On comparing coherent systems with homogeneous and heterogeneous components

Jorge Navarro[1]
Universidad de Murcia, Murcia, Spain.
E-mail: jorgenav@um.es.

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- Then $T = X_{i:n}$ for a $i \in \{1, \ldots, n\}$.

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Outline

Systems with IID components

- Samaniego's representation: $T = \phi(X_1, \ldots, X_n)$ with IID components with a continuous distribution F, then

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- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all t.

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

Comparisons of systems with IID components

Theorem (Kochar, Mukerjee and Samaniego, NRL 1999)

If T_1 and T_2 have IID components and signatures $\mathbf{s}_1$ and $\mathbf{s}_2$, then:
(i) If $\mathbf{s}_1 \leq_{ST} \mathbf{s}_2$, then $T_1 \leq_{ST} T_2$ for all cont. F.
(ii) If $\mathbf{s}_1 \leq_{HR} \mathbf{s}_2$, then $T_1 \leq_{HR} T_2$ for all cont. F.
(iii) If $\mathbf{s}_1 \leq_{LR} \mathbf{s}_2$, then $T_1 \leq_{LR} T_2$ for all abs. cont. F.

Systems with EXC components

- Navarro et al. (2008): $T=\phi(X_1,\dots,X_r)$ from $(X_1,\dots,X_n)$ EXC components ($r\le n$), then

$$F_T=s_1^{(n)}F_{1:n}+\cdots+s_n^{(n)}F_{n:n},$$

for some coefficients $s_1^{(n)},\dots,s_n^{(n)}$.

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- $\mathbf{s}^{(n)}$ does not depend on F and can be computed from (1.1).
- If $r = n$, then $\mathbf{s}^{(n)} = \mathbf{s}$.
- If $(X_1, \dots, X_n)$ has an abs. cont. joint dist. $\mathbf{F}$, then $s_i^{(n)} = \Pr(T = X_{i:n})$.

Comparisons of systems with EXC components

Theorem (Navarro et al., NRL 2008)

If T_1 and T_2 are semicoherent systems from $(X_1,\dots,X_n)$ EXC and with signatures of order n $\mathbf{s}_1^{(n)}$ and $\mathbf{s}_2^{(n)}$, then:
(i) If $\mathbf{s}_1^{(n)} \le_{ST} \mathbf{s}_2^{(n)}$, then $T_1 \le_{ST} T_2$ for all $\mathbf{F}$.
(ii) If $\mathbf{s}_1^{(n)} \le_{HR} \mathbf{s}_2^{(n)}$ and $X_{1:n} \le_{HR} \cdots \le_{HR} X_{n:n}$, then $T_1 \le_{HR} T_2$ for all $\mathbf{F}$.
(iii) If $\mathbf{s}_1^{(n)} \le_{HR} \mathbf{s}_2^{(n)}$ and

$$X_{1:n} \le_{MRL} \cdots \le_{MRL} X_{n:n}, \tag{1.2}$$

then $T_1 \le_{MRL} T_2$ for all $\mathbf{F}$.
(iv) If $\mathbf{s}_1^{(n)} \le_{LR} \mathbf{s}_2^{(n)}$ and $X_{1:n} \le_{LR} \cdots \le_{LR} X_{n:n}$, then $T_1 \le_{LR} T_2$ for all $\mathbf{F}$.

Comparisons of systems with EXC components

Theorem (Navarro and Rubio, NRL 2011)

If T_1 and T_2 are semicoherent systems from $(X_1, \dots, X_n)$ with signatures of order n $\mathbf{s}_1^{(n)}$ and $\mathbf{s}_2^{(n)}$, then:

(i) $\mathbf{s}_1^{(n)} \leq_{ST} \mathbf{s}_2^{(n)}$ if and only if $T_1 \leq_{ST} T_2$ for all EXC $\mathbf{F}$.

(ii) $\mathbf{s}_1^{(n)} \leq_{HR} \mathbf{s}_2^{(n)}$ if and only if $T_1 \leq_{HR} T_2$ for all EXC $\mathbf{F}$ such that

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(iii) $\mathbf{s}_1^{(n)} \leq_{LR} \mathbf{s}_2^{(n)}$ if and only if $T_1 \leq_{LR} T_2$ for all EXC $\mathbf{F}$ such that

$$X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}. \tag{1.4}$$

Table: Signatures of order 4 for all the systems with 1-4 components.

