e.g. Bernoulli distribution).

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Mixture representations

- Navarro et al. (2007), if T has EXC components, then

$$\overline{F}_T = \sum_{i=1}^{n} a_i \overline{F}_{1:i}. \qquad (1.2)$$

- $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is the minimal signature (or domination) of T (a_i does not depend on F but can be negative).
- A similar representation holds in terms of parallel system.
- In particular, for the OS:

$$\overline{F}_{i:n} = \sum_{j=n-i+1}^{n} (-1)^{j+i-n-1} \binom{j-1}{n-i} \binom{n}{j} \overline{F}_{1:j}. \qquad (1.3)$$

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Mixture representations-General case

- Recall that $T = \max_{1 \le j \le r} X_{P_j}$
- So: $\overline{F}_t(t) = P(T > t) = P(\cup_{j=1}^{r}\{X_{P_j} > t\})$
- By using the inclusion-exclusion formula, we have

$$\overline{F}_T = \sum_{j=1}^{r} \overline{F}_{P_j} - \sum_{i<j} \overline{F}_{P_i \cup P_j} + \ldots \pm \overline{F}_{1:n}.$$

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Stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$ stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow m_X(t) \leq m_Y(t)$, mean residual life order.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
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Stochastic orderings relations

$$\begin{array}{ccccc}
E(X_{s,t}) \le E(Y_{s,t}) & \Rightarrow & E(X_t) \le E(Y_t) & \Rightarrow & E(X) \le E(Y) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X \le_{DTM} Y & \Rightarrow & X \le_{MRL} Y & \Rightarrow & X \le_{M} Y \\
\Uparrow & & \Uparrow & & \Uparrow \\
X \le_{LR} Y & \Rightarrow & X \le_{HR} Y & \Rightarrow & X \le_{ST} Y \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_{s,t} \le_{ST} Y_{s,t} & \Rightarrow & X_t \le_{ST} Y_t & \Rightarrow & \overline{F}_X \le \overline{F}_Y
\end{array}$$

where $Z_t = (Z - t | Z > t)$ and $Z_{s,t} = (Z | s < Z < t)$ (see Navarro, Belzunce and Ruiz 1997, PEIS).

Stochastic comparisons using signatures

Theorem (Kochar, Mukerjee and Samaniego (1999))

Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signatures of the two coherent systems of order n, both based on components with IID lifetimes with common continuous distribution F. Let T_1 and T_2 be their respective lifetimes.
(a) If $\mathbf{s}_1 \leq_{ST} \mathbf{s}_2$, then $T_1 \leq_{ST} T_2$.
(b) If $\mathbf{s}_1 \leq_{HR} \mathbf{s}_2$, then $T_1 \leq_{HR} T_2$.
(c) If F is absolutely continuous and $\mathbf{s}_1 \leq_{LR} \mathbf{s}_2$, then $T_1 \leq_{LR} T_2$.

Mixed systems

- A **mixed system** of order n is a stochastic mixture of coherent systems of order n (Boland and Samaniego 2004).
- The mixed system which selects among n-component systems with signatures $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ according to the mixing distribution $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ will have signature $\sum_{i=1}^{k} p_i \mathbf{s}_i$.
- From (1.1), any probability vector in the simplex $\{s \in [0,1]^n : \sum_{i=1}^{n} s_i = 1\}$ determine a mixed system and viceversa.
- Representation and preservation theorems above are equally applicable to coherent and mixed systems.

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New results included in this talk

- Extensions of mixture representations, in two ways:
- Representations for not necessarily absolutely continuous joint distributions.
- Representations of $T = \phi(X_1, X_2, \ldots, X_k)$ in terms of $X_{1:n}, \ldots, X_{n:n}$ for $n > k$.
- Comparison results of systems with different size.

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The case $n = 2$

- There are 2 coherent systems: $X_{1:2}$ and $X_{2:2}$.
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Main result-exchangeable case

Theorem

If $(X_1, X_2, \ldots, X_n)$ is exchangeable and $T = \phi(X_1, X_2, \ldots, X_n)$, then

$$\overline{F}_T = \sum_{i=1}^{n} s_i \overline{F}_{i:n}, \qquad (2.1)$$

where $(s_1, s_2, \ldots, s_n)$ is the signature of T in IID cont. case.

