Comparisons of residual lifetimes of coherent systems under dependence

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[1]Supported by Ministerio de Economía y Competitividad under Grant MTM2012-34023-FEDER.

Residual lifetimes

- $X_1, \ldots, X_n$ component lifetimes with RF

$$\overline{F}_i(t) = \Pr(X_i > t).$$

- $T = \phi(X_1, \ldots, X_n)$ system lifetime with RF

$$\overline{F}_T(t) = \Pr(T > t).$$

- We assume $\overline{F}_i(t) = \Pr(X_i > t) > 0$ and $\overline{F}_T(t) > 0$ for $t \geq 0$.
- Component residual lifetimes $X_{i,t} = (X_i - t | X_i > t)$ with RF:

$$\overline{F}_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x | X_i > t) = \frac{\overline{F}_i(t+x)}{\overline{F}_i(t)}.$$

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System residual lifetimes

- We have two main options to define the system residual lifetime at time $t > 0$:
- The usual **residual lifetime** $T_t = (T - t|T > t)$ with RF

$$\overline{F}_t(x) = \Pr(T - t > x|T > t) = \frac{\overline{F}_T(t + x)}{\overline{F}_T(t)}.$$

- The **residual lifetime at the system level** $T_t^* = (T - t|X_1 > t, \dots, X_n > t)$ with RF

$$\overline{F}_t^*(x) = \Pr(T_t^* > x) = \frac{\Pr(T > t + x, X_1 > t, \dots, X_n > t)}{\Pr(X_1 > t, \dots, X_n > t)}$$

when $\Pr(X_1 > t, \dots, X_n > t) > 0$.

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when $\Pr(X_1>t,\dots,X_n>t)>0$.

System residual lifetimes

- Which one is the best system?
- Intuitively, it seems that T_t^* should be always better than T_t.
- It should be better to know that all the components are working at time t!
- For $T = \min(X_1, \ldots, X_n)$, $T_t =_{ST} T_t^*$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$.

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System residual lifetimes

- If $X_1, \ldots, X_n$ are independent, then

$$T_t = (T - t | T > t) \leq_{ST} T_t^* = (T - t | X_1 > t, \ldots, X_n > t); \quad (1)$$

see Pellerey and Petakos (IEEE Tr Rel, 2002) and Li and Lu (PEIS, 2003).

- Conditions on $(X_1, \ldots, X_n)$ to have (1) were given in Li, Pellerey and You (2013).
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Outline

- In this talk, conditions to get (1) are given when the dependence structure is known.
- The conditions are based on distorted distribution representations.
- Some conditions are obtained to get (1) for other usual stochastic orders.
- These conditions can also be applied to the case of independent components.
- Some illustrative examples are given.
- They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.
- Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!

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Generalized distorted distribution

- The **generalized distorted distribution** (GDD) associated to n DF $F_1, \dots, F_n$ and to an increasing continuous **multivariate distortion function** $Q : [0,1]^n \to [0,1]$ such that $Q(0,\dots,0) = 0$ and $Q(1,\dots,1) = 1$, is

$$F_Q(t) = Q(F_1(t), \dots, F_n(t)). \tag{2}$$

- For the RF we have

$$\overline{F}_Q(t) = \overline{Q}(\overline{F}_1(t), \dots, \overline{F}_n(t)), \tag{3}$$

where $\overline{F} = 1 - F$, $\overline{F}_Q = 1 - F_Q$ and $\overline{Q}(u_1, \dots, u_n) = 1 - Q(1-u_1, \dots, 1-u_n)$ is the **multivariate dual distortion function**; see Navarro et al. (MCAP 2015).

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Coherent systems-GENERAL case

- A **path set** of T is a set $P \subseteq \{1,\dots,n\}$ such that if all the components in P work, then the system works.
- A **minimal path set** of T is a path set which does not contains other path sets.
- If $P_1,\dots,P_m$ are the minimal path sets of T, then $T = \max_{j=1,\dots,m} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

$$\overline{F}_T(t) = \Pr\left(\max_{j=1,\dots,m} X_{P_j} > t\right) = \Pr\left(\cup_{j=1}^m \{X_{P_j} > t\}\right)$$
$$= \sum_{i=1}^m \overline{F}_{P_i}(t) - \sum_{i \neq j} \overline{F}_{P_i \cup P_j}(t) + \cdots \pm \overline{F}_{P_1 \cup \cdots \cup P_m}(t)$$

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Coherent system representation

- The copula representation for the RF of $(X_1, \ldots, X_n)$ is

$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)),$

where $\overline{F}_i(t) = \Pr(X_i > t)$ and K is the survival copula. Hence

$\overline{F}_{1:k}(t) = \Pr(X_1 > t, \ldots, X_k > t) = K(\overline{F}_1(t), \ldots, \overline{F}_r(t), 1, \ldots, 1).$

