# Minimal repair of failed components in coherent systems

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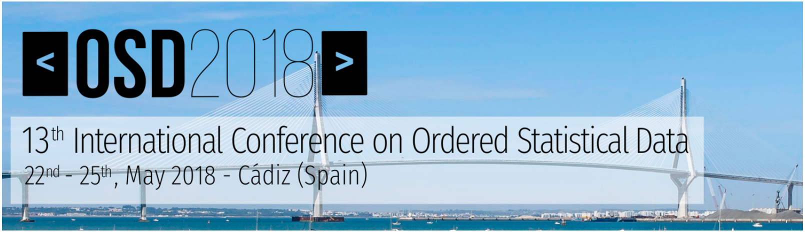


[1]Supported by Ministerio de Economía, Industria y Competitividad under Grant MTM2016-79943-P (AEI/FEDER).

# Relevation process and perfect repair

- Let $X$ and $Y$ be two nonnegative independent random variables with reliability functions $\bar{F}$ and $\bar{G}$. Then the reliability of $X+Y$ (convolution) $\bar{F} * \bar{G}(t) = \Pr(X+Y>t)$ is

$$\bar{F} * \bar{G}(t) = \int_t^\infty f(x)dx + \int_0^t \int_{t-x}^\infty g(y)f(x)dydx$$
$$= \bar{F}(t) + \int_0^t \bar{G}(t-x)f(x)dx,$$

where $f$ and $g$ are the respective pdf.

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where $f$ and $g$ are the respective pdf.
- Under a *perfect repair* in a cold standby, the unit $X$ is replaced when it fails by an independent unit $Y$ having the same distribution as $X$ (when new). Then

$$\bar{F}*\bar{F}(t)=\bar{F}(t)+\int_0^t \bar{F}(t-x)f(x)dx.$$

# Relevation process and minimal repair

- If $X$ and $Y$ are dependent, then the reliability of $X+Y$ is

$$\bar{F}\#\bar{G}(t)=\bar{F}(t)+\int_0^t \bar{G}_x(t-x)f(x)dx, \qquad (1)$$

where $\bar{G}_x(y)=\Pr(Y>y|X=x)$.

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- Under a *relevation process*, the unit $X$ is replaced when it fails at a time $x$ by a unit having reliability $\bar{G}$ but with the same age as $X$, that is, by $Y_x=(Y-x|Y>x)$ with reliability

$$\bar{G}_x(y)=\Pr(Y>y|X=x)=\Pr(Y-x>y|Y>x)=\frac{\bar{G}(x+y)}{\bar{G}(x)}$$

for $y\geq 0$. Hence,

$$\bar{F}\#\bar{G}(t)=\bar{F}(t)+\int_0^t \frac{\bar{G}(t)}{\bar{G}(x)}f(x)dx. \qquad (2)$$

# Relationships

## Proposition

*Under a relevation process, if $G$ is NBU (NWU), then*
$\bar{F} * \bar{G} \geq \bar{F} \# \bar{G} \quad (\leq)$.

## Proof.

If $G$ is NBU, then $\bar{G}(y) \geq \bar{G}(x+y)/\bar{G}(x)$ for $x, y \geq 0$. Then

$$\begin{aligned}\bar{F} * \bar{G}(t) &= \bar{F}(t) + \int_0^t \bar{G}(t-x) f(x) dx \\ &\geq \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(x)} f(x) dx. \\ &= \bar{F} \# \bar{G}(t).\end{aligned}$$

The inequality is reversed if $G$ is NWU. □

# Relevation process and minimal repair

- Under a *minimal repair*, the failed unit $X$ is replaced by a unit having the same reliability as $X$ but with the same age. Then

$$\bar{F}\#\bar{F}(t) = \bar{F}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f(x)dx = \bar{F}(t) - \bar{F}(t)\ln \bar{F}(t).$$

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- After $k$ replacements, the resulting reliability is

$$\bar{F}\#^k\bar{F}(t)=\bar{F}(t)\sum_{i=0}^{k}\frac{1}{i!}[-\ln\bar{F}(t)]^i,$$

where $\bar{F}\#^0\bar{F}=\bar{F}$, $\bar{F}\#^1\bar{F}=\bar{F}\#\bar{F}$, $\bar{F}\#^2\bar{F}=(\bar{F}\#\bar{F})\#\bar{F}$, ...

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- Note that $(\bar{F}\#\bar{F})\#\bar{F} \neq \bar{F}\#(\bar{F}\#\bar{F})$.

# Distorted distributions

- The **distorted distribution** associated to a distribution function $F$ and to an increasing continuous **distortion function** $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \tag{3}$$

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- For the reliability functions $\bar{F} = 1 - F$, $\bar{F}_q = 1 - F_q$, we have

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where $\bar{q}(u) = 1 - q(1-u)$ is the **dual distortion function**.

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- Note that $\bar{F} \#^k \bar{F}(t) = \bar{q}_k(\bar{F}(t))$ with

$$\bar{q}_k(u) = u \sum_{i=0}^{k} \frac{1}{i!}(-\ln u)^i. \tag{5}$$

# Coherent systems- General case

- The system lifetime $T$ of a coherent system can be written as

$$T = \phi(X_1, \ldots, X_n) = \max_{i=1,\ldots,r} \min_{j \in P_i} X_j,$$

where $P_1, \ldots, P_r$ are the minimal path sets of $\phi$.

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- Then, by using the inclusion-exclusion formula

$$\begin{aligned}
\bar{F}_T(t) &= \Pr(T > t) = \Pr\left(\max_{i=1,\ldots,r} \min_{j \in P_i} X_j > t\right) \\
&= \Pr\left(\cup_{i=1}^{r} \{\min_{j \in P_i} X_j > t\}\right) \\
&= \sum_{i=1}^{r} \Pr\left(\min_{j \in P_i} X_j > t\right) - \sum_{i<k} \Pr\left(\min_{j \in P_i \cup P_k} X_j > t\right) + \ldots \\
&\quad + (-1)^{r+1} \Pr\left(\min_{j \in P_1 \cup \cdots \cup P_r} X_j > t\right).
\end{aligned}$$

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- For $X_{1:i} = \min(X_1, \dots, X_i)$, we have

$$\begin{aligned}\Pr(X_{1:i} > t) &= \Pr(X_1 > t, \dots, X_i > t, X_{i+1} > -\infty, \dots, X_n > -\infty)\\ &= K(\bar{F}_1(t), \dots, \bar{F}_i(t), 1, \dots, 1).\end{aligned}$$

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- For $X_P = \min_{j \in P} X_j$ and $\bar{F}_P(t) = \Pr(X_P > t)$, we have

$$\bar{F}_P(t) = K_P(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

where $K_P(u_1, \ldots, u_n) = K(u_1^P, \ldots, u_n^P)$ and $u_j^P = u_j$ if $j \in P$ and $u_j^P = 1$ if $j \notin P$.