N	$T_N = \phi(X_1, X_2, X_3, X_4)$	$\mathbf{s}^{(4)}$
1	$X_{1:1} = X_1$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
2	$X_{1:2} = \min(X_1, X_2)$	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0)$
3	$X_{2:2} = \max(X_1, X_2)$	$(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$
4	$X_{1:3} = \min(X_1, X_2, X_3)$	$(\frac{3}{4}, \frac{1}{4}, 0, 0)$
5	$\min(X_1, \max(X_2, X_3))$	$(\frac{1}{4}, \frac{5}{12}, \frac{1}{3}, 0)$
6	$X_{2:3}$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$
7	$\max(X_1, \min(X_2, X_3))$	$(0, \frac{1}{3}, \frac{5}{12}, \frac{1}{4})$
8	$X_{3:3} = \max(X_1, X_2, X_3)$	$(0, 0, \frac{1}{4}, \frac{3}{4})$
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$	$(1, 0, 0, 0)$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
11	$\min(X_{2:3}, X_4)$	$(\frac{1}{4}, \frac{3}{4}, 0, 0)$
12	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$

14	$X_{2:4}$	$(0,1,0,0)$
15	$\max(\min(X_1,X_2),\min(X_1,X_3,X_4),\min(X_2,X_3,X_4))$	$(0,\frac{5}{6},\frac{1}{6},0)$
16	$\max(\min(X_1,X_2),\min(X_3,X_4))$	$(0,\frac{2}{3},\frac{1}{3},0)$
17	$\max(\min(X_1,X_2),\min(X_1,X_3),\min(X_2,X_3,X_4))$	$(0,\frac{2}{3},\frac{1}{3},0)$
18	$\max(\min(X_1,X_2),\min(X_2,X_3),\min(X_3,X_4))$	$(0,\frac{1}{2},\frac{1}{2},0)$
19	$\min(\max(X_1,X_2),\max(X_2,X_3),\max(X_3,X_4))$	$(0,\frac{1}{2},\frac{1}{2},0)$
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23	$X_{3:4}$	$(0,0,1,0)$
24	$\max(X_1,\min(X_2,X_3,X_4))$	$(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})$
25	$\max(X_1,\min(X_2,X_3),\min(X_3,X_4))$	$(0,\frac{1}{6},\frac{7}{12},\frac{1}{4})$
26	$\max(X_{2:3},X_4)$	$(0,0,\frac{3}{4},\frac{1}{4})$
27	$\min(\max(X_1,X_2,X_3),\max(X_2,X_3,X_4))$	$(0,0,\frac{1}{2},\frac{1}{2})$
28	$X_{4:4}=\max(X_1,X_2,X_3,X_4)$	$(0,0,0,1)$

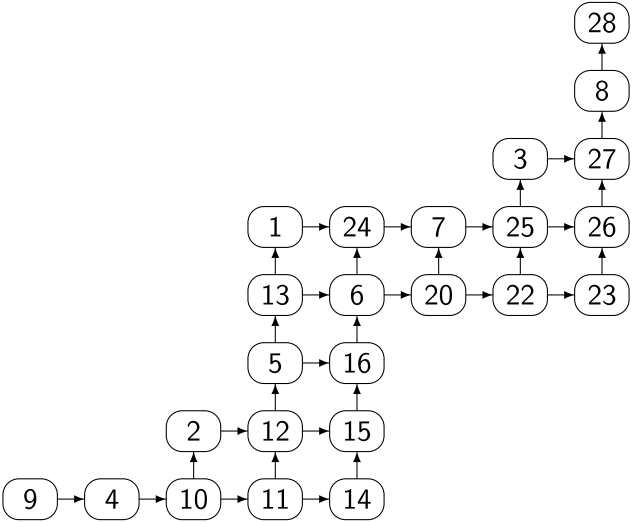

Figure: ST orderings for EXC **F** (IID case).