- Then the system reliability can be written as

$$\overline{F}_T(t) = \overline{Q}_{\phi,K}(\overline{F}_1(t), \ldots, \overline{F}_n(t)).$$

- If the components are ID, then $\overline{F}_T(t) = \overline{q}_{\phi,K}(\overline{F}(t))$.
- If the components are IND, then $\overline{Q}_{\phi,K}$ is a multinomial.
- If the components are IID, then $\overline{q}_{\phi,K}(u) = \sum_{i=1}^{n} a_i u^i$, where $(a_1, \ldots, a_n)$ is the minimal signature.

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Representations for the system residual lifetimes

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- Then

$$\overline{F}_t(x) = \frac{\overline{Q}(\overline{F}_1(t)\overline{F}_{1,t}(x), \dots, \overline{F}_n(t)\overline{F}_{n,t}(x))}{\overline{Q}(\overline{F}_1(t), \dots, \overline{F}_n(t))},$$

where $\overline{F}_{i,t}(x) = \overline{F}_i(t+x)/\overline{F}_i(t)$.

- Therefore

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$$\overline{F}^*_t(x) = \frac{\Pr(T > t + x, X_1 > t, \dots, X_n > t)}{\Pr(X_1 > t, \dots, X_n > t)}.$$

- As $T = \max_{j=1,\dots,m} X_{P_j}$ for the minimal path sets $P_1, \dots, P_m$, then

$$\overline{F}^*_t(x) = \frac{\Pr(\max_{j=1,\dots,m} X_{P_j} > t + x, X_1 > t, \dots, X_n > t)}{K(\overline{F}_1(t), \dots, \overline{F}_n(t))}.$$

- Therefore

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$$\overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).$$

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Comparison results-DD

- If q_1 and q_2 are two DF,

$$q_1(F) \leq_{ord} q_2(F) \text{ for all } F?$$

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$$F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)?$$

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Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
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Example 1: Parallel system with two ID components

- $T = \max(X_1, X_2)$ where X_1 and X_2 have DF F.
- Then $\overline{F}_T(t) = \overline{q}(\overline{F}(t))$ where

$$\overline{q}(u) = \overline{Q}(u, u) = 2u - K(u, u).$$

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$$\Psi(u) = [c - K(c,c)][K(cu,c) - K(cu,cu)] + v[K(cu,c) - uK(c,c)] \geq 0. \qquad (5)$$

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- If K is the Clayton copula

$$K(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta > 0,$$

then

$$\psi(u) = \left(u^{-\theta}c^{-\theta} + c^{-\theta} - 1\right)^{-1/\theta} - \left(u^{-\theta}c^{-\theta} + u^{-\theta}[c^{-\theta} - 1]\right)^{-1/\theta}.$$

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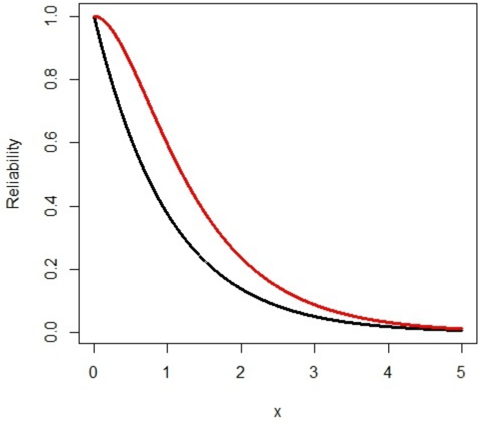

Figure: Reliability functions of T_t (black) and T_t^* (red) when $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 2$.

Example 1: Gumbel-Barnett Archimedean copula

- If K is the Gumbel-Barnett Archimedean copula

$$K(u,v) = uv \exp\left[-\theta(\ln u)(\ln v)\right], \quad \theta \in (0,1], \qquad (6)$$

  then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0,1]$ when $\theta = 1$.
- Therefore T_t and T_t^* are not ST ordered (for all F and t).
- These conditions lead to a Gumbel bivariate exponential with a negative correlation.
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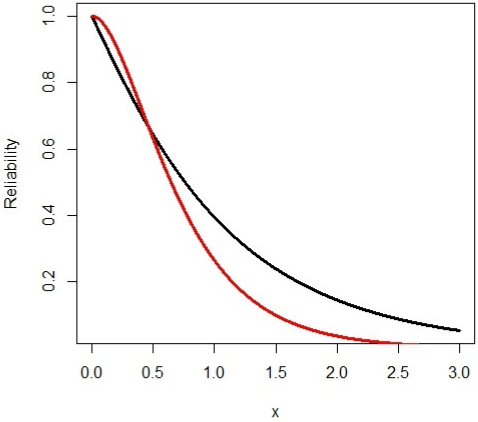

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- By plotting the ratio $g(u) = \overline{q}_t(u)/\overline{q}_t^*(u)$ for $t = 1$ we see that it is first decreasing in $(0, u_0)$ and then increasing in $(u_0, 1]$ for a $u_0 \in (0, 1)$.