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where $K_P(u_1, \ldots, u_n) = K(u_1^P, \ldots, u_n^P)$ and $u_j^P = u_j$ if $j \in P$ and $u_j^P = 1$ if $j \notin P$.
- Therefore, from the inclusion-exclusion representation

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

where $\bar{Q}$ is a multivariate dual distortion function.

## Coherent systems, particular cases

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- If $\bar{F}_1 = \cdots = \bar{F}_n = \bar{F}$ (ID), then

$$\bar{F}_T(t) = \bar{Q}(\bar{F}(t), \ldots, \bar{F}(t)) = \bar{q}(\bar{F}(t)),$$

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  where $\bar{q}(u) = \bar{Q}(u, \ldots, u)$ is a distortion function.
- In the IID case, $\bar{F}_T = a_1 \bar{F}_{1:1} + \cdots + a_n \bar{F}_{1:n} = \bar{q}(\bar{F}(t))$ where

$$\bar{q}(u) = a_1 u + \cdots + a_n u^n$$

  where $\boldsymbol{a} = (a_1, \ldots, a_n)$ is the minimal signature of the system.

# An example: Parallel system (active redundancy)

- If $T = X_{2:2} = \max(X_1, X_2)$, then $P_1 = \{1\}$, $P_2 = \{2\}$, and

$$\begin{aligned} \bar{F}_{2:2}(t) &= \Pr(\{X_1 > t\} \cup \{X_2 > t\}) \\ &= \Pr(X_1 > t) + \Pr(X_2 > t) - \Pr(X_1 > t, X_2 > t) \\ &= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \\ &= \bar{Q}_{2:2}(\bar{F}_1(t), \bar{F}_2(t)), \end{aligned}$$

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- If $X_1, X_2$ are ID$\sim \bar{F}$, then $\bar{F}_{2:2}(t) = \bar{q}_{2:2}(\bar{F}(t))$, with

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- If $X_1, X_2$ are IID, then $\bar{q}_{2:2}(u) = 2u - u^2$.

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- As $\bar{F}_{1:2}(t) = \bar{F}^2(t)$ and $f_{1:2}(t) = 2\bar{F}(t)f(t)$, we have

$$\bar{F}_{2:2}(t) = \bar{F}^2(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} 2\bar{F}(x) f(x) dx = 2\bar{F}(t) - \bar{F}^2(t).$$

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- The sequential order statistics can be obtained in a similar way.

# Coherent systems and relevation transform

- If $T = X_{2:2}$ and $(X_1, X_2)$ are dependent and abs. cont., $p_1 = \Pr(X_1 < X_2)$ and $p_2 = \Pr(X_2 < X_1)$, then

$$\bar{F}_{2:2}(t) = p_1 \Pr(X_{2:2} > t | X_1 < X_2) + p_2 \Pr(X_{2:2} > t | X_2 < X_1)$$
$$= p_1 \bar{F}_1^{(X_1 < X_2)} \# \bar{G}_1(t) + p_2 \bar{F}_2^{(X_2 < X_1)} \# \bar{G}_2(t),$$

where

$$\bar{F}_1^{(X_1 < X_2)}(t) = \Pr(X_1 > t | X_1 < X_2),$$
$$\bar{F}_2^{(X_2 < X_1)}(t) = \Pr(X_2 > t | X_2 < X_1),$$
$$\bar{G}_{1,x}(y) = \Pr(X_2 - x > y | X_1 = x, X_2 > x)$$

and

$$\bar{G}_{2,x}(y) = \Pr(X_1 - x > y | X_2 = x, X_1 > x).$$

# Coherent systems and relevation transform

- As $\boldsymbol{f}(x,y) = f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x),\bar{F}_2(y))$, then

$$\begin{aligned} p_1 &= \Pr(X_1 < X_2) \\ &= \int_0^\infty \int_x^\infty f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x),\bar{F}_2(y))dydx \\ &= \int_0^\infty f_1(x)\partial_1 K(\bar{F}_1(x),\bar{F}_2(x))dx \end{aligned}$$

when $\lim_{y\to\infty} \partial_1 K(\bar{F}_1(x),\bar{F}_2(y)) = 0$.

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  when $\lim_{y\to\infty} \partial_1 K(\bar{F}_1(x),\bar{F}_2(y)) = 0$.
- Analogously,

$$p_2 = \Pr(X_2 < X_1) = \int_0^\infty f_2(x)\partial_2 K(\bar{F}_1(x),\bar{F}_2(x))dx.$$

  when $\lim_{x\to\infty} \partial_2 K(\bar{F}_1(x),\bar{F}_2(y)) = 0$.

# Coherent systems and relevation transform

- The joint density of $(X_1, X_2|X_1 < X_2)$ is $\boldsymbol{h}(x,y) = \boldsymbol{f}(x,y)/p_1$ for all $x \le y$ (0 otherwise). Then the marginal density of $(X_1|X_1 < X_2)$ is

$$h_1(x) = \frac{1}{p_1}\int_x^\infty \boldsymbol{f}(x,y)dy = \frac{1}{p_1} f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(x)).$$

# Coherent systems and relevation transform

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$$h_1(x) = \frac{1}{p_1}\int_x^\infty \boldsymbol{f}(x,y)dy = \frac{1}{p_1} f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(x)).$$

- Hence, the conditional density of $(X_2|X_1 = x, X_2 > x)$ is

$$h_{2|1}(y|x) = \frac{\boldsymbol{h}(x,y)}{h_1(x)} = \frac{f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))}{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))}.$$

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- Then the reliability function $\bar{G}_{1,x}$ is given by

$$\bar{G}_{1,x}(y) = \int_{x+y}^\infty h_{2|1}(z|x) dz = \frac{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x+y))}{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))}. \quad (6)$$

# Coherent systems and relevation transform

- Therefore, from (1), we obtain

$$\begin{aligned}
\bar{F}_1^{(X_1<X_2)} \# \bar{G}_1(t) &= \bar{F}_1^{(X_1<X_2)}(t) + \int_0^t \bar{G}_{1,x}(t-x) h_1(x) dx \\
&= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1} \int_0^t \frac{\partial_1 K(\bar{F}_1(x), \bar{F}_2(t))}{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))} f_1(x) \partial_1 K(\bar{F}_1(x), \bar{F}_2(x)) dx \\
&= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1} \int_0^t f_1(x) \partial_1 K(\bar{F}_1(x), \bar{F}_2(t)) dx \\
&= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1} \left[ \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \right].
\end{aligned}$$

