Order statistics and related concepts

Jorge Navarro[1],
Universidad de Murcia, Spain
E-mail: jorgenav@um.es

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Order statistics

- OS IID case

Notation

- $X_1, \dots, X_n$ IID random variables.
- $X_1, \dots, X_n$ exchangeable (EXC), i.e., for any σ

$$(X_1, \dots, X_n) =_{ST} (X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

- $(X_1, \dots, X_n)$ an arbitrary random vector with joint distribution

$$\mathbf{F}(x_1, \dots, x_n) = \Pr(X_1 \le x_1, \dots, X_n \le x_n)$$

and with joint reliability

$$\overline{\mathbf{F}}(x_1, \dots, x_n) = \Pr(X_1 > x_1, \dots, X_n > x_n).$$

- Let $X_{1:n}, \dots, X_{n:n}$ be the associated OS.
- Let $F_{i:n}(t) = \Pr(X_{i:n} \le t)$ be the distribution function (DF).

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Generalized mixture representations

- In the IID case:

$$F_{i:n}(t) = \sum_{j=i}^{n} \binom{n}{j} F^j(t) \overline{F}^{n-j}(t),$$

where $F(t) = \Pr(X_i \leq t) = 1 - \overline{F}(t)$.

- Also in the IID case:

$$F_{i:n}(t) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} F_{j:j}(t) = q_{i:n}(F(t)), \quad (1.1)$$

where $F_{j:j}(t) = F^j(t)$ and $q_{i:n}(u)$ is an increasing polinomial.

- In the EXC case the left hand side of (1.1) holds with $F_{j:j}(t) = \mathbf{F}(\underbrace{t, \dots, t}_{j}, \underbrace{\infty, \dots, \infty}_{n-j})$.
- Some coefficients in (1.1) are negative.

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- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$ stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$, mean residual life order.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{LR} Y \Leftrightarrow (X|s < X < t) \leq_{ST} (Y|s < Y < t)$ for $s < t$.

Stochastic orderings

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Stochastic orderings relationships

$$
\begin{array}{ccccc}
E(X_{s,t}) \leq E(Y_{s,t}) & \Rightarrow & E(X_t) \leq E(Y_t) & \Rightarrow & E(X) \leq E(Y) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X \leq_{DTM} Y & \Rightarrow & X \leq_{MRL} Y & \Rightarrow & X \leq_{M} Y \\
\Uparrow & & \Uparrow & & \Uparrow \\
X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{ST} Y \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_{s,t} \leq_{ST} Y_{s,t} & \Rightarrow & X_t \leq_{ST} Y_t & \Rightarrow & \overline{F}_X \leq \overline{F}_Y
\end{array}
$$

where $Z_t = (Z - t | Z > t)$ and $Z_{s,t} = (Z | s < Z < t)$ (see Navarro, Belzunce and Ruiz, PEIS, 1997).

Stochastic aging classes

- X is Increasing (Decreasing) Hazard rate IHR (DHR) if h is increasing.
- X is IHR $\Leftrightarrow (X - s | X > s) \geq_{ST} (X - t | X > t)$ for all $s < t$.
- X is New Better (Worse) than Used NBU (NWU) if $\Leftrightarrow X \geq_{ST} (X - t | X > t)$ for all $t > 0$.
- X is Increasing (Decreasing) Likelihood Ratio ILR (DLR) if f is log-concave (log-convex).
- X is ILR $\Leftrightarrow (X - s | X > s) \geq_{LR} (X - t | X > t)$ for all $s < t$.
- $ILR \Rightarrow IHR \Rightarrow NBU$.

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Ordering properties for OS

- In the IID case:

$$X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}.$$

- In the I case:

$$X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}.$$

- In the general case:

$$X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}.$$

- In the IID case:

$$F \quad IHR \Rightarrow F_{i:n} \quad IHR$$

$$F \quad NBU \Rightarrow F_{i:n} \quad NBU, \text{ and}$$

$$F \quad ILR \Rightarrow F_{i:n} \quad ILR.$$

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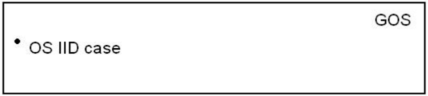
Generalized Order statistics (GOS)



Generalized Order statistics (GOS)