Note that $s_i \neq P(T = X_{i:n})$ but that

$$s_i = \frac{1}{n!} \sum_{\sigma} 1(\sigma \in A_i)$$

$A_i = \{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n},$ when $x_{\sigma(1)} < \ldots < x_{\sigma(1)}\}.$

Main result-exchangeable case-Proof

- From (1.3): $(\overline{F}_{1:n},\ldots,\overline{F}_{n:n})' = A_n(\overline{F}_{1:1},\ldots,\overline{F}_{1:n})'$
- A_n is a triangular matrix.
- So $|A_n| \neq 0$ and A_n^{-1} exists.
- From (1.2): $\overline{F}_T$ can be written as a linear combination of $\overline{F}_{1:i}$, $i = 1, 2, \ldots, n$.
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Representations of systems with different size

- Recall that IID case: $2\overline{F}_{1:1} = \overline{F}_{1:2} + \overline{F}_{2:2}$.
- So: $\overline{F}_{1:1} = \frac{1}{2}\overline{F}_{1:2} + \frac{1}{2}\overline{F}_{2:2}$.
- In general, as $\overline{F}_1 + \ldots + \overline{F}_n = \overline{F}_{1:n} + \ldots + \overline{F}_{n:n}$, then

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Representations of order n

Theorem

If $(X_1, X_2, \ldots, X_n)$ is exchangeable and $T = \phi(X_1, X_2, \ldots, X_k)$ $(k < n)$, then

$$\overline{F}_T = \sum_{i=1}^{n} s_i(n) \overline{F}_{i:n} \quad (3.2)$$

where the vector $\mathbf{s}(n) = (s_1(n), s_2(n), \ldots, s_n(n))$ does not depend on F. $\mathbf{s}(n)$ is called the signature of order n of T.

Representations of order n-Proof

- Recall that $(\overline{F}_{1:n},\dots,\overline{F}_{n:n})' = A_n(\overline{F}_{1:1},\dots,\overline{F}_{1:n})'$
- A_n is a triangular matrix.
- So $|A_n| \neq 0$ and A_n^{-1} exists.
- From (1.2): $\overline{F}_T$ can be written as a linear combination of $\overline{F}_{1:i}$, $i = 1, 2, \dots, k$.
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Relations between signatures.

- If $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ is the signature of order n of T, then T is equal in law to the mixed system with $(n+1)$-components with signature vector

$$\mathbf{s}(n+1) = \left(\frac{ns_1}{n+1}, \frac{s_1 + (n-1)s_2}{n+1}, \frac{2s_2 + (n-2)s_3}{n+1}, \ldots, \frac{ns_n}{n+1} \right) \tag{3.3}$$

- Repeated application of (3.3) leads to the general expression for $\mathbf{s}(m)$ as a function of $\mathbf{s}(n)$ $(n < m)$.
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Table: Signatures of order 4 of coherent systems of order 1-4.

	$T = \Phi(X_1, X_2, X_3, X_4)$	Signature
1	$X_{1:1} = X_1$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
2	$X_{1:2} = \min(X_1, X_2)$ (2-series)	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0)$
3	$X_{2:2} = \max(X_1, X_2)$ (2-parallel)	$(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$
4	$X_{1:3} = \min(X_1, X_2, X_3)$ (3-series)	$(\frac{3}{4}, \frac{1}{4}, 0, 0)$
5	$\min(X_2, \max(X_1, X_3))$	$(\frac{1}{4}, \frac{5}{12}, \frac{1}{3}, 0)$
6	$X_{2:3}$ (2-out-of-3)	$(0, \frac{1}{2}, \frac{1}{2}, 0)$
7	$\max(X_2, \min(X_1, X_3))$	$(0, \frac{1}{3}, \frac{5}{12}, \frac{1}{4})$
8	$X_{3:3} = \max(X_1, X_2, X_3)$ (3-parallel)	$(0, 0, \frac{1}{4}, \frac{3}{4})$

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	$T = \Phi(X_1, X_2, X_3, X_4)$	Signature
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$ (series)	$(1, 0, 0, 0)$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
11	$\min(X_{2:3}, X_4)$	$(\frac{1}{4}, \frac{3}{4}, 0, 0)$
12	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$
14	$X_{2:4}$ (2-out-of-4)	$(0, 1, 0, 0)$
15	$\max(\min(X_1, X_2), \min_{i=1,3,4}(X_i), \min_{i=2,3,4}(X_i))$	$(0, \frac{5}{6}, \frac{1}{6}, 0)$
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$
17	$\max(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_3, X_4))$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$
18	$\max(\min(X_1, X_2), \min(X_2, X_3), \min(X_3, X_4))$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$

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20	$\min(\max(X_1, X_2), \max(X_1, X_3), \max(X_2, X_3, X_4))$	$(0, \frac{1}{3}, \frac{2}{3}, 0)$
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	$(0, \frac{1}{3}, \frac{2}{3}, 0)$
22	$\min(\max(X_1, X_2), \max_{i=1,3,4}(X_i), \max_{i=2,3,4}(X_i))$	$(0, \frac{1}{6}, \frac{5}{6}, 0)$
23	$X_{3:4}$ (3-out-of-4)	$(0, 0, 1, 0)$
24	$\max(X_1, \min(X_2, X_3, X_4))$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$
25	$\max(X_1, \min(X_2, X_3), \min(X_3, X_4))$	$(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$
26	$\max(X_{2:3}, X_4)$	$(0, 0, \frac{3}{4}, \frac{1}{4})$
27	$\min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4))$	$(0, 0, \frac{1}{2}, \frac{1}{2})$
28	$X_{4:4} = \max(X_1, X_2, X_3, X_4)$ (parallel)	$(0, 0, 0, 1)$

Assumptions

- In the general case, we have:

$$X_{1:n} \leq_{ST} X_{2:n} \leq_{ST} \cdots \leq_{ST} X_{n:n} \quad (4.1)$$

- However, the similar relations for the HR-order:

$$X_{1:n} \leq_{HR} X_{2:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}, \quad (4.2)$$

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- and the LR-order:

$$X_{1:n} \leq_{LR} X_{2:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}, \quad (4.4)$$

are not necessarily true in the exchangeable case; see Navarro and Shaked (JAP 2006), Navarro and Hernandez (Metrika 2008) and Navarro (JSPI 2008).