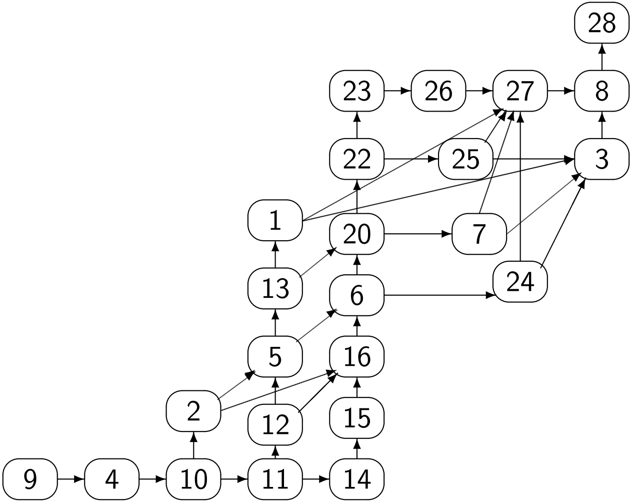

Figure: HR (MRL) orderings for EXC **F** under (1.3) (resp. (1.2)).

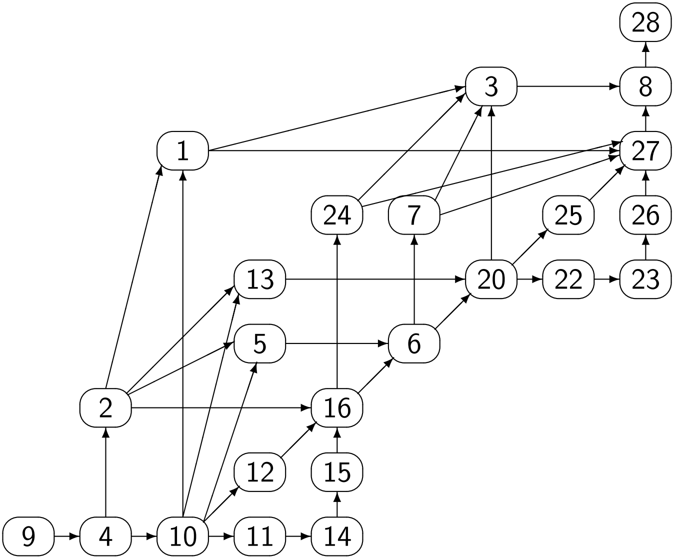

Figure: LR orderings for EXC **F** under (1.4).

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- The **structural signature** $(s_1, \ldots, s_n)$ with

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  for $i=1,\dots,n$, (which does not depend on $\mathbf{F}$).
- However, if $(X_1,\dots.X_n)$ is not EXC, then

$$F_T\neq w_1F_{1:n}+\dots+w_nF_{n:n}.$$

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- The **distorted distribution** (DD) associated to a distribution function (DF) F and to a **distortion function** q (i.e., to an increasing continuous function $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$) is

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- For the reliability functions (RF) $\overline{F} = 1 - F$, $\overline{F}_q = 1 - F_q$, we have

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \tag{2.2}$$

where $\overline{q}(u) = 1 - q(1-u)$ is called the **dual distortion function** in Hürlimann (2004, N Am Actuarial J).

Multivariate distortion functions

- The **generalized distorted distribution** (GDD) associated to n DF $F_1, \ldots, F_n$ and to an increasing continuous **multivariate distortion function** $Q : [0,1]^n \to [0,1]$ such that $Q(0,\ldots,0) = 0$ and $Q(1,\ldots,1) = 1$, is

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- Q and $\overline{Q}$ are continuous aggregation functions.

Coherent systems- General case

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- Then, by using the inclusion-exclusion formula

$$\begin{aligned}
\overline{F}_T(t) &= \Pr(T > t) = \Pr\left(\max_{i=1,\dots,r} \min_{j\in P_i} X_j > t\right) \\
&= \Pr\left(\cup_{i=1}^{r} \{\min_{j\in P_i} X_j > t\}\right) \\
&= \sum_{i=1}^{r} \Pr\left(\min_{j\in P_i} X_j > t\right) - \sum_{i<k} \Pr\left(\min_{j\in P_i\cap P_k} X_j > t\right) + \dots \\
&\quad + (-1)^{r+1} \sum_{i=1}^{r} \Pr\left(\min_{j\in P_1\cap\dots\cap P_r} X_j > t\right).
\end{aligned}$$

Coherent systems- General case

- If $\Pr(X_1 > x_1, \dots, X_n > x_n) = \mathbf{K}(\overline{F}_1(x_1), \dots, \overline{F}_n(x_n))$, where $\mathbf{K}$ is the survival copula, then:

$$\begin{aligned}\Pr(X_{1:i} > t) &= \Pr(X_1 > t, \dots, X_i > t, X_{i+1} > -\infty, \dots, X_n > -\infty) \\ &= \mathbf{K}(\overline{F}_1(t), \dots, \overline{F}_i(t), 1, \dots, 1).\end{aligned}$$

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- If $\overline{F}_P(t) = \Pr(X_P > t)$ for $X_P = \min_{j \in P} X_j$, then

$$\overline{F}_P(t) = K_P(\overline{F}_1(t), \dots, \overline{F}_n(t))$$

where $K_P(u_1, \dots, u_n) = \mathbf{K}(u_1^P, \dots, u_n^P)$ and $u_j^P = u_j$ if $j \in P$ and $u_j^P = 1$ if $j \notin P$.

