Representaciones basadas en cópulas para distribuciones condicionadas y sistemas

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- For the reliability functions (RF) $\bar{F} = 1 - F$, $\bar{F}_q = 1 - F_q$, we have

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \tag{1.2}$$

where $\bar{q}(u) = 1 - q(1-u)$ is the **dual distortion function**; see Hürlimann (N Am Actuarial J, 2004).

Multivariate distortion functions

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where $\bar{F} = 1 - F$, $\bar{F}_Q = 1 - F_Q$ and $\bar{Q}(u_1, \ldots, u_n) = 1 - Q(1 - u_1, \ldots, 1 - u_n)$ is the **multivariate dual distortion function**; see Navarro et al. (ASMBI, 2014).

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- Q and $\bar{Q}$ are continuous aggregation functions.

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- Coherent systems $T = \phi(X_1, \ldots, X_n)$.

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- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all t.
- Then

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

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- Navarro et al. ASMBI, 2013 and Navarro and Gomis ASMBI, 2016.

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holds if and only if the function

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is decreasing in $[0,1]^n$.

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- Navarro and del Águila (Metrika, 2017).

Bounds for distorted distributions

- If T has the RF $\bar{q}(\bar{F}(t))$, then:

$$\bar{q}(\bar{F}(t)) = \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)}\bar{F}(t) \leq \bar{F}(t) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

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- These bounds are sharp.

Bounds for distorted distributions

- Moreover, if $T \geq 0$, then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = \int_0^\infty \frac{\bar{q}(\bar{F}(t))}{\bar{F}(t)} \bar{F}(t)dt$$
$$\leq \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u} \int_0^\infty \bar{F}(t)dt = E(X) \sup_{u \in (0,1]} \frac{\bar{q}(u)}{u},$$

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- Moreover, if $T \geq 0$, then

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$$\leq \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u} \int_0^\infty \bar{F}(t)dt = E(X) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u},$$

where X has the RF $\bar{F}$.

- Analogously

$$E(T) \geq E(X) \inf_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

Bounds for distorted distributions

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- Analogously

$$E(T) \geq E(X) \inf_{u\in(0,1]} \frac{\bar{q}(u)}{u}.$$

- These bounds are sharp.

Bounds for distorted distributions

Proposition

If T has the RF $\bar{q}(\bar{F}(t))$, then:

$$\inf_{u\in(0,1)} \frac{\bar{q}(u)-u}{2u(1-u)} \leq \frac{E(T)-E(X)}{\Delta_F} \leq \sup_{u\in(0,1)} \frac{\bar{q}(u)-u}{2u(1-u)} \tag{1.7}$$

when $0=\inf\{x : F(x)>0\}$ *and the Gini mean difference dispersion measure*

$$\Delta_F = 2\int_0^\infty F(x)(1-F(x))dx$$

is positive. The bounds are sharp.

Copula represenattion

- Let (X, Y) be a bivariate random vector. Then

$$\boldsymbol{F}(x, y) = \Pr(X \leq x, Y \leq y) = C(F(x), G(y)),$$

where $F(x) = \Pr(X \leq x)$ and $G(y) = \Pr(Y \leq y)$ are the marginal DF and C is a copula (i.e., C a continuous distribution function with uniform marginals over $(0, 1)$).

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- The joint reliability function can be represented as

$$\bar{\boldsymbol{F}}(x,y)=\Pr(X>x,Y>y)=\hat{C}(\bar{F}(x),\bar{G}(y)),$$

where $\bar{F}=1-F$ and $\bar{G}=1-G$ and $\hat{C}$ is also a copula, called *survival copula*.

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- $\hat{C}$ is determined by C (and vice versa) by

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

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- Case I: $(Y|X \leq x)$ when $F(x) = \Pr(X \leq x) > 0$.
- Case II: $(Y|X > x)$ when $\bar{F}(x) = \Pr(X > x) > 0$.
- Case III: $(Y|X = x)$ when $f(x) = F'(x) > 0$.

Conditional distributions. Case I.

- For $(Y|X \leq x)$ we have:

$$\Pr(Y \leq y|X \leq x) = \frac{\Pr(X \leq x, Y \leq y)}{\Pr(X \leq x)} = \frac{C(F(x), G(y))}{F(x)} = q_1(G(y)),$$

where the distortion function is given by

$$q_1(u) = \frac{C(F(x), u)}{F(x)}.$$

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$$q_1(u) = \frac{C(F(x), u)}{F(x)}.$$

- Analogously, its reliability can be written as

$$\Pr(Y > y|X \leq x) = \frac{F(x) - C(F(x), G(y))}{F(x)} = \bar{q}_1(\bar{G}(y)),$$

where the dual distortion function is given by

$$\bar{q}_1(u) = 1 - q_1(1-u) = \frac{F(x) - C(F(x), 1-u)}{F(x)} = \frac{u - \hat{C}(\bar{F}(x), u)}{F(x)}.$$

Conditional distributions. Case II.