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Stochastic comparisons using signatures

Theorem

Let $\mathbf{s}_1(n)$ and $\mathbf{s}_2(n)$ be the signatures of order n of two coherent or mixed systems of order n_1 and n_2, both based on components with IID or EXC lifetimes with the same joint distribution. Let T_1 and T_2 be their respective lifetimes.
(a) If $\mathbf{s}_1(n) \leq_{ST} \mathbf{s}_2(n)$, then $T_1 \leq_{ST} T_2$.
(b) If $\mathbf{s}_1(n) \leq_{HR} \mathbf{s}_2(n)$ and (4.2) hold, then $T_1 \leq_{HR} T_2$.
(c) If $\mathbf{s}_1(n) \leq_{HR} \mathbf{s}_2(n)$ and (4.3) hold, then $T_1 \leq_{MRL} T_2$.
(d) If $\mathbf{s}_1(n) \leq_{LR} \mathbf{s}_2(n)$ and (4.4) hold, then $T_1 \leq_{LR} T_2$.

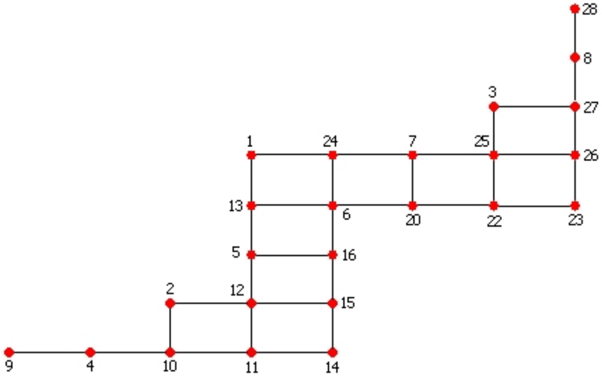

Figure: Comparisons based on the ST-order.

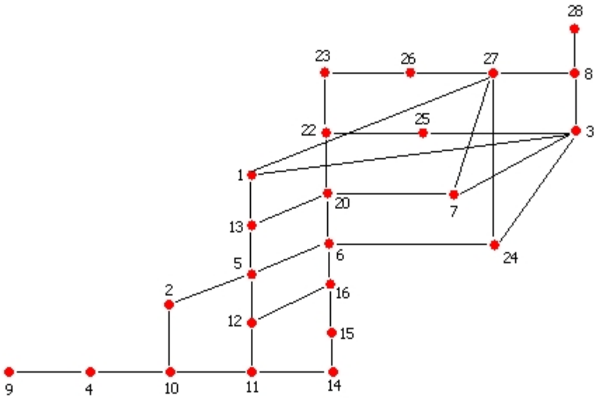

Figure: Comparisons based on the HR or MRL-orders.

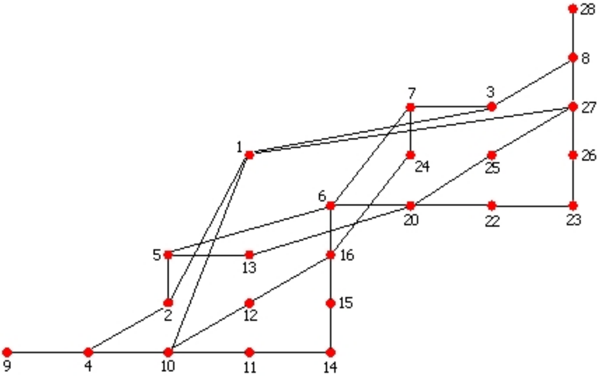

Figure: Comparisons based on the LR-order.

Conclusions

- The mixture representations based on order statistics are good tools to study systems.
- The new representations allow us to manage both the general exchangeable case and the case of systems with different size.
- Some new ordering results are obtained but we need to assume that the order statistics are HR, MRL or LR ordered.

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Open questions

- Conditions to have $X_{i:n} \leq_{HR,MRL,LR} X_{i+1:n}$.
- Conditions to have $X_{1:i} \geq_{HR,MRL,LR} X_{1:i+1}$ (some have been obtained already).
- Conditions to have $X_{i:i} \leq_{HR,MRL,LR} X_{i+1:i+1}$.
- Representations in the non-symmetric case (INID or general cases).
- Ordering results for generalized mixtures.
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