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- Therefore, from the inclusion-exclusion representation

$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \dots, \overline{F}_n(t))$$

where $\overline{Q}$ is a multivariate dual distortion function.

Coherent systems- particular cases

- If $\overline{F}_1 = \cdots = \overline{F}_n = \overline{F}$, then

$$\overline{F}_T(t) = \overline{Q}(\overline{F}(t), \ldots, \overline{F}(t)) = \overline{q}(\overline{F}),$$

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- If $X_1, \ldots, X_n$ are independent, then $\overline{Q}$ is a polynomial and it is called *structure reliability function* in Barlow and Proschan (1975).
- In particular, in the IID case, $\overline{F}_T = a_1 \overline{F}_{1:1} + \cdots + a_n \overline{F}_{1:n}$ and

$$\overline{q}(u) = a_1 u + \cdots + a_n u^n$$

  where $\mathbf{a} = (a_1, \ldots, a_n)$ is the minimal signature of the system (see, e.g. Navarro et al., ASMBI 2013).

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- Navarro et al. ASMBI, 2013 and Navarro and Gomis ASMBI, 2016.

Example 1

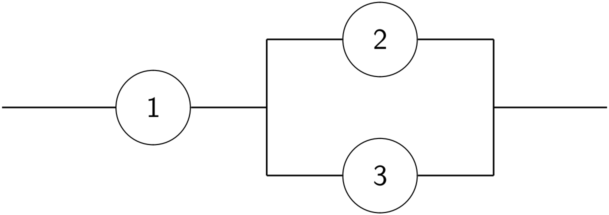

Figure: System with lifetime $T = \min(X_1, \max(X_2, X_3))$.

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- The minimal path sets are $\{1, 2\}$ and $\{1, 3\}$

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- Hence, the system reliability is

$$\begin{aligned}\overline{F}_T(t) &= \Pr(\{\min(X_1,X_2)>t\}\cup\{\min(X_1,X_3)>t\})\\ &= \Pr(X_{\{1,2\}}>t)+\Pr(X_{\{1,3\}}>t)-\Pr(X_{\{1,2,3\}}>t)\\ &= K(\overline{F}(t),\overline{F}(t),1)+K(\overline{F}(t),1,\overline{F}(t))-K(\overline{F}(t),\overline{F}(t),\overline{F}(t))\\ &= \overline{q}(\overline{F}(t))\end{aligned}$$

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where $\overline{q}(u)=K(u,u,1)+K(u,1,u)-K(u,u,u)$.

- If $K(u_1,u_2,u_3)=u_1u_2u_3\,(1+\alpha(2-u_1-u_2)(1-u_3))$, for $\alpha\in[-0.5,0.5]$, then

$$\overline{q}_\alpha(u)=u^2+u^2\left(1+\alpha(1-u)^2\right)-u^3\left(1+2\alpha(1-u)^2\right).$$

Example 1

- If we want to compare $T = \min(X_1, \max(X_2, X_3))$ and X_1 in the HR order we plot $\overline{q}_\alpha(u)/u$ in $(0, 1)$ for $\alpha = -0.5, -0.25, 0, 0.25, 0.5$.

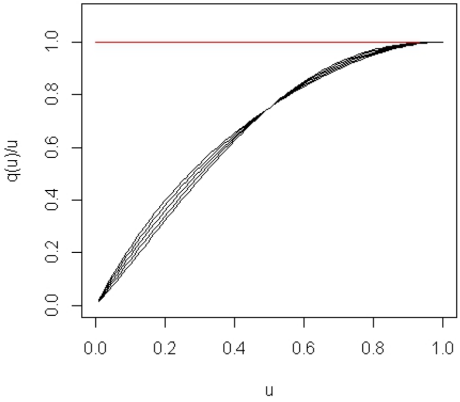

Figure: Ratio of the dual distortion functions of T and X_1 when $\alpha = -0.5, -0.25, 0, 0.25, 0.5$.