- Analogously, for $(Y|X > x)$ we have:

$$\Pr(Y > y|X > x) = \frac{\Pr(X > x, Y > y)}{\Pr(X > x)} = \frac{\hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = \bar{q}_2(\bar{G}(y)),$$

where

$$\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}.$$

Conditional distributions. Case II.

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where

$$\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}.$$

- Analogously, its distribution function can be written as

$$\Pr(Y \le y|X > x) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = q_2(G(y)),$$

where

$$q_2(u) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), 1-u)}{\bar{F}(x)} = \frac{u - C(F(x), u)}{\bar{F}(x)}.$$

Conditional distributions. Case III.

- Let us consider now $(Y|X = x)$ when $f(x) > 0$ and $\boldsymbol{F}$ is absolutely continuous. Then the joint density is

$$\boldsymbol{f}(x, y) = f(x)g(y)\partial_2\partial_1 C(F(x), G(y)).$$

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- Hence $f_{Y|X=x}(y|x) = \boldsymbol{f}(x, y)/f(x)$ and

$$\begin{aligned} \Pr(Y \leq y|X = x) &= \int_{-\infty}^{y} \frac{\boldsymbol{f}(x, z)}{f(x)} dz \\ &= \int_{-\infty}^{y} g(z)\partial_2\partial_1 C(F(x), G(z)) dz \\ &= \partial_1 C(F(x), G(y)) \\ &= q_3(G(y)), \end{aligned} \tag{2.1}$$

where $q_3(u) = \partial_1 C(F(x), u)$.

Conditional distributions. Case III.

- Analogously, its reliability function can be written as

$$\Pr(Y > y|X = x) = \bar{q}_3(\bar{G}(y)), \tag{2.2}$$

where

$$\bar{q}_3(u) = 1 - q_3(1-u) = 1 - \partial_1 C(F(x), 1-u) = \partial_1 \hat{C}(\bar{F}(x), u).$$

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- $(Y|X = x, Y \leq y)$.
- The same holds for $(X_1, \ldots, X_n)$ when $n > 2$.

Comparisons between Y and $(Y|X \le x)$

Proposition

(i) $Y \ge_{ST} (Y|X \le x)$ $(\le_{ST})$ *for all* F, G *if and only if* $C(F(x), u) \ge uF(x)$ $(\le)$ *for all* $u \in (0,1)$.

(ii) $Y \ge_{HR} (Y|X \le x)$ $(\le_{HR})$ *for all* F, G *if and only if* $\hat{C}(\bar{F}(x), u)/u$ *is decreasing (increasing) in* u *in* $(0,1)$.

(iii) $Y \ge_{RHR} (Y|X \le x)$ $(\le_{RHR})$ *for all* F, G *if and only if* $C(F(x), u)/u$ *is decreasing (increasing) in* u *in* $(0,1)$.

(iv) $Y \ge_{LR} (Y|X \le x)$ $(\le_{LR})$ *for all* F, G *if and only if* $C(F(x), u)$ *is concave (convex) in* u *in the interval* $(0,1)$.

(v) $Y \ge_{MRL} (Y|X \le x)$ $(\le_{MRL})$ *for all* F, G *if* $\hat{C}(\bar{F}(x), u)/u$ *is bathtub (upside-down bathtub) in* u *in* $(0,1)$ *and* $E(Y) \ge E(Y|X \le x)$ $(\le)$.

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- The following conditions are equivalent:
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- Note that

$$\Pr(Y \leq y) = F(x)\Pr(Y \leq y|X \leq x) + \bar{F}(x)\Pr(Y \leq y|X > x).$$

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Proposition

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Comparisons between $(Y|X \leq x_1)$ and $(Y|X \leq x_2)$

- $(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)$ holds for all F, G if, and only if,

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- Hence, (X, Y) is LTD$(Y|X)$ (LTI$(Y|X)$) for all F, G if and only if $C(u, v)/u$ is decreasing (increasing) in u for all v (see Nelsen p. 192, Th. 5.2.5).