- For an arbitrary DF F, GOS $X_{1:n}^{GOS}, \ldots, X_{n:n}^{GOS}$ based on F can be obtained (Kamps, 1995, B. G. Teubner Stuttgart, p.49) via the quantile transformation

$$X_{r:n}^{GOS} = F^{-1}(U_{r:n}^{GOS}), \quad r = 1, \ldots, n,$$

where $(U_{1:n}^{*}, \ldots, U_{n:n}^{*})$ has the joint PDF

$$g^{GOS}(u_1, \ldots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1 - u_i)^{m_i} \right) (1 - u_n)^{k-1}$$

for $0 \leq u_1 \leq \ldots \leq u_n < 1$, $n \geq 2$, $k \geq 1$, $\gamma_1, \ldots, \gamma_n > 0$ and $m_i = \gamma_i - \gamma_{i+1} - 1$.

- If $\gamma_1, \ldots, \gamma_n$ are pairwise different, then

$$F_{r:n}^{GOS}(t) = 1 - c_{r-1} \sum_{i=1}^{r} \frac{a_{i,r}}{\gamma_i} (1 - F(t))^{\gamma_i} = q_{r:n}^{GOS}(F(t)) \tag{1.2}$$

with the constants

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad a_{i,r} = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n$$

where the empty product $\prod_{\emptyset}$ is defined to be 1.

Ordering properties for GOS

- For the GOS we have:

$$X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}$$

Cramer, Kamps and Raqab (2003, Applicationes Mathematicae) and Hu and Zhuang (2005, Statist Probab Lett).

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Particular cases of GOS

- The GOS include:
- OS, IID case ($m_1 = \cdots = m_{n-1} = 0$ and $k = 1$).
- kRV, k-th record values ($m_1 = \cdots = m_{n-1} = -1$ and $k = 1, 2, \ldots$).
- RV, record values ($m_1 = \cdots = m_{n-1} = -1$ and $k = 1$).
- SOS, Sequential Order Statistics under the Proportional Hazard Rate (PHR) model, i.e., with $\overline{F}_r = \overline{F}^{\alpha_r}$ for $r = 1, \ldots, n$ ($\gamma_r = (n - r + 1)\alpha_r$ and $k = \alpha_n$).

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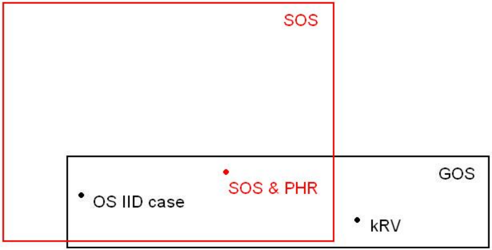
- SOS, Sequential Order Statistics under the Proportional Hazard Rate (PHR) model, i.e., with $\overline{F}_r = \overline{F}^{\alpha_r}$ for $r = 1, \ldots, n$ ($\gamma_r = (n - r + 1)\alpha_r$ and $k = \alpha_n$).

Sequential Order statistics (SOS)



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- $\overline{F}_1, \ldots, \overline{F}_n$.
- $Y_1^{(1)}, \ldots, Y_n^{(1)}$ IID $\sim \overline{F}_1$.
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- OS (IID case) are SOS when $\overline{F}_1 = \cdots = \overline{F}_n$.
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Ordering properties for SOS

- The SOS are not necessarily HR ordered; see Navarro and Burkschat (2011, Naval Res Log).
- For the SOS:

$$\overline{F}_1, \dots, \overline{F}_n \quad IHR \nRightarrow F^{SOS}_{r:n} \quad IHR$$

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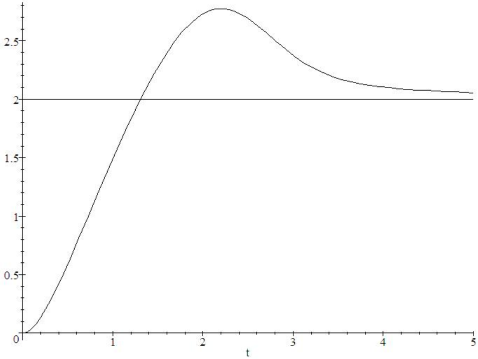

Figure: Hazard rate functions $h_{1:2}^{SOS}$ (constant line) and $h_{2:2}^{SOS}$ for the SOS obtained from $\overline{F}_1(t) = e^{-t}$ (exponential) and $\overline{F}_2(t) = e^{-t^2}$ (Weibull). The SOS are not HR ordered and $h_{2:2}^{*}$ is not monotone.