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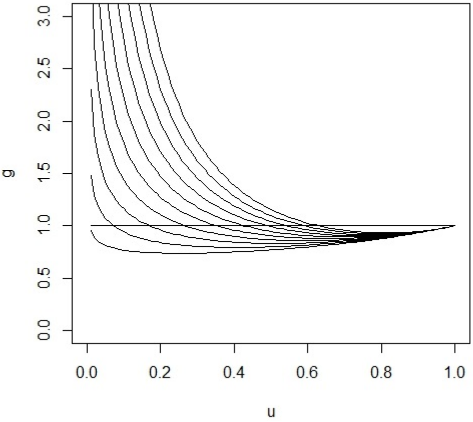

Figure: Ratio $g(u) = \overline{q}_t(u)/\overline{q}^*_t(u)$ for $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 0.1, 0.2, \ldots, 1$ (from the bottom to the top).

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- For example, if $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 1$, then

$$E(T_t) = 1.05615 > E(T_t^*) = 0.77366.$$

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Example 2: Parallel system with two INID components

- $T = \max(X_1, X_2)$, X_1, X_2 IND with DF F_1 and F_2.
- Then

$$\overline{Q}(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \overline{Q}^*_t(u_1, u_2),$$

$$\overline{Q}_t(u_1, u_2) = \frac{c_1 u_1 + c_2 u_2 - c_1 c_2 u_1 u_2}{c_1 + c_2 - c_1 c_2},$$

where $c_1 = \overline{F}_1(t)$ and $c_2 = \overline{F}_2(t)$.

- $T_t \leq_{HR} T^*_t$ holds for all F_1, F_2 if and only if

$$\frac{\overline{Q}(u,v)}{\overline{Q}_t(u,v)} = \frac{(u + v - uv)(c_1 + c_2 - c_1 c_2)}{c_1 u + c_2 - c_1 c_2 uv}$$

is decreasing in u and v in the set $[0,1]^2$.

- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS,2003) is not correct.

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$$\overline{Q}(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \overline{Q}^*_t(u_1, u_2),$$

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where $c_1 = \overline{F}_1(t)$ and $c_2 = \overline{F}_2(t)$.

- $T_t \leq_{HR} T^*_t$ holds for all F_1, F_2 if and only if

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is decreasing in u and v in the set $[0, 1]^2$.

- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS,2003) is not correct.

Example 2: Parallel system with two INID components

- $T = \max(X_1, X_2)$, X_1, X_2 IND with DF F_1 and F_2.
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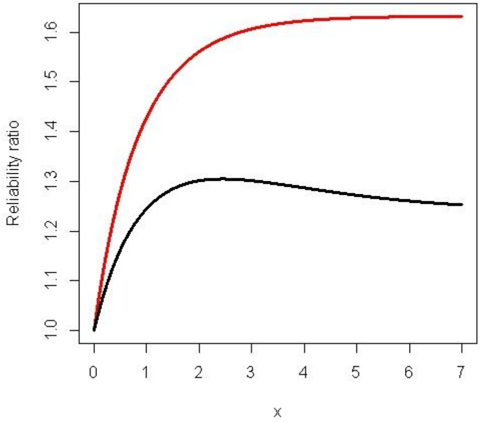

Figure: Ratio $\overline{F}_t^*/\overline{F}_t$ for $t = 1$, $\overline{F}_1(x) = e^{-x}$ and $\overline{F}_2(x) = e^{-x/2}$ (black) or $\overline{F}_2(x) = e^{-x}$ (red).

Example 2: Parallel system with two INID components

- If X_1, X_2 are IID with DF F, then $T_t \leq_{HR} T_t^*$ holds for all F since

$$\frac{\overline{q}(u)}{\overline{q}_t(u)} = \frac{2-u}{2-u\overline{F}(t)}\left(2-\overline{F}(t)\right)$$

is decreasing in u in the set $[0,1]$.

- Even more, $T_t \leq_{LR} T_t^*$ holds for all F since

$$\frac{\overline{q}'(u)}{\overline{q}'_t(u)} = \frac{1-u}{1-u\overline{F}(t)}\left(2-\overline{F}(t)\right)$$

is a decreasing function in $[0,1]$ for all $t > 0$.

