Importance indices and replacement policies in coherent systems with dependent components

Jorge Navarro[1]
Universidad de Murcia, Murcia, Spain.
E-mail: jorgenav@um.es.

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- Navarro, Fernández-Martínez, Fernández-Sánchez and Arriaza. Relationships between importance measures and redundancy in systems with dependent components. To appear in *Probability in the Engineering and Informational Sciences*. Published online first May 2019. DOI: 10.1017/S0269964819000159.

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- A path set P is a **minimal path set** if it does not contain other path sets.
- If $P_1,\dots,P_r$ are the minimal path sets, then

$$\phi(x_1,\dots,x_n) = \max_{i=1,\dots,r} \min_{j \in P_i} x_j.$$

Coherent systems-General case

- The system lifetime can be written as

$$T = \phi(X_1, \dots, X_n) = \max_{i=1,\dots,r} \min_{j \in P_i} X_j,$$

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- Then, by using the inclusion-exclusion formula, the system reliability function $\bar{F}_T(t) = \Pr(T > t)$ can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)) \qquad (1.1)$$

where $\bar{F}_1, \dots, \bar{F}_n$ are the component reliability functions and $\bar{Q}$ is a distortion (continuous aggregation) function (see, e.g., Navarro, Metrika 2018).

Distortion functions

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- From (1.1), the system distribution $F_T = 1 - \bar{F}$ can be written as

$$F_T(t) = Q(F_1(t),\dots,F_n(t)) \tag{1.2}$$

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  where $F_1,\dots,F_n$ are the component distribution functions and $Q(u_1,\dots,u_n) = 1 - \bar{Q}(1-u_1,\dots,1-u_n)$ is also a distortion function.
- Q and $\bar{Q}$ depend on the minimal path sets (system structure) and on the copula of $(X_1,\dots,X_n)$ (the dependence structure).

Coherent systems-particular cases

- If $\bar{F}_1 = \cdots = \bar{F}_n = \bar{F}$ (ID), then

$$\bar{F}_T(t) = \bar{Q}(\bar{F}(t), \ldots, \bar{F}(t)) = \bar{q}(\bar{F}(t)),$$

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- If $X_1, \dots, X_n$ are independent, then $\bar{Q}$ is a polynomial called **structure reliability function** in Barlow and Proschan (1975).
- In particular, in the IID case, we have

$$\bar{q}(u) = a_1 u + \cdots + a_n u^n$$

  where $\boldsymbol{a} = (a_1, \dots, a_n)$ is the minimal signature of the system (see, e.g., Navarro, Metrika 2018).

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- If the components are dependent, these expressions lead to different concepts.
- The Birnbaum's importance index was extended recently in Miziuła and Navarro (IEEE Tr. Rel. 2019).
- The new definition is based on expression (1.1).
- Thus, the *Birnbaum importance index* of the *i*th component is

$$I_B(i; p_1, \dots, p_n) = \partial_i \bar{Q}(p_1, \dots, p_n), \qquad (1.3)$$

where $\partial_i \bar{Q}$ represents the partial derivative of $\bar{Q}$ with respect to its *i*th variable (we assume that this derivative exists).

Birnbaum's importance index

- Hence we say that component i is *more important* than component j (shortly written as $i \geq_{mi} j$) if

$$I_B(i; p_1, \dots, p_n) \geq I_B(j; p_1, \dots, p_n)$$

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- It represents how an increment in the reliability of the ith component, increments the system reliability.
- Note I_B does not depend on the component reliability functions.
- If the components are independent, this measure coincides with the classical Birnbaum measure.