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- As it is increasing for $\alpha = -0.5, -0.25, 0, 0.25, 0.5$, then $T \leq_{HR} X_1$ for all F.

Example 1

- If we want to compare $T = \min(X_1, \max(X_2, X_3))$ for different values of α, we should study $g = \overline{q}_\beta / \overline{q}_\alpha$ for $-0.5 \leq \alpha < \beta \leq 0.5$.

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- A straightforward calculation shows that g is strictly decreasing in $(0, u_0)$ and strictly increasing in $(u_0, 1)$ for

$$u_0 = \frac{13}{8} - \frac{1}{8}\sqrt{57} \cong 0.681270.$$

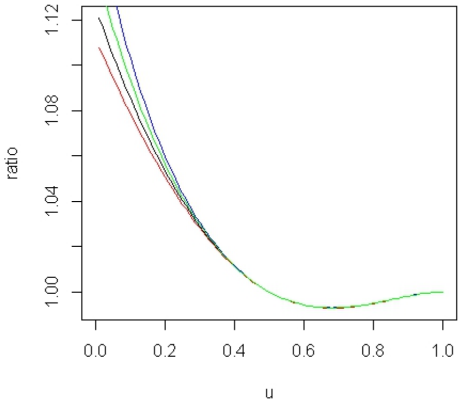

Figure: Ratio $\overline{q}_\beta/\overline{q}_\alpha$ of the dual distortion functions of T when $(\alpha, \beta) = (-0.5, -0.25)$ (blue), $(-0.25, 0)$ (green), $(0, 0.25)$ (black) and $(0.25, 0.5)$ (red).

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- Therefore $T_\alpha \leq_{MRL} T_\beta$ for all F such that $E(T_\alpha) \leq E(T_\beta)$.
- If $X_i \equiv Exp(\mu)$, then

$$E(T) = \frac{2\mu}{3} + \frac{\mu}{60}\alpha$$

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- These systems are not ST ordered since g takes values greater and smaller than 1.

Comparisons IID case-Navarro (Test, 2016)

- T_1 with minimal signature $(p_1, \ldots, p_n)$ IID$\sim F$ comp.

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$$\sum_{i=1}^{n} (q_i - p_i) x^i \geq 0 \text{ for all } x \in (0,1).$$

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$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j-i)(p_j q_i - p_i q_j) x^{i+j-2} \geq 0 \text{ for all } x \in (0,1).$$

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- $T_1 \leq_{LR} T_2$ holds for all F if and only if

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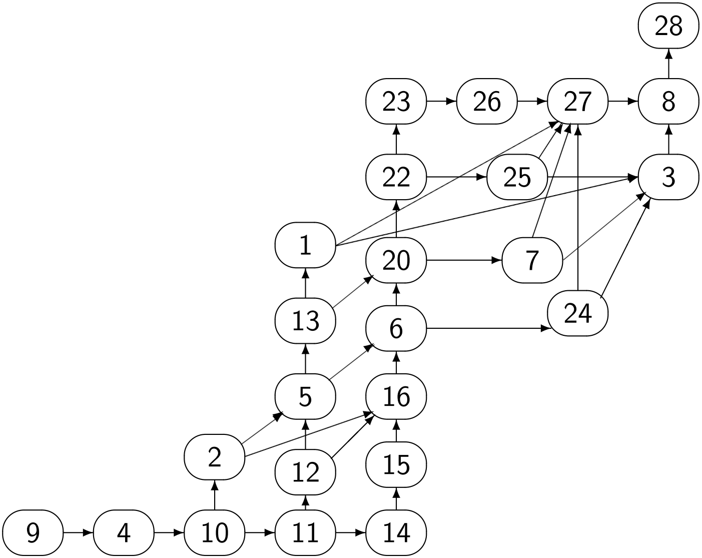

Figure: HR orderings for IID components from signatures.