J. Navarro, E-mail: jorgenav@um.es

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Example 3: Coherent system with DID components

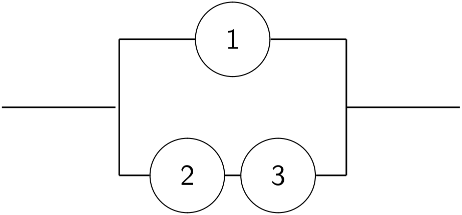

Figure: System in Example 3.

Example 3: Coherent system with DID components

- $T = \max(X_1, \min(X_2, X_3))$, X_1, X_2, X_3 DID with DF F.
- Then $P_1 = \{1\}$, $P_2 = \{2, 3\}$ and

$$\overline{q}(u) = u + K(1, u, u) - K(u, u, u).$$

- Therefore $\overline{q}_t(u) = \overline{q}(cu)/\overline{q}(c)$ and

$$\overline{q}_t^*(u) = \frac{K(cu, c, c) + K(c, cu, cu) - K(cu, cu, cu)}{K(c, c, c)},$$

  where $c = \overline{F}(t)$.
- We assume a Farlie-Gumbel-Morgenstern (FGM) copula

$$K(u, v, w) = uvw(1 + \theta(1 - u)(1 - v)(1 - w)), \quad \theta \in [-1, 1].$$

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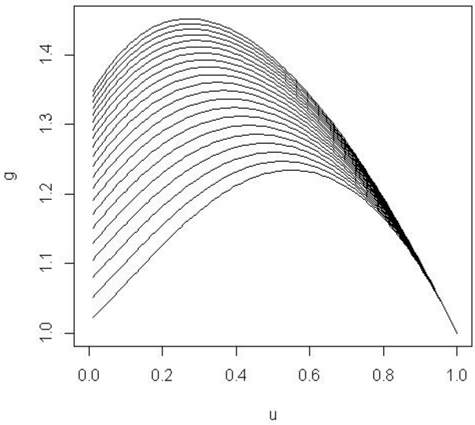

Figure: Ratio $g(u) = \overline{q}_t^*(u)/\overline{q}_t(u)$ for $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = -1, -0.9, \ldots, 1$ (from the bottom to the top).

Example 3: Coherent system with DID components

- As $g(u) = \overline{q}_t^*(u)/\overline{q}_t(u) \geq 1$, then $T_t \leq_{ST} T_t^*$.
- As $g(u) = \overline{q}_t^*(u)/\overline{q}_t(u) \geq 1$ is not monotone, then T_t and T_t^* are not HR ordered.

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Further results

- For more results see Navarro and Durante (2015).
- Case 3: $T^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t} = (T - t | H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t})$ where the (past) history of the system can be represented as

$$H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t} = \{X_{i_1} = t_1, \dots, X_{i_r} = t_r, X_j > t \text{ for } j \notin \{i_1, \dots, i_r\}\},$$

where $0 < r < n$, $0 < t_1 < \dots < t_r < t$, $\Pr(H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}) > 0$ and the event $H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}$ implies $T > t$.
- This case can also be represented as

$$\Pr(T - t > x | H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}) = \overline{Q}^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}(\overline{F}_{1,t}(x), \dots, \overline{F}_{n,t}(x)).$$

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The references of this talk

- Navarro (2015). Comparisons of the residual lifetimes of coherent systems under different assumptions. Submitted.
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Recent references on coherent systems

- Navarro and Spizzichino (2010). Comparisons of series and parallel systems with components sharing the same copula. Appl Stoch Mod Bus Ind 26, 775–791.
- Navarro and Gomis (2015). Comparisons in the mean residual life order of coherent systems with identically distributed components. To appear in Applied Stochastic Models in Business and Industry. DOI: 10.1002/asmb.2121.
- Navarro, Pellerey and Di Crescenzo (2015). Orderings of coherent systems with randomized dependent components. European Journal of Operational Research 240, 127–139.
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Recent references on Distorted distributions

- Navarro, del Aguila, Sordo and Suarez-Llorens (2013). Stochastic ordering properties for systems with dependent identically distributed components. Appl Stoch Mod Bus Ind 29, 264–278.
- Navarro, del Aguila, Sordo, Suarez-Llorens (2015). Preservation of stochastic orders under the formation of generalized distorted distributions. Applications to coherent systems. To appear in Methodology and Computing in Applied Probability. DOI: 10.1007/s11009-015-9441-z.

References

- For the more references, please visit my personal web page:

  https : //webs.um.es/jorgenav/

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