Properties

- Iyer (1992) extended the Barlow-Proschan importance index of the ith component by

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$$I_{BP}(i) = \int_0^\infty I_B(i; \bar{F}_1(t), \dots, \bar{F}_n(t))\, dF_i(t).$$

Properties

- **Theorem 1.1:** For any $i \in \{1, \ldots, n\}$, we have

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- **Corollary 1.1:** If $\bar{F}_1 = \ldots = \bar{F}_n = \bar{F}$, then

$$I_{BP}(i) = \int_0^1 I_B(i; p, \ldots, p)\ dp$$

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Properties

- **Proposition 1.1:** Let f_T and $f_1, \dots, f_n$ be probability density functions of T and $X_1, \dots, X_n$, respectively. Then

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- **Proposition 1.2:** If $f_1 = \ldots = f_n = f$, then

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- **Proposition 3:** If $f_1 = \ldots = f_n = f$, then $T \leq_{lr} X_1$ ($T \geq_{lr} X_1$) holds for any f if and only if $\sum_{i=1}^{n} I_B\big(i; p\big)$ is increasing (decreasing) in p in the interval $(0,1)$.

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- Redundancy and replacement policies are very important procedures to improve the system reliability.
- Many results have been obtained in the literature (especially in the case of independent components).
- In practice, there exist different redundancy options:

Active redundancy

- In an **active redundancy**, the *i*th component with lifetime X_i is reinforced by adding and additional component Y_i connected with it by a given (good) structure.

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- The lifetime of the resulting unit is $Z_i = \max(X_i, Y_i) \geq X_i$.
- If X_i, Y_i are IID, then the reliability function of Z_i is

$$\Pr(Z_i > t) = \Pr(X_i > t) + \Pr(Y_i > t) - \Pr(X_i > t)\Pr(Y_i > t) = \bar{q}_{2:2}(\bar{F}_i(t)), \tag{2.1}$$

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- For some models, it can also be represented as in (2.1).

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- The reliability function of Z_i^{**} is

$$\begin{aligned}\Pr(Z_i^{**} > t) &= \Pr(X_i > t) + \int_0^t \Pr(Y_{i,x} > t - x)\, dF_i(x) \\ &= \bar{F}_i(t) + \int_0^t \frac{\bar{F}_i(t)}{\bar{F}_i(x)} dF_i(x) \\ &= \bar{F}_i(t) - \bar{F}_i(t) \ln \bar{F}_i(t) \\ &= \bar{q}_{MR}(\bar{F}_i(t)),\end{aligned}$$

where $\bar{q}_{MR}(u) = u - u \ln u \geq u$ is also a distortion function.

General formulation

- From now on we only consider redundancy mechanisms whose reliability function can be represented as $\bar{q}(\bar{F}_i)$ for a distortion function $\bar{q}$ such that $\bar{q}(u) \geq u$ for all $u \in [0, 1]$.

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- We assume that this function is the same for all the components.
- Then, from (1.1), the reliability of the resulting system T_i is

$$\begin{aligned} R_i(t) &= \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_{i-1}(t), \bar{q}(\bar{F}_i(t)), \bar{F}_{i+1}(t), \dots, \bar{F}_n(t)) \\ &= \bar{Q}_i(\bar{F}_1(t), \dots, \bar{F}_n(t)), \end{aligned} \tag{2.2}$$

where $\bar{Q}_i(u_1, \dots, u_n) = \bar{Q}(u_1, \dots, u_{i-1}, \bar{q}(u_i), u_{i+1}, \dots, u_n)$.

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  where $\bar{Q}_i(u_1, \dots, u_n) = \bar{Q}(u_1, \dots, u_{i-1}, \bar{q}(u_i), u_{i+1}, \dots, u_n)$.
- Hence we can compare the different options just by comparing the different distortion functions $\bar{Q}_1, \dots, \bar{Q}_n$.

Main result

- **Theorem 2.1:** In a coherent system with ID components, if component $i \geq_{mi} j$, then $T_i \geq_{st} T_j$ for any redundancy function $\bar{q}$ and any reliability $\bar{F}$.