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(iii) $(Y|X \leq x)$ is RHR-increasing (decreasing) in x for all F, G if and only if $C(u_2, v)/C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iv) $(Y|X \leq x)$ is LR-increasing (decreasing) in x for all F, G if and only if $\partial_2 C(u_2, v)/\partial_2 C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

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Comparisons between $(Y|X > x_1)$ and $(Y|X > x_2)$

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(v) $(Y|X>x)$ is MRL-increasing (decreasing) in x for all F, G if and only if $\hat{C}(u_2,v)/\hat{C}(u_1,v)$ is upside-down bathtub (bathtub) in v for all $0<u_1\le u_2<1$ and $E(Y|X>x)$ is increasing (decreasing) in x.

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- $(Y|X=x_1) \leq_{ST} (Y|X=x_2)$ holds for all F, G if, and only if,

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- It is a positive (negative) dependence property.
- SI$(Y|X)$ (SD$(Y|X)$) implies LTD$(Y|X)$ (LTI$(Y|X)$) and RTI$(Y|X)$ (RTD$(Y|X)$) and so PQD (NQD).

(i) $(Y|X=x)$ is ST-increasing (decreasing) in x for all F, G if and only if $C(u,v)$ (or $\hat{C}(u,v)$) is concave (convex) in u for all v.

(ii) $(Y|X=x)$ is HR-increasing (decreasing) in x for all F, G if and only if $\partial_1\hat{C}(u_2,v)/\partial_1\hat{C}(u_1,v)$ is increasing (decreasing) in v for all $0<u_1\le u_2<1$.

(iii) $(Y|X=x)$ is RHR-increasing (decreasing) in x for all F, G if and only if $\partial_1 C(u_2,v)/\partial_1 C(u_1,v)$ is increasing (decreasing) in v for all $0<u_1\le u_2<1$.

(iv) $(Y|X=x)$ is LR-increasing (decreasing) in x for all F, G if and only if $\partial_2\partial_1 C(u_2,v)/\partial_2\partial_1 C(u_1,v)$ is increasing (decreasing) in v for all $0<u_1\le u_2<1$.

(v) $(Y|X=x)$ is MRL-increasing (decreasing) in x for all F, G if and only if $\partial_1\hat{C}(u_2,v)/\partial_1\hat{C}(u_1,v)$ is upside-down bathtub (bathtub) in v for all $0<u_1\le u_2<1$ and $E(Y|X=x)$ is increasing (decreasing) in x.

Dependence

- The following conditions are equivalent:
  - (i) (X, Y) is $LTD(X|Y)$.
  - (ii) $(X|Y \leq y)$ is ST-increasing.
  - (iii) $Y \geq_{RHR} (Y|X \leq x)$ for all x.
  - (iv) $Y \leq_{RHR} (Y|X > x)$ for all x.
  - (v) $(Y|X \leq x) \leq_{RHR} (Y|X > x)$ for all x.
  - (vi) $C(u, v)/v$ is decreasing in v for all u.

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- Analogous equivalences can be stated for the respective negative notions.

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  (i) (X, Y) is $RTI(X|Y)$.
  (ii) $(X|Y > y)$ is ST-increasing.
  (iii) $Y \geq_{HR} (Y|X \leq x)$ for all x.
  (iv) $Y \leq_{HR} (Y|X > x)$ for all x.
  (v) $(Y|X \leq x) \leq_{HR} (Y|X > x)$ for all x.
  (vi) $\hat{C}(u, v)/v$ is decreasing in v for all u.

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  (iii) $Y \leq_{LR} (Y|X > x)$ for all x.
  (iv) $(Y|X \leq x) \leq_{LR} (Y|X > x)$ for all x.
  (v) $C(u, v)$ is concave in v for all u.
  (vi) $\hat{C}(u, v)$ is concave in v for all u.

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  (v) $C(u, v)$ is concave in v for all u.
  (vi) $\hat{C}(u, v)$ is concave in v for all u.
- Analogous results can be stated for $(Y|X)$ and for the respective negative notions.

Dependence

- (X, Y) is *Left Corner Set Decreasing* $(LCSD)$ if $\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2.

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- (X,Y) is *Left Corner Set Decreasing* ($LCSD$) if $\Pr(X\le x_1, Y\le y_1|X\le x_2, Y\le y_2)$ is decreasing in x_2 and y_2.
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- The following conditions are equivalent:
  - (i) $LCSD$.
  - (ii) $(Y|X\le x)$ is RHR-increasing.
  - (iii) C is TP_2.
- The following conditions are equivalent:
  - (i) $RCSI$.
  - (ii) $(Y|X> x)$ is HR-increasing.
  - (iii) $\hat{C}$ is TP_2.