# Coherent systems and relevation transform

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$$\bar{F}_1^{(X_1<X_2)}\#\bar{G}_1(t) = \bar{F}_1^{(X_1<X_2)}(t) + \int_0^t \bar{G}_{1,x}(t-x)h_1(x)dx$$

$$= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1}\int_0^t \frac{\partial_1 K(\bar{F}_1(x),\bar{F}_2(t))}{\partial_1 K(\bar{F}_1(x),\bar{F}_2(x))} f_1(x)\partial_1 K(\bar{F}_1(x),\bar{F}_2(x))dx$$

$$= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1}\int_0^t f_1(x)\partial_1 K(\bar{F}_1(x),\bar{F}_2(t))dx$$

$$= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1}\left[\bar{F}_2(t) - K(\bar{F}_1(t),\bar{F}_2(t))\right].$$

- In a similar way (by the symmetry), we get

$$\bar{F}_2^{(X_2<X_1)}\#\bar{G}_2(t) = \bar{F}_2^{(X_2<X_1)}(t) + \frac{1}{p_2}\left[\bar{F}_1(t) - K(\bar{F}_1(t),\bar{F}_2(t))\right]. \quad (7)$$

# Coherent systems and relevation transform

- Finally, we get

$$\begin{aligned}
\bar{F}_{2:2}(t) &= p_1 \bar{F}_1^{(X_1<X_2)} \# \bar{G}_1(t) + p_2 \bar{F}_2^{(X_2<X_1)} \# \bar{G}_2(t) \\
&= p_1 \bar{F}_1^{(X_1<X_2)}(t) + p_2 \bar{F}_2^{(X_2<X_1)}(t) + \bar{F}_1(t) + \bar{F}_2(t) \\
&\quad - 2K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= p_1 \Pr(X_{1:2} > t | X_1 < X_2) + p_2 \Pr(X_{1:2} > t | X_2 < X_1) \\
&\quad + \bar{F}_1(t) + \bar{F}_2(t) - 2K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= \Pr(X_{1:2} > t) + \bar{F}_1(t) + \bar{F}_2(t) - 2K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= \bar{Q}_{2:2}(\bar{F}_1(t), \bar{F}_2(t)),
\end{aligned}$$

where $\bar{Q}_{2:2}(u, v) = u + v - K(u, v)$ (as above).

# Coherent systems and relevation transform

- If $\boldsymbol{F}$ is exchangeable (EXC), then $\bar{F}_{2:2} = \bar{F}_{1:2} \# \bar{G}$, where

$$\bar{G}_x(y) = \Pr(X_2 - x > y | X_1 = x, X_2 > x) = \frac{\Pr(X_2 > x + y | X_1 = x)}{\Pr(X_2 > x | X_1 = x)}.$$

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- Then, from (6), we get

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- Hence, from (1), we have

$$\begin{aligned}\bar{F}_{1:2} \# \bar{G}(t) &= \bar{F}_{1:2}(t) + \int_0^t \bar{G}_x(t-x) f_{1:2}(x) dx \\ &= \bar{F}_{1:2}(t) + \int_0^t \frac{\partial_1 K(\bar{F}(x), \bar{F}(t))}{\partial_1 K(\bar{F}(x), \bar{F}(x))} f_{1:2}(x) dx.\end{aligned}$$

# Coherent systems and relevation transform

- Note that $\bar{F}_{1:2}(x) = K(\bar{F}(x), \bar{F}(x))$ and

$$\begin{aligned} f_{1:2}(x) &= f(x)\partial_1 K(\bar{F}(x), \bar{F}(x)) + f(x)\partial_2 K(\bar{F}(x), \bar{F}(x)) \\ &= 2f(x)\partial_1 K(\bar{F}(x), \bar{F}(x)). \end{aligned}$$

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- Therefore, from (1),

$$\begin{aligned} \bar{F}_{1:2}\#\bar{G}(t) &= \bar{F}_{1:2}(t) + 2\int_0^t \partial_1 K(\bar{F}(x), \bar{F}(t))f(x)dx \\ &= K(\bar{F}(t), \bar{F}(t)) - 2K(\bar{F}(t), \bar{F}(t)) + 2K(1, \bar{F}(t)) \\ &= 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t)) \\ &= \bar{q}_{2:2}(\bar{F}(t)), \end{aligned}$$

where $\bar{q}_{2:2}(u) = 2u - K(u, u)$ (as above).

# Coherent systems and relevation transform

Another approach for the general case is

$$\bar{F}_{2:2} = \bar{F}_{1:2} \# \bar{G}, \tag{8}$$

where

$$\begin{aligned}\bar{G}_x(y) &= p_1(x) \Pr(X_2 - x > y | X_1 = x, X_2 > x) \\ &\quad + p_2(x) \Pr(X_1 - x > y | X_2 = x, X_1 > x) \\ &= p_1(x) \frac{\Pr(X_2 > x + y | X_1 = x)}{\Pr(X_2 > x | X_1 = x)} + p_2(x) \frac{\Pr(X_1 > x + y | X_2 = x)}{\Pr(X_1 > x | X_2 = x)},\end{aligned}$$

$p_1(x) = \Pr(X_1 < X_2 | X_{1:2} = x)$ and $p_2(x) = \Pr(X_2 < X_1 | X_{1:2} = x)$.

# Minimal repair of systems: Cases.

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# Minimal repair of systems: Cases.

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- **Case I:** To repair the component which fails first.
- **Case II:** To repair the component which leads to the system failure.
- **Case III:** To repair a fixed component (e.g., to repair the $i$th component).
- Which option is the best one?

# Case III

- If we repair the $i$th component, the resulting system $T_{III}^{(i)}$ has the following reliability

$$\bar{F}_{T_{III}^{(i)}}(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_{i-1}(t), \bar{q}_1(\bar{F}_i(t)), \bar{F}_{i+1}(t), \ldots, \bar{F}_n(t))$$

where $\bar{q}_1(u) = u - u \log u$ was obtained in (5).

# Case III

- If we repair the $i$th component, the resulting system $T_{III}^{(i)}$ has the following reliability

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where $\bar{q}_1(u) = u - u \log u$ was obtained in (5).

- If the components are ID, then $\bar{F}_{T_{III}^{(i)}}(t) = \bar{q}_{III}^{(i)}(\bar{F}(t))$, where

$$\bar{q}_{III}^{(i)}(u) = \bar{Q}(u, \dots, u, \bar{q}_1(u), u, \dots, u) \qquad (9)$$

and $\bar{q}_1$ is placed at the $i$th position.