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- **Lemma 2.1:** Let $G : [0,1]^n \to \mathbb{R}$ be a Lipschitz continuous function and let D_k be the set where $\partial_k G$ exists for $k = 1, \dots, n$. Then, the following conditions are equivalent:
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- Theorem 2.1 does not hold if we do not assume homogeneous components (we need at least $X_i =_{st} X_j$).
- The following proposition proves that condition $i \geq_{mi} j$ can be considered as a strong condition.

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- If $i \geq_{mi} j$, then $i \geq_{wmi} j$ but the reverse property is not true.

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- **Proposition 2.1:** In a coherent system with independent components and distortion function $\bar{Q}$, if

$$I_B(i; u, \dots, u) \geq I_B(j; u, \dots, u) \tag{2.4}$$

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- If the components are IID, then $T_1 \geq_{st} T_2$ for all $\bar{F}$ and all $\bar{q}$.

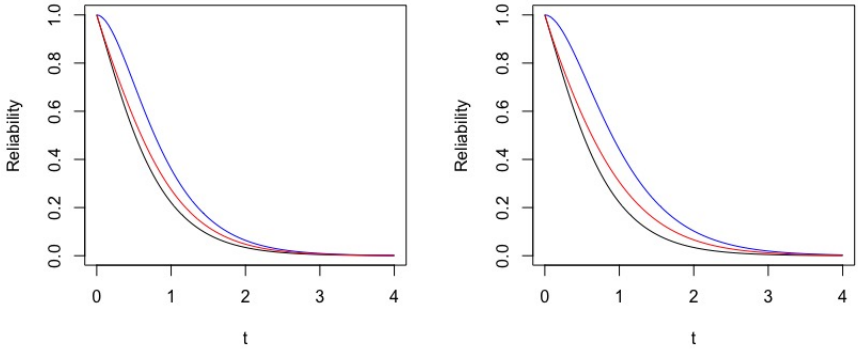

Figure: Reliability functions for T (black), T_1 (blue) and T_2 (red) for IID standard exponential components with active redundancy (left) and minimal repair (right).

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$$\begin{array}{ccccc} X \leq_{lr} Y & \Rightarrow & X \leq_{hr} Y & \Rightarrow & X \leq_{mrl} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{rhr} Y & \Rightarrow & X \leq_{st} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

Homogeneous components

- If a system has possibly dependent ID components, then

$$R_i(t) = \bar{Q}(\bar{F}(t), \dots, \bar{F}(t), \bar{q}(\bar{F}(t)), \bar{F}(t), \dots, \bar{F}(t)) = \bar{q}_i(\bar{F}(t)), \qquad (2.5)$$

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- **Proposition 2.2:** Let $\bar{q}_i$ and $\bar{q}_j$ be the distortion functions obtained in (2.5). Then:
  - (i) $T_i \leq_{st} T_j$ for all F if and only if $\overline{q}_i \leq \overline{q}_j$ in $(0,1)$.
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  - (v) If there exists $u_0 \in (0,1]$ such that $\overline{q}_j/\overline{q}_i$ is decreasing in $(0,u_0)$ and increasing in $(u_0,1)$, then $T_i \leq_{mrl} T_j$ for all F such that $E(T_i) \leq E(T_j)$.

Heterogeneous components

- In the general case, the reliability function can be written as

$$R_i(t) = \bar{Q}_i(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \tag{2.6}$$

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for all $\mathbf{u} = (u, \ldots, u) \in [0,1]^n$ and $p \in [\bar{q}_{2:2}(u), 1]$ (where $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 1)$), then $T_i \geq_{st} T_j$ for all $\bar{q} \geq \bar{q}_{2:2}$.

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- For example, it can be applied to minimal repairs since $\bar{q}_{MR}(u) = u - u \ln u \geq \bar{q}_{2:2} = 2u - u^2$.