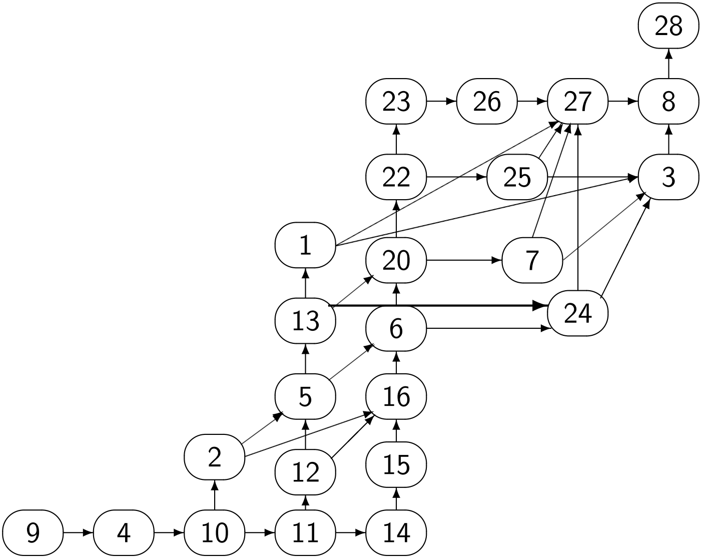

Figure: All the HR orderings for IID components.

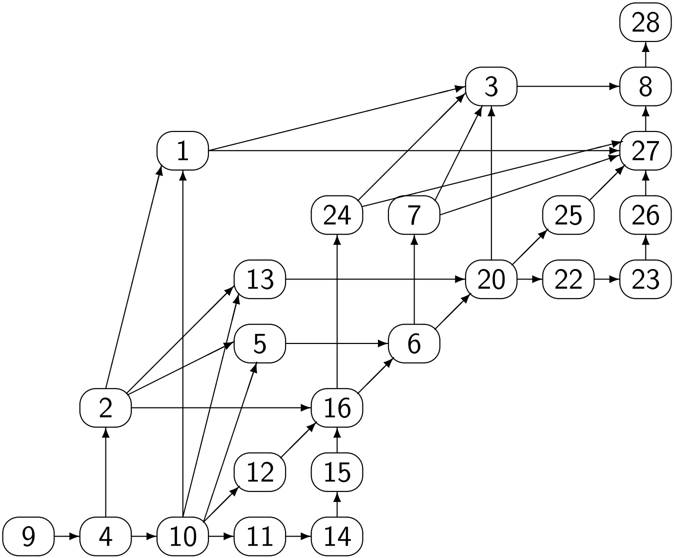

Figure: LR orderings for IID components from signatures.

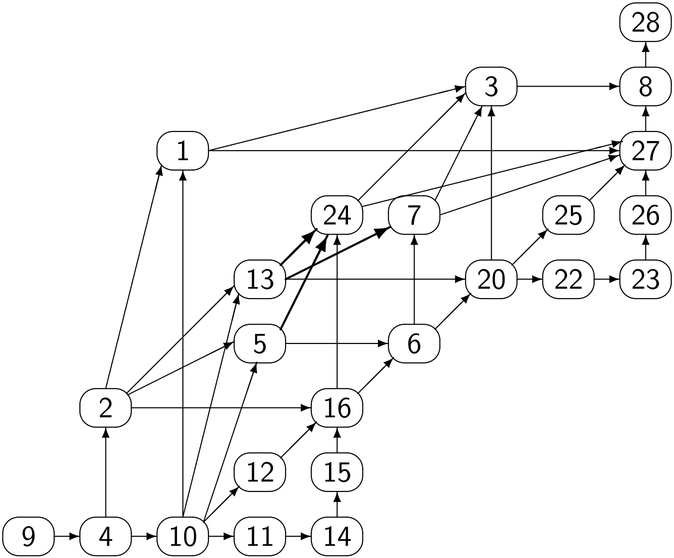

Figure: All the LR orderings for IID components.

Coherent systems- General case

- From the preceding section, we have

$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \dots, \overline{F}_n(t)),$$

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$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \dots, \overline{F}_n(t)),$$

where $\overline{Q}$ is a multivariate dual distortion function.

- Therefore we can use the following results obtained in Navarro et al. (Methodology and Computing in Applied Probability, 2016) and in Navarro and del Águila (Metrika, 2017) to compare generalized distorted distributions.

Comparisons of GDD

- If T_i has RF $\overline{Q}_i(\overline{F}_1,\dots,\overline{F}_r)$, $i=1,2$, then:

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$$F_1 \geq_{ST} \cdots \geq_{ST} F_n$$

holds if and only if $\overline{Q}_1 \leq \overline{Q}_2$ in
$D = \{(u_1, \dots, u_n) \in [0,1]^n : u_1 \geq \cdots \geq u_n\}$.