# Case III

- If we repair the $i$th component, the resulting system $T^{(i)}_{III}$ has the following reliability

$$\bar{F}_{T^{(i)}_{III}}(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_{i-1}(t), \bar{q}_1(\bar{F}_i(t)), \bar{F}_{i+1}(t), \dots, \bar{F}_n(t))$$

  where $\bar{q}_1(u) = u - u \log u$ was obtained in (5).
- If the components are ID, then $\bar{F}_{T^{(i)}_{III}}(t) = \bar{q}^{(i)}_{III}(\bar{F}(t))$, where

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  and $\bar{q}_1$ is placed at the $i$th position.
- Comparison results for this kind of replacements were presented in the talk by Antonio Arriaza.

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# Case I

- In this case we repair the component that fails first.
- Its lifetime is $X = X_{1:n}$.
- Then the broken component is minimally repaired and the resulting system has the same structure as $T$ but we know that all the components are working and have age $X$.
- Hence its reliability is

$$\bar{F}_{T_I}(t) = \bar{F}_{1:n} \# \bar{G}(t), \tag{10}$$

where

$$\bar{G}_x(y) = \Pr(T - x > y | X_1 > x, \dots, X_n > x)$$
$$= \frac{\Pr(T > x + y, X_1 > x, \dots, X_n > x)}{\Pr(X_1 > x, \dots, X_n > x)}$$

when $X = x$.

# Case I. General representation

- In Proposition 3 of Navarro (Stat. Papers 2016) is proved that

$$\bar{G}_x(t) = \bar{Q}_x(\bar{F}_{1,x}(t), \dots, \bar{F}_{n,x}(t)),$$

where $\bar{F}_{i,x}(t) = \Pr(X_i - x > t | X_i > x) = \bar{F}_i(t + x)/\bar{F}_i(x)$ for $i = 1, \dots, n$ and $\bar{Q}_x$ is a distortion function.

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- Hence, from (1), we have,

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:n}(t) + \int_0^t \bar{G}_x(t - x) f_{1:n}(x) dx \\ &= \bar{F}_{1:n}(t) + \int_0^t \bar{Q}_x(\bar{F}_{1,x}(t - x), \dots, \bar{F}_{n,x}(t - x)) f_{1:n}(x) dx.\end{aligned}$$

# Case I: Example 1.

- If $T = X_{1:n}$, then

$$\bar{G}_x(t) = \Pr(X_{1:n} - x > y | X_{1:n} > x) = \frac{\Pr(X_{1:n} > x + y)}{\Pr(X_{1:n} > x)} = \frac{\bar{F}_{1:n}(x + y)}{\bar{F}_{1:n}(x)}.$$

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- Therefore

$$\begin{aligned}
\bar{F}_{T_I}(t) &= \bar{F}_{1:n}(t) + \int_0^t \bar{G}_x(t-x) f_{1:n}(x)dx \\
&= \bar{F}_{1:n}(t) + \int_0^t \frac{\bar{F}_{1:n}(t)}{\bar{F}_{1:n}(x)} f_{1:n}(x)dx \\
&= \bar{F}_{1:n}(t) - \bar{F}_{1:n}(t) \log \bar{F}_{1:n}(t) \\
&= \bar{F}_{1:n} \# \bar{F}_{1:n}(t) \\
&= \bar{q}_1(\bar{F}_{1:n}(t)).
\end{aligned}$$

# Case I: Example 1.

- Hence, $T_I \geq_{ST} T_{III}^{(i)}$ $(\leq_{ST})$ holds for all $F_1, \dots, F_n$ iff

$$\bar{q}_1(K(u_1, \dots, u_n)) \geq K(u_1, \dots, \bar{q}_1(u_i), \dots, u_n) \quad (\leq),$$

where $\leq_{ST}$ is the (usual) stochastic order.

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where $\leq_{ST}$ is the (usual) stochastic order.

- In particular, if the components are independent, then

$$\bar{q}_1(u_1 \ldots u_n) = u_1 \ldots u_n(1 - \log[u_1 \ldots u_n]) \geq u_1 \ldots u_n(1 - \log u_i).$$

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- So $T_I \geq_{ST} T_{III}^{(i)}$ holds for all $F_1, \dots, F_n$.

# Case I: Example 2.

If $T = X_{2:2}$, then $\bar{F}_{T_I} = \bar{F}_{1:2} \# \bar{G}$ where

$$\bar{G}_x(y) = \Pr(T - x > y | X_1 > x, X_2 > x)$$

$$= \frac{\Pr(\max(X_1, X_2) > x + y, X_1 > x, X_2 > x)}{\Pr(X_1 > x, X_2 > x)}$$

$$= \frac{\Pr(X_1 > x + y, X_2 > x) + \Pr(X_2 > x + y, X_1 > x) - \Pr(X_{1:2} > x + y)}{\Pr(X_1 > x, X_2 > x)}$$

$$= \frac{K(\bar{F}_1(x + y), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(x + y)) - K(\bar{F}_1(x + y), \bar{F}_2(x + y))}{K(\bar{F}_1(x), \bar{F}_2(x))}$$

$$= \bar{Q}_x(\bar{F}_{1,x}(y), \bar{F}_{2,x}(y)),$$

where $\bar{F}_{1,x}(y) = \bar{F}_1(x + y)/\bar{F}_1(x)$, $\bar{F}_{2,x}(y) = \bar{F}_2(x + y)/\bar{F}_2(x)$,

$$\bar{Q}_x(u_1, u_2) = \frac{K(u_1 \bar{F}_1(x), \bar{F}_2(x)) + K(\bar{F}_1(x), u_2 \bar{F}_2(x)) - K(u_1 \bar{F}_1(x), u_2 \bar{F}_2(x))}{K(\bar{F}_1(x), \bar{F}_2(x))}.$$

# Case I: Example 2.

Hence, from (1) and (10),

$$\begin{aligned}
\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \int_0^t \bar{Q}_x(\bar{F}_{1,x}(t-x), \bar{F}_{2,x}(t-x)) f_{1:2}(x)dx \\
&= \bar{F}_{1:2}(t) + \int_0^t \frac{K(\bar{F}_1(t), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(t)) - \bar{F}_{1:2}(t)}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx \\
&= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\log(\bar{F}_{1:2}(t)) \\
&\quad + \int_0^t \frac{K(\bar{F}_1(t), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx.
\end{aligned}$$