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- (X,Y) is *Left Corner Set Decreasing* (*LCSD*) if $\Pr(X \le x_1, Y \le y_1 | X \le x_2, Y \le y_2)$ is decreasing in x_2 and y_2.
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- The following conditions are equivalent:
  - (i) *LCSD*.
  - (ii) $(Y|X \le x)$ is RHR-increasing.
  - (iii) C is TP_2.
- The following conditions are equivalent:
  - (i) *RCSI*.
  - (ii) $(Y|X > x)$ is HR-increasing.
  - (iii) $\hat{C}$ is TP_2.
- Similar results can be obtained for *Left Corner Set Increasing* (*LCSI*) and *Right Corner Set Decreasing* (*RCSD*).

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  (i) $SI_{HR}(X|Y)$.
  (ii) $(Y|X>x)$ is LR-increasing.
  (iii) $\partial_2 \hat{C}$ is TP_2.

Dependence relationships

$$\begin{array}{ccccc}
SI_{LR}(X|Y) & \Rightarrow & SI_{RHR}(X|Y) & \Rightarrow & LCSD \\
\Downarrow & & \Downarrow & & \Downarrow \\
SI_{HR}(X|Y) & \Rightarrow & SI_{ST}(X|Y) & \Rightarrow & LTD(X|Y) \\
\Downarrow & & \Downarrow & & \Downarrow \\
RCSI & \Rightarrow & RTI(X|Y) & \Rightarrow & PQD
\end{array}$$

Bounds for $(Y|X \leq x)$

Proposition

If (X, Y) has the copulas C and $\hat{C}$ and $Y \geq 0$, then:

$$\inf_{u \in (0,1]} \frac{C(F(x), u)}{uF(x)} \leq \frac{\Pr(Y \leq y | X \leq x)}{G(y)} \leq \sup_{u \in (0,1]} \frac{C(F(x), u)}{uF(x)}$$

$$\inf_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \leq \frac{\Pr(Y > y | X \leq x)}{\bar{G}(y)} \leq \sup_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)}$$

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$$\inf_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \leq \frac{E(Y | X \leq x)}{E(Y)} \leq \sup_{u \in (0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)}.$$

Bounds for $(Y|X > x)$

Proposition

If (X, Y) *has the copulas* C *and* $\hat{C}$ *and* $Y \geq 0$*, then:*

$$\inf_{u\in(0,1]} \frac{u - C(F(x), u)}{u\bar{F}(x)} \leq \frac{\Pr(Y \leq y|X > x)}{G(y)} \leq \sup_{u\in(0,1]} \frac{u - C(F(x), u)}{u\bar{F}(x)},$$

$$\inf_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u\bar{F}(x)} \leq \frac{\Pr(Y > y|X > x)}{\bar{G}(y)} \leq \sup_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u\bar{F}(x)}$$

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$$\inf_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u\bar{F}(x)} \leq \frac{E(Y|X > x)}{E(Y)} \leq \sup_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u\bar{F}(x)}.$$

Bounds for $(Y|X = x)$

Proposition

If (X, Y) has the copulas C and $\hat{C}$ and $Y \geq 0$, then:

$$\inf_{u\in(0,1]} \frac{\partial_1 C(F(x), u)}{u} \leq \frac{\Pr(Y \leq y|X = x)}{G(y)} \leq \sup_{u\in(0,1]} \frac{\partial_1 C(F(x), u)}{u},$$

$$\inf_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u} \leq \frac{\Pr(Y > y|X = x)}{\bar{G}(y)} \leq \sup_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u}$$

and

$$\inf_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u} \leq \frac{E(Y|X = x)}{E(Y)} \leq \sup_{u\in(0,1]} \frac{\partial_1 \hat{C}(\bar{F}(x), u)}{u}.$$

Bounds based on Δ_G

Proposition

If (X,Y) has the copulas C and $\hat{C}$, $0=\inf\{y : G(y)>0\}$, $p=\bar{F}(x)$ and $\Delta_G := 2\int_0^\infty G(y)(1-G(y))dy > 0$, then:

$$\inf_{u\in(0,1)} \frac{up-\hat{C}(p,u)}{2u(1-u)(1-p)} \le \frac{E(Y|X\le x)-E(Y)}{\Delta_G} \le \sup_{u\in(0,1)} \frac{up-\hat{C}(p,u)}{2u(1-u)(1-p)},$$

$$\inf_{u\in(0,1)} \frac{\hat{C}(p,u)-up}{2u(1-u)p} \le \frac{E(Y|X>x)-E(Y)}{\Delta_G} \le \sup_{u\in(0,1)} \frac{\hat{C}(p,u)-up}{2u(1-u)p}$$

and

$$\inf_{u\in(0,1)} \frac{\partial_1\hat{C}(p,u)-u}{2u(1-u)} \le \frac{E(Y|X=x)-E(Y)}{\Delta_G} \le \sup_{u\in(0,1)} \frac{\partial_1\hat{C}(p,u)-u}{2u(1-u)}.$$

Example 1

- If (X, Y) has the Clayton-Oakes copula:

$$C(u,v) = \frac{uv}{u+v-uv}$$

for $0 \leq u, v \leq 1$, then

$$q_1(u) = \frac{C(F(x),u)}{F(x)} = \frac{u}{F(x)+u-uF(x)}.$$

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- $\rho_S = -39 + 4\pi^2 \cong 0.478417$ and $\tau_K = 1/3$.