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- $T_1 \leq_{HR} T_2$ for all $\overline{F}_1,\dots,\overline{F}_r$ such that

$$F_1 \geq_{HR} \cdots \geq_{HR} F_n \tag{3.1}$$

holds if and only if the function

$$H(v_1,\dots,v_n)=\frac{\overline{Q}_2(v_1,v_1v_2,\dots,v_1\dots v_n)}{\overline{Q}_1(v_1,v_1v_2,\dots,v_1\dots v_n)} \tag{3.2}$$

is decreasing in $[0,1]^n$.

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- A similar result is obtained for the RHR order.

Example 2

- $T = X_{2:2} = \max(X_1, X_2)$, X_1, X_2 INID.

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- Then $X_{1:2} \leq_{HR} X_{2:2}$ holds for all $\overline{F}_1, \overline{F}_2$ since

$$\frac{\overline{Q}(u_1, u_2)}{u_1 u_2} = \frac{u_1 + u_2 - u_1 u_2}{u_1 u_2} = \frac{1}{u_1} + \frac{1}{u_2} - 1$$

  is decreasing in $(0, 1)^2$.

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- If we want to compare $T = X_{2:2} = \max(X_1, X_2)$ with X_1, we should study

$$\frac{\overline{Q}(u_1, u_2)}{u_1} = \frac{u_1 + u_2 - u_1 u_2}{u_1} = 1 + \frac{u_2}{u_1} - u_2 = 1 + u_2 \left(\frac{1}{u_1} - 1 \right).$$

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- As it is is decreasing in $(0,1)^2$, then $X_2 \leq_{HR} T$ for all $\overline{F}_1, \overline{F}_2$ such that $X_2 \leq_{HR} X_1$.
- That is, if $X_2 \leq_{HR} X_1$, then

$$X_{1:2} \leq_{HR} X_2 \leq_{HR} X_{2:2}$$

and

$$X_{1:2} \leq_{HR} X_2 \leq_{HR} X_1 \nleq_{HR} X_{2:2}.$$

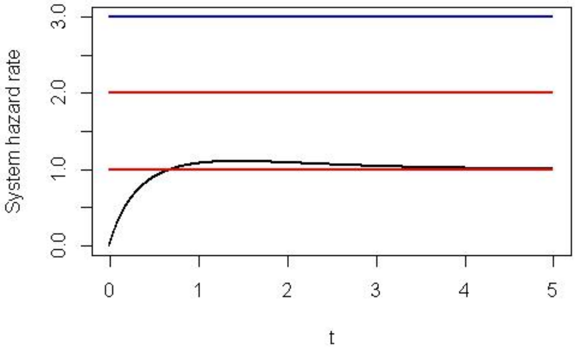


Figure: Hazard rate functions of X_i (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when $X_i \equiv Exp(\mu = 1/i)$, $i = 1, 2$.

Further examples

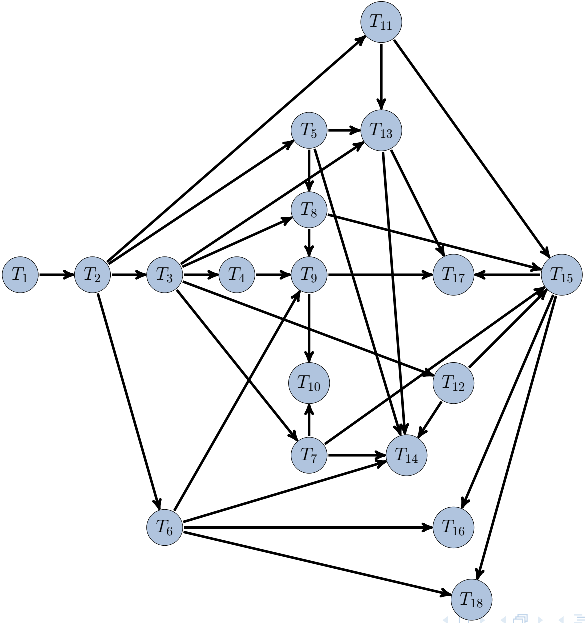
- By using the preceding techniques, we have ordered all the coherent systems with 1-3 independent components in Navarro and del Aguila (Metrika, 2017) in both cases (i.e., with and without ordered components).

Table: Dual distortions functions of systems with 1-3 INID components.