# Case I: Example 2.

If the components are IID, then

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}^2(t) + 2\bar{F}^2(t)\log(\bar{F}(t)) + \int_0^t \frac{\bar{F}(t)\bar{F}(x) + \bar{F}(x)\bar{F}(t)}{\bar{F}^2(x)} 2f(x)\bar{F}(x)dx \\ &= \bar{F}^2(t) + 2\bar{F}^2(t)\log(\bar{F}(t)) + 4\bar{F}(t)\int_0^t f(x)dx \\ &= \bar{F}^2(t) + 2\bar{F}^2(t)\log(\bar{F}(t)) + 4\bar{F}(t)F(t) \\ &= \bar{q}_I(\bar{F}(t))\end{aligned}$$

where

$$\bar{q}_I(u) = u^2 + 2u^2\log(u) + 4u(1-u) = 4u - 3u^2 + 2u^2\log(u).$$

# Case I: Example 2.

- In the IID case, for this system we have $\bar{Q}(u,v) = u + v - uv$.
- Therefore $\bar{F}_{T_{III}^{(i)}}(t) = \bar{q}_{III}^{(i)}(\bar{F}(t))$ with

$$\begin{aligned}\bar{q}_{III}^{(i)}(u) &= \bar{Q}(u, \bar{q}_1(u)) = u + \bar{q}_1(u) - u\bar{q}_1(u)\\ &= 2u - u^2 - u\log u + u^2 \log u.\end{aligned}$$

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- Hence $\bar{q}_I \leq \bar{q}_{III}^{(i)}$ for $i = 1, 2$.
- So, $T_I \leq_{ST} T_{III}^{(i)}$ holds for all $F$, that is, in this system, it is better to replace a fixed component than to replace the first failure.

# Case I. General representation, ID components.

**Theorem**

*Let $T$ be the lifetime of a coherent system with ID components having a common reliability $\bar{F}$. Then*

$$\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t)) \tag{11}$$

*for all $t \geq 0$ and a distortion function $\bar{q}_I$.*

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*for all $t \geq 0$ and a distortion function $\bar{q}_I$.*

- The distortion function $\bar{q}_I$ depends on the structure of the system and on the underlying survival copula $K$ but does not depend on $\bar{F}$.

# Case I. General representation, ID components.

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*Let $T$ be the lifetime of a coherent system with ID components having a common reliability $\bar{F}$. Then*

$$\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t)) \tag{11}$$

*for all $t \geq 0$ and a distortion function $\bar{q}_I$.*

- The distortion function $\bar{q}_I$ depends on the structure of the system and on the underlying survival copula $K$ but does not depend on $\bar{F}$.
- Sometimes, it is not easy to compute $\bar{q}_I$.

# Case I. General representation, IID components.

**Theorem**

*Let $T$ be the lifetime of a coherent system with IID components having a common reliability $\bar{F}$. Then $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ where*

$$\bar{q}_I(u) = n \sum_{i=1}^{n-1} \frac{a_i}{n-i} u^i + \left(1 - n \sum_{i=1}^{n-1} \frac{a_i}{n-i}\right) u^n - n a_n u^n \log u \quad (12)$$

*and $(a_1, \ldots, a_n)$ is the minimal signature of the system.*

# Case II. General representation.

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- Note that, in case I, we just repair the first failure and so, for example, if the components are exchangeable, then all the components have the same probability of being repaired.
- However, we must note that case II is not always available in practice for all the systems (e.g., in a plain).
- In this case it is not easy to obtain the reliability $\bar{F}_{T_{II}}$ of the resulting system lifetime $T_{II}$. Let us see a simple example.

# Case II. Example 1, IID case.

- If $T = X_{2:2}$ and the components are IID, then, from (1), we have

$$\bar{F}_{T_{II}}(t) = \bar{F}_T \# \bar{F}(t) = \bar{F}_T(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_T(x) dx,$$

where $\bar{F}_T(t) = 2\bar{F}(t) - \bar{F}^2(t)$ and $f_T(t) = 2(1 - \bar{F}(t))f(t)$.

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- Hence

$$\begin{aligned}\bar{F}_{T_{II}}(t) &= 2\bar{F}(t) - \bar{F}^2(t) + 2\bar{F}(t)\int_0^t \frac{1-\bar{F}(x)}{\bar{F}(x)} f(x)dx \\ &= 2\bar{F}(t) - \bar{F}^2(t) - 2\bar{F}(t)(F(t) + \log \bar{F}(t)) \\ &= \bar{F}^2(t) - 2\bar{F}(t)\log \bar{F}(t) \\ &= \bar{q}_{II}(\bar{F}(t))\end{aligned}$$

with $\bar{q}_{II}(u) = u^2 - 2u\log u$.

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- A straightforward calculation leads to $\bar{q} \leq \bar{q}_I \leq \bar{q}_{III}^{(i)} \leq \bar{q}_{II}$.

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- Therefore, the best option in this system is to repair the component which is critical for the system.
- The second best option is to replace a fixed component and, of course, the three options are better than the original system $T$.

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- So $T \leq_{ST} T_I \leq_{ST} T_{III}^{(i)} \leq_{ST} T_{II}$ for all $\bar{F}$ and $i = 1, 2$.
- Therefore, the best option in this system is to repair the component which is critical for the system.
- The second best option is to replace a fixed component and, of course, the three options are better than the original system $T$.
- They are also better than a parallel system with three components (active redundancy).

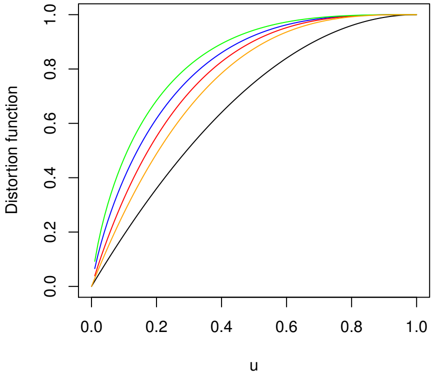


Figure: Distortion functions for a parallel system with 2 IID components (black), in case I (red), in case II (green), in case III (blue) and with 3 IID components (orange).