Ordering properties for SOS

- Conditions for the HR, MRL and LR ordering of SOS were given in Navarro and Burkschat (2011, Naval Res Log).
- For example:

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Let $X_{1:n}^{SOS}, \ldots, X_{n:n}^{SOS}$ be the SOS based on $\overline{F}_1, \ldots, \overline{F}_n$ having hazard rate function $h_1, \ldots, h_n$. If h_k/h_{k+1} is increasing for $k = 1, \ldots, i$, then $X_{i:n}^{SOS} \leq_{HR} X_{i+1:n}^{SOS}$.

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Let f_i be log-concave for $i = 1, 2, \ldots, r$ and $h_{j+1} - h_j$ be decreasing for $j = 1, 2, \ldots, r-1$. Then $X_{r:n}^{SOS}$ is ILR.

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- For more results, please go to PS9.

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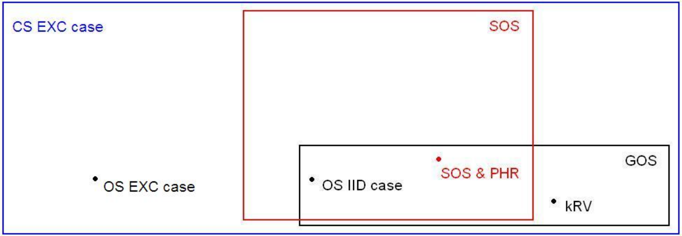
Coherent systems (CS)



Coherent systems- Exchangeable case

- Coherent systems $\phi = \phi(x_1, \ldots, x_n) \in \{0, 1\}$ where $x_i \in \{0, 1\}$, the structure function ϕ is nondecreasing and strictly increasing in x_i for at least one point $(x_1, \ldots, x_n)$, for $i = 1, \ldots, n$.
- If $X_1, \ldots, X_n$ are the component lifetimes, then there exist ϕ such that the system lifetime $T = \phi(X_1, \ldots, X_n)$.
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- Samaniego (IEEE TR, 1985), IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} p_i \overline{F}_{i:n}(t), \tag{2.1}$$

where $p_i = \Pr(T = X_{i:n})$.

- $\mathbf{p} = (p_1, \ldots, p_n)$ is the signature of the system.
- p_i does not depend on $\overline{F}$ and

$$p_i = \frac{\left|\{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)}\}\right|}{n!} \tag{2.2}$$

- Navarro and Rychlik (JMVA, 2007), (2.1) holds for EXC absolutely continuous joint distribution.
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Mixed systems

- A **mixed system** of order n is a stochastic mixture of coherent systems of order n (Boland and Samaniego, 2004).
- From (2.1), in the EXC case, all the mixed systems of order n can be written as mixtures of $X_{1:n}, \ldots, X_{n:n}$.
- The vector with the coefficients in that representation is called the signature of the mixed system.
- Conversely, any probability vector in the simplex $\{\mathbf{c} \in [0,1]^n : \sum_{i=1}^n c_i = 1\}$ determines a mixed system with reliability

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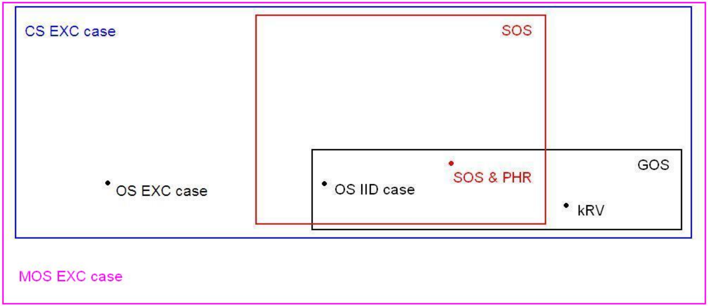
Mixtures of Order Statistics (MOS), EXC case



Generalized mixture representations

- Navarro, Ruiz and Sandoval (CSTM, 2007), if T has EXC components, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t). \qquad (2.3)$$

- $\mathbf{a} = (a_1, \ldots, a_n)$ is the minimal signature of T.
- a_i only depends on ϕ but can be negative and so (2.3) is a generalized mixture.
- In the IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}^i(t) = q_\phi(\overline{F}(t)), \qquad (2.4)$$

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Generalized mixture representations

- A **path set** of T is a set such that if all the components in P work, then the system works.
- A **minimal path set** of T is a path set which does not contains other path sets.
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\overline{F}_T(t) &= \Pr\left(\max_{j=1,\ldots,r} X_{P_j} > t\right) \\
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&= \sum_{i=1}^{r} \overline{F}_{P_i}(t) - \sum_{i \neq j} \overline{F}_{P_i \cup P_j}(t) + \cdots \pm \overline{F}_{P_1 \cup \ldots P_r}(t).
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- If K is the survival copula of $(X_1,\dots,X_n)$, then

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$$\overline{F}_P(t)=K(\mathbf{z}_P)$$

where $\mathbf{z}_P=(z_1,\dots,z_n)$, $z_i=\overline{F}_i(t)$ for $i\in P$ and $z_i=1$ for $i\notin P$.