N	$T=\psi(X_1,X_2,X_3)$	$\overline{Q}(u_1,u_2,u_3)$
1	$X_{1:3}=\min(X_1,X_2,X_3)$	$u_1u_2u_3$
2	$\min(X_2,X_3)$	u_2u_3
3	$\min(X_1,X_3)$	u_1u_3
4	$\min(X_1,X_2)$	u_1u_2
5	$\min(X_3,\max(X_1,X_2))$	$u_1u_3+u_2u_3-u_1u_2u_3$
6	$\min(X_2,\max(X_1,X_3))$	$u_1u_2+u_2u_3-u_1u_2u_3$
7	$\min(X_1,\max(X_2,X_3))$	$u_1u_2+u_1u_3-u_1u_2u_3$
8	X_3	u_3
9	X_2	u_2
10	X_1	u_1

Table: Dual distortions functions of systems with 1-3 INID components.

N	$T=\psi(X_1,X_2,X_3)$	$\overline{Q}(u_1,u_2,u_3)$
11	$X_{2:3}$	$u_1u_2+u_1u_3+u_2u_3-2u_1u_2u_3$
12	$\max(X_3,\min(X_1,X_2))$	$u_3+u_1u_2-u_1u_2u_3$
13	$\max(X_2,\min(X_1,X_3))$	$u_2+u_1u_3-u_1u_2u_3$
14	$\max(X_1,\min(X_2,X_3))$	$u_1+u_2u_3-u_1u_2u_3$
15	$\max(X_2,X_3)$	$u_2+u_3-u_2u_3$
16	$\max(X_1,X_3)$	$u_1+u_3-u_1u_3$
17	$\max(X_1,X_2)$	$u_1+u_2-u_1u_2$
18	$X_{3:3}=\max(X_1,X_2,X_3)$	$u_1+u_2+u_3-u_1u_2-u_1u_3$ $-u_2u_3+u_1u_2u_3$


T_1
T_2
T_3
T_4
T_5
T_6
T_7
T_8
T_9
T_{10}
T_{11}
T_{12}
T_{13}
T_{14}
T_{15}
T_{16}
T_{17}
T_{18}

Example 3

- $T = X_{2:2} = \max(X_1, X_2)$, X_1, X_2 DEP$\sim K$.

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$$\overline{F}_{1:2}(t) = K(\overline{F}_1(t), \overline{F}_2(t)) = \overline{Q}_{1:2}(\overline{F}_1(t), \overline{F}_2(t)),$$

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where $\overline{Q}_{1:2}(u_1, u_2) = K(u_1, u_2)$.

- Then $X_{1:2} \leq_{HR} X_{2:2}$ holds for all $\overline{F}_1, \overline{F}_2$ if and only if

$$\frac{\overline{Q}(u_1, u_2)}{K(u_1, u_2)} = \frac{u_1 + u_2 - K(u_1, u_2)}{K(u_1, u_2)} = \frac{u_1 + u_2}{K(u_1, u_2)} - 1$$

is decreasing in $(0, 1)^2$.

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  is decreasing in $(0, 1)^2$.
- This property is not necessarily true for all K (see Navarro, Torrado and del Águila 2017).

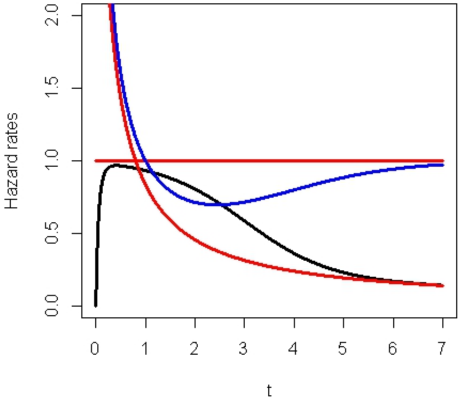

Figure: Hazard rate functions of X_i (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when $\overline{F}_1(t) = \exp(-t)$ (Exponential), $\overline{F}_2(t) = 1/(1+5t)$ (Pareto) and $K(u_1, u_2) = u_1 u_2/(u_1 + u_2 - u_1 u_2)$ (Clayton-Oakes).

Our Main References: Coherent systems

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- Navarro and Gomis (2016). Comparisons in the mean residual life order of coherent systems with identically distributed components. Applied Stochastic Models in Business and Industry 32, 33–47.

References

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- Thank you for your attention!!