# Case II. Example 1, exchangeable case.

- If the components are EXC, then $\bar{F}_{T_{II}}(t) = \bar{F}_T \# \bar{G}(t)$, where

$$\begin{aligned}\bar{G}_x(y) &= \Pr(X_2 - x > y | X_1 \le x, X_2 > x) \\ &= \frac{\Pr(X_1 \le x, X_2 > x + y)}{\Pr(X_1 \le x, X_2 > x)} \\ &= \frac{\Pr(X_2 > x + y) - \Pr(X_1 > x, X_2 > x + y)}{\Pr(X_2 > x) - \Pr(X_1 > x, X_2 > x)} \\ &= \frac{\bar{F}(x+y) - K(\bar{F}(x), \bar{F}(x+y))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))}.\end{aligned}$$

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&= \frac{\Pr(X_1 \le x, X_2 > x + y)}{\Pr(X_1 \le x, X_2 > x)} \\
&= \frac{\Pr(X_2 > x + y) - \Pr(X_1 > x, X_2 > x + y)}{\Pr(X_2 > x) - \Pr(X_1 > x, X_2 > x)} \\
&= \frac{\bar{F}(x + y) - K(\bar{F}(x), \bar{F}(x + y))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))}.
\end{aligned}$$

- Hence, from (1), we have

$$\bar{F}_{T_{II}}(t) = \bar{F}_T(t) + \int_0^t \frac{\bar{F}(t) - K(\bar{F}(x), \bar{F}(t))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))} f_T(x) dx.$$

# Case II. Example 1, exchangeable case.

As $\bar{F}_T(t) = 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t))$, then

$$f_T(t) = 2(1 - \partial_1 K(\bar{F}(t), \bar{F}(t)))f(t)$$

and

$$\begin{aligned}
\bar{F}_{T_{II}}(t) &= \bar{F}_T(t) + 2\int_0^t \frac{\bar{F}(t) - K(\bar{F}(x), \bar{F}(t))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))}(1 - \partial_1 K(\bar{F}(x), \bar{F}(x)))f(x)dx \\
&= \bar{F}_T(t) + 2\int_{\bar{F}(t)}^1 \frac{\bar{F}(t) - K(v, \bar{F}(t))}{v - K(v, v)}(1 - \partial_1 K(v, v))dv \\
&= \bar{q}_{II}(\bar{F}(t))
\end{aligned}$$

with

$$\bar{q}_{II}(u) = 2u - K(u, u) + 2\int_u^1 \frac{u - K(v, u)}{v - K(v, v)}(1 - \partial_1 K(v, v))dv. \quad (13)$$

# Case II. Example 1, general case.

- In the general case, we get $\bar{F}_{T_{II}}(t) = \bar{F}_T \# \bar{G}(t)$, where

$$\begin{aligned}
\bar{G}_x(y) &= p_1(x) \Pr(X_2 - x > y | X_1 \le x, X_2 > x) \\
&\quad + p_2(x) \Pr(X_1 - x > y | X_2 \le x, X_1 > x) \\
&= p_1(x) \frac{\bar{F}_2(x+y) - K(\bar{F}_1(x), \bar{F}_2(x+y))}{\bar{F}_2(x) - K(\bar{F}_1(x), \bar{F}_2(x))} \\
&\quad + p_2(x) \frac{\bar{F}_1(x+y) - K(\bar{F}_1(x+y), \bar{F}_2(x))}{\bar{F}_1(x) - K(\bar{F}_1(x), \bar{F}_2(x))},
\end{aligned}$$

$p_1(x) := \Pr(X_1 < X_2 | T = x)$ and
$p_2(x) := \Pr(X_2 < X_1 | T = x) = 1 - p_1(x)$ for $x, y \ge 0$.

# Case II. Example 1, general case.

- In the general case, we get

$$\begin{aligned}\bar{F}_{T_{II}}(t) &= \bar{F}_T(t) + \int_0^t \bar{G}_x(t-x) f_T(x)dx \\ &= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \\ &+ \int_0^t [1 - \partial_2 K(\bar{F}_1(x), \bar{F}_2(x))] \frac{\bar{F}_2(t) - K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_2(x) - K(\bar{F}_1(x), \bar{F}_2(x))} f_2(x)dx \\ &+ \int_0^t [1 - \partial_1 K(\bar{F}_1(x), \bar{F}_2(x))] \frac{\bar{F}_1(t) - K(\bar{F}_1(t), \bar{F}_2(x))}{\bar{F}_1(x) - K(\bar{F}_1(x), \bar{F}_2(x))} f_1(x)dx.\end{aligned}$$

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\begin{aligned}
\bar{F}_{T_{II}}(t) &= \bar{F}_T(t) + \int_0^t \bar{G}_x(t-x) f_T(x)dx \\
&= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \\
&+ \int_0^t [1 - \partial_2 K(\bar{F}_1(x), \bar{F}_2(x))] \frac{\bar{F}_2(t) - K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_2(x) - K(\bar{F}_1(x), \bar{F}_2(x))} f_2(x)dx \\
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\end{aligned}
$$

- Of course, in the exchangeable case, we have $\Pr(X_1 < X_2 | T = x) = \Pr(X_2 < X_1 | T = x) = 1/2$ and (13).

# Case II. Representation, exchangeable case.

**Theorem**

*If the components have an absolutely continuous exchangeable joint reliability with a common reliability $\bar{F}$, then the reliability function of $T_{II}$ can be written as*

$$\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t)) \tag{14}$$

*for all $t \geq 0$ and for a distortion function $\bar{q}_{II}$ which does not depend on $\bar{F}$.*

# Case II. Representation, IID case.

## Theorem

*If the components are IID$\sim \bar{F}$, then $\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t))$, where*

$$\bar{q}_{II}(u) = \sum_{i=1}^{n} c_i u^i + \sum_{i=1}^{n} d_i u^i \log u \qquad (15)$$

*for some coefficients $c_i, d_i$, $i = 1, \ldots, n$ which only depend on the structure of the system.*

The proof is based on Samaniego's representation

$$\bar{F}_T = s_1 \bar{F}_{1:n} + \cdots + s_n \bar{F}_{n:n}$$

where $s_i = \Pr(T = X_{i:n})$ and $\boldsymbol{s} = (s_1, \ldots, s_n)$ is the signature of $T$.

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# Case II. Example 2, IID case.

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- The signature of the system is $\boldsymbol{s} = (0, 2/3, 1/3)$.
- It can be computed from the permutations given in Table 1.

- Table 1: Repairing options for the system in Example 2.

| $j$ | $A_j$ | $H_j$ | $\lvert A_j \rvert$ | $T$ | $i_j$ | $T_j$ |
|---|---|---|---|---|---|---|
| 1 | $(1, i_2, i_3)$ | $X_1 < X_{i_2} < X_{i_3}$ | 2 | $T = X_{i_2}$ | 2 | $\min(X_2, X_3)$ |
| 2 | $(i_1, 1, i_3)$ | $X_{i_1} < X_1 < X_{i_3}$ | 2 | $T = X_1$ | 2 | $X_1$ |
| 3 | $(i_1, i_2, 1)$ | $X_{i_1} < X_{i_2} < X_1$ | 2 | $T = X_1$ | 3 | $X_1$ |

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- Let $T = \max(X_1, \min(X_2, X_3))$ with IID components.
- The signature of the system is $\boldsymbol{s} = (0, 2/3, 1/3)$.
- It can be computed from the permutations given in Table 1.
- This table contains the sets $A_j$ of permutations which leads to the same relevation transform, the numbers $i_j$ of component failures which cause the system failure and the expressions of the repaired system lifetimes $T_j$ for each $j = 1, 2, 3$.
- Table 1: Repairing options for the system in Example 2.