- Therefore

$$\overline{F}_T(t)=Q_{\phi,K}(\overline{F}_1(t),\dots,\overline{F}_n(t)).$$

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Ordering results for systems-EXC case

Theorem (Navarro et al., NRL 2008)

If $T_1 = \phi_1(X_1, \ldots, X_n)$ *and* $T_2 = \phi_2(X_1, \ldots, X_n)$ *have signatures* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$, $(X_1, \ldots, X_n)$ *is EXC, then:*
(i) *If* $\mathbf{p} \leq_{ST} \mathbf{q}$, *then* $T_1 \leq_{ST} T_2$.
(ii) *If* $\mathbf{p} \leq_{HR} \mathbf{q}$ *and* $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ *holds, then* $T_1 \leq_{HR} T_2$.
(iii) *If* $\mathbf{p} \leq_{HR} \mathbf{q}$ *and* $X_{1:n} \leq_{MRL} \cdots \leq_{MRL} X_{n:n}$ *holds, then* $T_1 \leq_{MRL} T_2$.
(iv) *If* $\mathbf{p} \leq_{LR} \mathbf{q}$ *and* $X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}$ *holds, then* $T_1 \leq_{LR} T_2$.

Aging classes results for coherent systems

- If $X_1, \ldots, X_n$ are IID

$$X_1 \quad NBU \Rightarrow T \quad NBU,$$

but

$$X_1 \quad IHR \not\Rightarrow T \quad IHR$$

and

$$X_1 \quad ILR \not\Rightarrow T \quad ILR.$$

Distorted Distributions (DD)

- The distorted distributions are a way to model distortion risk measures developed from research on premium principles, see Wang (1996, ASTIN Bull).
- The *distorted distribution* associated to F and to an increasing right continuous *distortion function* $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \tag{3.1}$$

- Some authors assume that q is continuous and strictly increasing. Then F and F_q have the same support.
- For the reliability functions we have

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \tag{3.2}$$

where $\overline{q}(u) = 1 - q(1-u)$ is the *dual distortion function*; see Hürlimann (2004, N Am Actuarial J).
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Particular cases of Distorted Distributions (DD)

- The OS in the IID case are DD (1.1).
- The GOS (Records, k-Records, etc.) are DD (1.2).
- The SOS in the PHR case are DD.
- The CS (OS) in the ID case (includes the EXC case) are DD (2.5).
- The SOS in the general case are DD but q and F are quite complicate.
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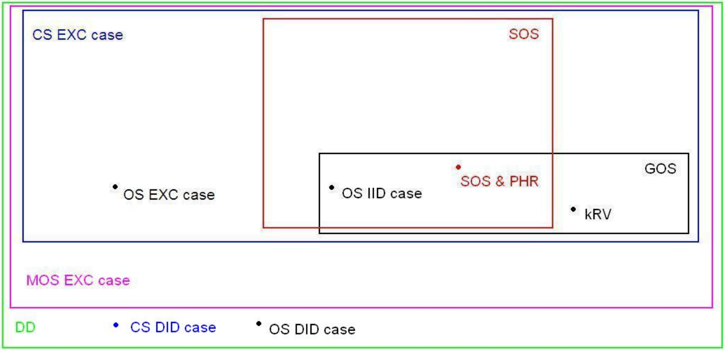

Ordering results for distorted distributions

- Conditions to get ordering results for DD were given in Navarro, del Aguila, Sordo and Suarez-Llorens (to appear in ASMBI, DOI: 10.1002/asmb.1917).
- For example:

Theorem

Let $F_1 = q_1(F)$ and let $F_2 = q_2(F)$. Then we have the following properties:
(i) $F_1 \leq_{ST} F_2$ ($\geq_{ST}$) for all F if and only if $q_1(u)/q_2(u) \geq 1$ ($\leq$) in $(0,1)$.
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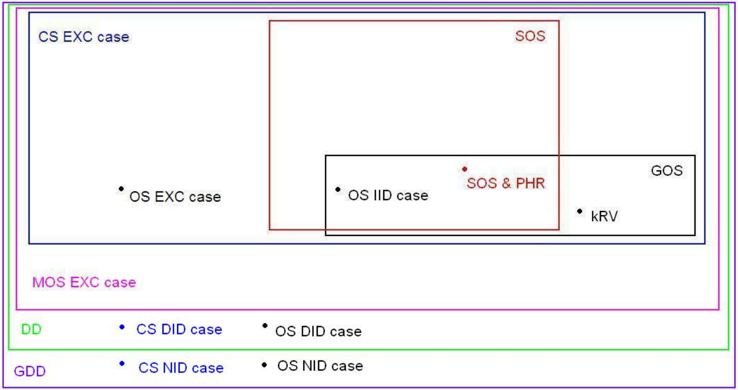

Example

- $X_{2:3}$ has the path sets $P_1 = \{1,2\}$, $P_2 = \{1,3\}$, and $P_3 = \{2,3\}$.
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- In the EXC case $q(u) = q_{2:3}^{EXC}(u) = 3K(u,u,1) - 2K(u,u,u)$.
- In the IID case $q(u) = q_{2:3}^{IID}(u) = 3u^2 - 2u^3$.

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Connectivity problems in Networks

- A **graph** (or network) is an ordered pair $G = (V, E)$ comprising a set V of nodes together with a set E of edges, which are 2-element subsets of V.
- A **directed graph** is an ordered pair $G = (V, E)$ comprising a set V of nodes together with a set E of edges, which are elements of $V \times V$.
- Let us assume that in a graph (directed graph) the nodes cannot fail but the edges can fail. Let $X_1, \ldots, X_n$ be the edges lifetimes.
- Suppose that we want to study a given connectivity problem (e.g. the connection of all the nodes). Let T_N be the lifetime of the network for this connectivity problem. Then

$$T_N = \phi(X_1, \ldots, X_n)$$

for a coherent system ϕ.

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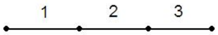
All nodes connection problem in a network

Network



Path sets:

?

Coherent system

All nodes connection problem in a network

Network

Path sets:

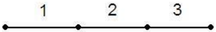
{1,2,3}

Coherent system

?

All nodes connection problem in a network

Network



Path sets:

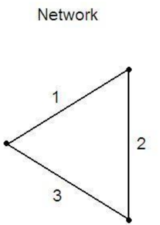
{1,2,3}

Coherent system

Series system

$X_{1:3}$

All nodes connection problem in a network



Path sets:

?

Coherent system

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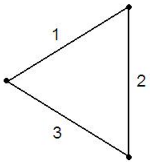
Network



Path sets:

{1,2}

{1,3}

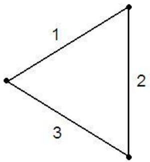
{2,3}

Coherent system

?

All nodes connection problem in a network

Network



Path sets:

{1,2}

{1,3}

{2,3}

Coherent system

2-out-of-3

$X_{2:3}$

All nodes connection problem in a network

Network

?

Path sets:

{1}

{2}

{3}

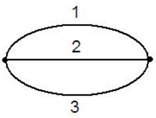
Coherent system

Parallel system

$X_{3:3}$

All nodes connection problem in a network

Network



Path sets:

{1}

{2}

{3}

Coherent system

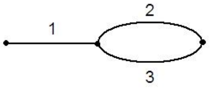
Parallel system

$X_{3:3}$

All nodes connection problem in a network

Network

Coherent system



Path sets:

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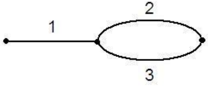
All nodes connection problem in a network

Network



Coherent system

Path sets:

{1,2}

{1,3}

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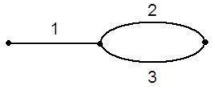
Network



Path sets:

{1,2}

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Coherent system

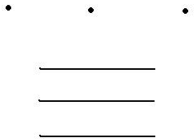
$\min(X_1, \max(X_2, X_3))$

All nodes connection problem in a network

What is the best way to connect three nodes with three edges?

Network

Coherent system



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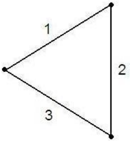

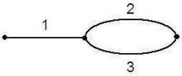

Coherent system

2-out-of-3

$X_{2:3} \geq \min(X_1, \max(X_2, X_3))$

References

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