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- We can apply an active redundancy with $\bar{q}_{2:2}(u) = 2u - u^2$.
- If the components are IID and $\bar{q} = \bar{q}_{2:2}$, then

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- Let us consider $T = \max(X_1, \min(X_2, X_3))$.
- If the components are independent, then

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- Even more, as

$$\partial_1\bar{Q}(u, u, u) = 1 - u^2 \geq u - u^2 = \partial_2\bar{Q}(u, u, u),$$

$1 \geq_{wmi} 2$ and $T_1 \geq_{st} T_2$ for all $\bar{F}$ and all $\bar{q}$.

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- Analogously, as $\bar{q}_1'/\bar{q}_2'$ is decreasing, then $T_1 \geq_{lr} T_2$ for all $\bar{F}$ and $\bar{q} = \bar{q}_{2:2}$.
- If the components are just independent, then T_1 and T_2 are not st-ordered for all $\bar{F}_1, \bar{F}_2, \bar{F}_3$ since $\bar{Q}_1$ and $\bar{Q}_2$ are not ordered in $[0,1]^3$.

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- If the components are ID, then

$$\bar{q}_1(u)=\bar{Q}(\bar{q}_{2:2}(u),u,u)=2u-u^2+\frac{u-2u^2+u^3}{2-u};$$

$$\bar{q}_2(u)=\bar{q}_3(u)=\bar{Q}(u,\bar{q}_{2:2}(u),u)=u+\frac{u(1-u)(2-u)}{3-3u+u^2}.$$

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- As $r = \bar{q}_1/\bar{q}_2$ is decreasing, $T_1 \geq_{hr} T_2$ holds for all $\bar{F}$ and $\bar{q} = \bar{q}_{2.2}$.
- The respective reliability functions for $\bar{F}(t) = \exp(-t)$ can be seen in the following figure.

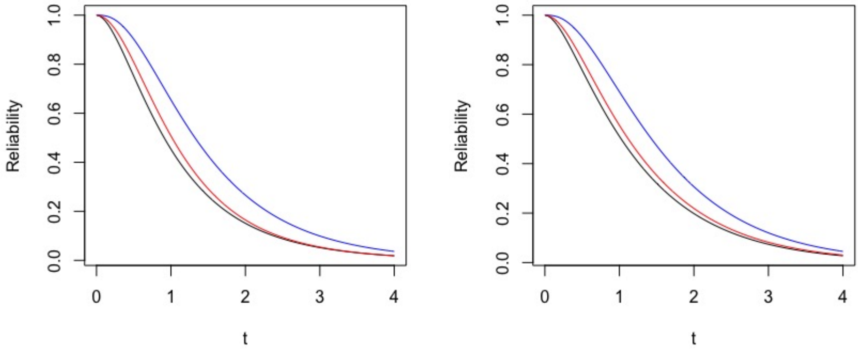

Figure: Reliability functions for T (black), T_1 (blue) and T_2 (red) for IID exponential components (left) and DID (right) with a Clayton copula.

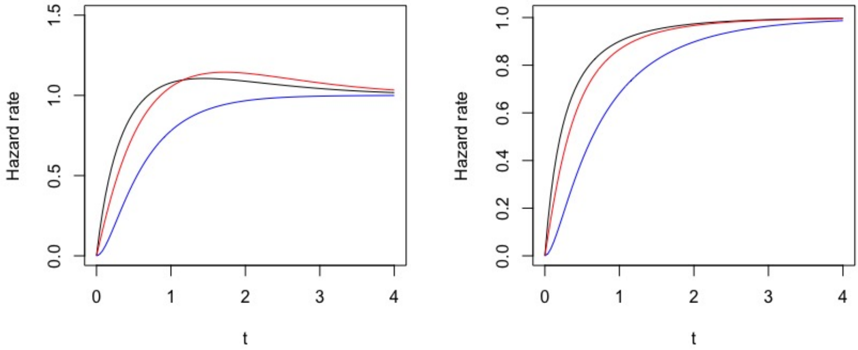

Figure: Hazard rate functions for T (black), T_1 (blue) and T_2 (red) for IID exponential components (left) and DID (right) with a Clayton copula.

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References

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- Thank you for your attention!!