| $j$ | $A_j$ | $H_j$ | $\|A_j\|$ | $T$ | $i_j$ | $T_j$ |
|---|---|---|---|---|---|---|
| 1 | $(1, i_2, i_3)$ | $X_1 < X_{i_2} < X_{i_3}$ | 2 | $T = X_{i_2}$ | 2 | $\min(X_2, X_3)$ |
| 2 | $(i_1, 1, i_3)$ | $X_{i_1} < X_1 < X_{i_3}$ | 2 | $T = X_1$ | 2 | $X_1$ |
| 3 | $(i_1, i_2, 1)$ | $X_{i_1} < X_{i_2} < X_1$ | 2 | $T = X_1$ | 3 | $X_1$ |

# Case II. Example 2, IID case.

- Hence

$$\Pr(T_{II} > t) = \frac{1}{3}\sum_{j=1}^{3}\Pr(T_{II} > t|H_j)$$

for the events $H_j$ given in Table 1.

# Case II. Example 2, IID case.

- Hence

$$\Pr(T_{II} > t) = \frac{1}{3}\sum_{j=1}^{3} \Pr(T_{II} > t | H_j)$$

for the events $H_j$ given in Table 1.

- The first one can be obtained as

$$\Pr(T_{II} > t | H_1) = \bar{F}_{i_1:3} \# \bar{G}_1(t) = \bar{F}_{2:3} \# \bar{G}_1(t),$$

where

$$\begin{aligned} \bar{G}_{1,x}(y) &= \Pr(T_1 - x > y | X_{2:3} = x, H_1) \\ &= \Pr(\min(X_2, X_3) - x > y | X_1 < x < X_2, x < X_3) \\ &= \frac{\bar{F}^2(x+y)}{\bar{F}^2(x)} \end{aligned}$$

since the components are IID.

# Case II. Example 2, IID case.

- Therefore, from (1), we have

$$\Pr(T_{II} > t | H_1) = \bar{F}_{2:3}(t) + \int_0^t \frac{\bar{F}^2(t)}{\bar{F}^2(x)} f_{2:3}(x) dx,$$

where $\bar{F}_{2:3}(t) = 3\bar{F}^2(t) - 2\bar{F}^3(t)$ and
$f_{2:3}(t) = 6(\bar{F}(t) - \bar{F}^2(t))f(t)$.

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- Hence

$$\begin{aligned}\Pr(T_{II} > t|H_1) &= \bar{F}_{2:3}(t) + 6\bar{F}^2(t) \int_0^t \frac{\bar{F}(x) - \bar{F}^2(x)}{\bar{F}^2(x)} f(x)dx \\ &= \bar{F}_{2:3}(t) + 6\bar{F}^2(t) \int_0^t \left(\frac{1}{\bar{F}(x)} - 1\right) f(x)dx \\ &= \bar{F}_{2:3}(t) + 6\bar{F}^2(t) \left(-\log \bar{F}(t) - F(t)\right) \\ &= -3\bar{F}^2(t) + 4\bar{F}^3(t) - 6\bar{F}^2(t) \log \bar{F}(t).\end{aligned}$$

# Case II. Example 2, IID case.

- For $H_2$, we have $\Pr(T_{II} > t|H_2) = \bar{F}_{2:3} \# \bar{F}(t)$ and

$$\Pr(T_{II} > t|H_2) = \bar{F}_{2:3}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_{2:3}(x)dx = 3\bar{F}(t) - 3\bar{F}^2(t) + \bar{F}^3(t).$$

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- This case is equivalent to a parallel system with 3 IID components (i.e. $\bar{F}_{3:3} = \bar{F}_{2:3}\#\bar{F}$).

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- This case is equivalent to a parallel system with 3 IID components (i.e. $\bar{F}_{3:3} = \bar{F}_{2:3}\#\bar{F}$).
- Finally, for $H_3$, we have $\Pr(T_{II} > t|H_3) = \bar{F}_{3:3}\#\bar{F}(t)$ and

$$\begin{aligned}\Pr(T_{II} > t|H_3) &= \bar{F}_{3:3}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_{3:3}(x)dx \\ &= -\frac{3}{2}\bar{F}(t) + 3\bar{F}^2(t) - \frac{1}{2}\bar{F}^3(t) - 3\bar{F}(t)\log\bar{F}(t).\end{aligned}$$

# Case II. Example 2, IID case.

- For $H_2$, we have $\Pr(T_{II} > t|H_2) = \bar{F}_{2:3}\#\bar{F}(t)$ and

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- Finally, for $H_3$, we have $\Pr(T_{II} > t|H_3) = \bar{F}_{3:3}\#\bar{F}(t)$ and

$$\Pr(T_{II} > t|H_3) = \bar{F}_{3:3}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_{3:3}(x)dx$$
$$= -\frac{3}{2}\bar{F}(t) + 3\bar{F}^2(t) - \frac{1}{2}\bar{F}^3(t) - 3\bar{F}(t)\log\bar{F}(t).$$

- Hence

$$\Pr(T_{II} > t) = \frac{1}{2}\bar{F}(t) - \bar{F}^2(t) + \frac{3}{2}\bar{F}^3(t) - \bar{F}(t)\log\bar{F}(t) - 2\bar{F}^2(t)\log\bar{F}(t)$$

# Case II. Example 2, IID case.

- Note that $\Pr(T_{II} > t) = \bar{q}_{II}(\bar{F}(t))$, where

$$\bar{q}_{II}(u) = \frac{1}{2}u - u^2 + \frac{3}{2}u^3 - u\log u - 2u^2\log u.$$

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- The distortion function for case I can be obtained from (12) as

$$\bar{q}_I(u) = \frac{3}{2}u + 3u^2 - \frac{7}{2}u^3 + 3u^3 \log u.$$

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- A straightforward calculation shows that $\bar{q}_I \leq \bar{q}_{II}$.

# Case II. Example 2, IID case.

- Note that $\Pr(T_{II} > t) = \bar{q}_{II}(\bar{F}(t))$, where

$$\bar{q}_{II}(u) = \frac{1}{2}u - u^2 + \frac{3}{2}u^3 - u\log u - 2u^2\log u.$$

- The distortion function for case I can be obtained from (12) as

$$\bar{q}_I(u) = \frac{3}{2}u + 3u^2 - \frac{7}{2}u^3 + 3u^3\log u.$$

- A straightforward calculation shows that $\bar{q}_I \leq \bar{q}_{II}$.
- So $T_I \leq_{ST} T_{II}$ holds for all $\bar{F}$.

# Case II. Order statistics, IID case.

## Proposition

*If $T = X_{i:n}$ for a fixed $i \in \{2, \ldots, n\}$ and the components are IID, then $\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t))$, where*

$$\bar{q}_{II}(u) = \binom{n}{n-i+1} u^{n-i+1} - i\binom{n}{i} u^{n-i+1} \log u + u^{n-i+1} \sum_{k=n-i+2}^{n} (-1)^{k-n+i-1} \frac{k}{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} + \sum_{k=n-i+2}^{n} (-1)^{k-n+i} \frac{n-i+1}{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} u^k.$$

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- For example, if we know that the system does not fail with the first component failure, we can consider to repair the system at the second component failure with a minimal repair of the broken component at this point.
- If the components are exchangeable, the reliability function of the repaired system is $\bar{F}_{(2)}(t) = \bar{F}_{2:n} \# \bar{G}(t)$, where

$$\bar{G}_x(y) = \frac{1}{n} \sum_{i=1}^{n} \Pr(T_i - x > y | X_i \leq x, X_j > t \text{ for all } j \neq i)$$

and $T_i$ is the lifetime of the system obtained from $T$ when the $i$th component is broken. A similar expression can be obtained if the system is repaired at the $j$th failure for $j = 3, 4, \ldots$.

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where $\bar{F}_{1:n} = \bar{F}^n$, $(\bar{G}_{1:n})_x(y) = \bar{F}_x^n(y) = \frac{\bar{F}^n(x+y)}{\bar{F}^n(x)}$ is the reliability of $Y_{1:n}$, where $Y_1, \dots, Y_n$ are IID$\sim \bar{F}_x(y) = \frac{\bar{F}(x+y)}{\bar{F}(x)}$ and $\bar{G}_y(z) = \bar{q}_T(\bar{F}_y(z)) = \sum_{i=1}^n a_i \bar{F}_y(z)$ when $Y_{1:n} = y$.

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- Therefore

$$\bar{F}_I^{(2)}(t) = \bar{F}^n(t) - n\bar{F}^n(t)\log \bar{F}(t) + \frac{n^2 a_n}{2}\bar{F}^n(t)\log^2 \bar{F}(t)$$
$$+ n^2 \sum_{i=1}^{n-1} a_i \frac{\bar{F}^n(t)\log \bar{F}(t)}{n-i} + n^2 \sum_{i=1}^{n-1} a_i \frac{\bar{F}^i(t) - \bar{F}^n(t)}{(n-i)^2}$$

# Comparisons for distorted distributions.

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- $T_1 \leq_{RHR} T_2$ for all $\bar{F}$ if and only if $q_2/q_1$ increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all $\bar{F}$ if and only if $\bar{q}_2'/\bar{q}_1'$ decreases in $(0, 1)$.

# Comparisons for replacement policies.

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- For example, if $T = \min(X_1, \max(X_2, \min(X_3, X_4)))$ and with 4 IID components, then the respective distortion functions of cases I, II and II are

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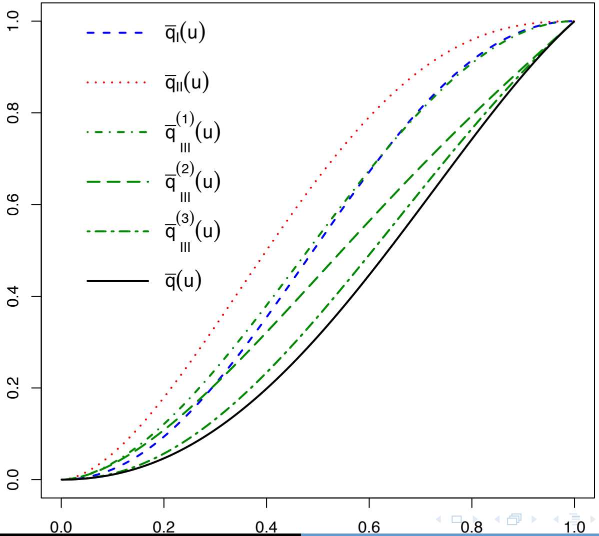

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- If $T = \max(X_1, \min(X_2, X_3))$ and with 3 IID components, then the respective distortion functions of cases I, II and II are plotted in the following figure.
- They prove that

$$T \leq_{ST} T_{III}^{(2)} \leq_{ST} T_I \leq_{ST} T_{II} \leq_{ST} T_{III}^{(1)}.$$

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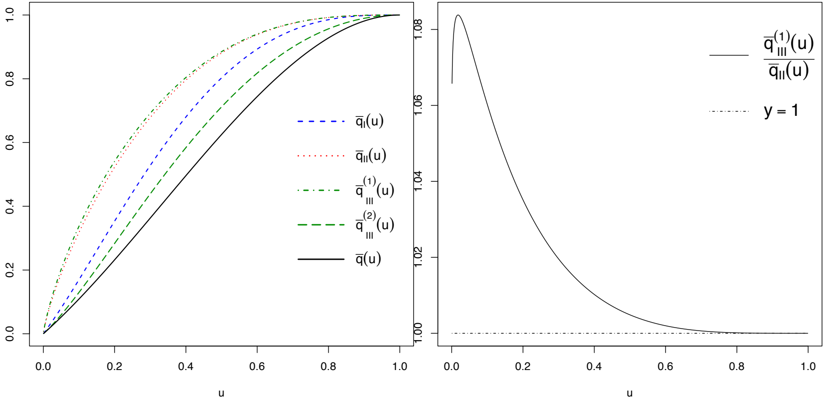
$\bar{q}_{I}(u)$
$\bar{q}_{II}(u)$
$\bar{q}^{(1)}_{III}(u)$
$\bar{q}^{(2)}_{III}(u)$
$\bar{q}(u)$
u
$\frac{\bar{q}^{(1)}_{III}(u)}{\bar{q}_{II}(u)}$
y = 1
u

# Main references

[1] Navarro J, Arriaza A, Suárez-Llorens A (2017). Minimal repair of failed components in coherent systems. Submitted.
[2] Arriaza A, Navarro J, Suárez-Llorens A (2017). Stochastic comparisons of replacement policies in coherent systems under minimal repair. Submitted.
[3] Navarro J, Arriaza A, Suárez-Llorens A (2017). R-script to compute the dual distortion functions of systems under minimal repair. Rodin University of Cádiz repository 10498/19935. http://rodin.uca.es. URL http://hdl.handle.net/10498/19935.
[4] Navarro J. (2016). Distribution-free comparisons of residual lifetimes of coherent systems based on copula properties. To appear in Statistical Papers. Published online first June 2016.

# References

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- Thank you for your attention!!