Example 1

- We can obtain the following bounds:

$$G(y) \leq \Pr(Y \leq y|X \leq x) \leq \frac{1}{F(x)}G(y)$$

and

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- Then $E(Y|X \leq x) \leq E(Y)$ when $Y \geq 0$.
- Analogously, for $f(x) > 0$, we have

$$0 \leq \Pr(Y \leq y | X = x) \leq G(y)$$

when $F(x) \geq 1/2$ and

$$0 \leq \Pr(Y \leq y | X = x) \leq \frac{1}{4F(x)\bar{F}(x)} G(y)$$

when $F(x) < 1/2$.

Coherent systems- General case

- **System** $\phi : \{0,1\}^n \to \{0,1\}$, $x_i = 0,1$ state of the ith components, $\phi(x_1,\ldots,x_n)$ state of the system.

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- A path set P is a **minimal path set** if it does not contain other path sets.

Coherent systems- General case

- The system lifetime can be written as

$$T = \phi(X_1, \dots, X_n) = \max_{i=1,\dots,r} \min_{j \in P_i} X_j,$$

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- Then, by using the inclusion-exclusion formula

$$\begin{aligned}\bar{F}_T(t) &= \Pr(T > t) = \Pr(\max_{i=1,\dots,r} \min_{j \in P_i} X_j > t) \\ &= \Pr(\cup_{i=1}^{r} \{\min_{j \in P_i} X_j > t\}) \\ &= \sum_{i=1}^{r} \Pr(\min_{j \in P_i} X_j > t) - \sum_{i<k} \Pr(\min_{j \in P_i \cap P_k} X_j > t) + \dots \\ &\quad + (-1)^{r+1} \Pr(\min_{j \in P_1 \cap \dots \cap P_r} X_j > t).\end{aligned}$$

Coherent systems- General case

- If $\Pr(X_1 > x_1, \dots, X_n > x_n) = \boldsymbol{K}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n))$, where $\boldsymbol{K}$ is the survival copula, then:

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- For $X_{1:i} = \min(X_1, \ldots, X_i)$, we have

$$\Pr(X_{1:i} > t) = \Pr(X_1 > t, \ldots, X_i > t, X_{i+1} > -\infty, \ldots, X_n > -\infty)$$
$$= \boldsymbol{K}(\bar{F}_1(t), \ldots, \bar{F}_i(t), 1, \ldots, 1).$$

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- For $X_P = \min_{j \in P} X_j$ and $\bar{F}_P(t) = \Pr(X_P > t)$, we have

$$\bar{F}_P(t) = K_P(\bar{F}_1(t), \dots, \bar{F}_n(t))$$

  where $K_P(u_1, \dots, u_n) = \boldsymbol{K}(u_1^P, \dots, u_n^P)$ and $u_j^P = u_j$ if $j \in P$ and $u_j^P = 1$ if $j \notin P$.
- Therefore, from the inclusion-exclusion representation

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t))$$

  where $\bar{Q}$ is a multivariate dual distortion function.

Coherent systems, particular cases

- If $\bar{F}_1 = \cdots = \bar{F}_n = \bar{F}$, then

$$\bar{F}_T(t) = \bar{Q}(\bar{F}(t), \ldots, \bar{F}(t)) = \bar{q}(\bar{F}(t)),$$

where $\bar{q}(u) = \bar{Q}(u, \ldots, u)$ is a distortion function.

Coherent systems, particular cases

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- If $X_1, \ldots, X_n$ are independent, then $\bar{Q}$ is a polynomial called *structure reliability function* in Barlow and Proschan (1975).
- In particular, in the IID case, $\bar{F}_T = a_1 \bar{F}_{1:1} + \cdots + a_n \bar{F}_{1:n}$ and

$$\bar{q}(u) = a_1 u + \cdots + a_n u^n$$

  where $\boldsymbol{a} = (a_1, \ldots, a_n)$ is the minimal signature of the system (see, e.g. Navarro et al., ASMBI 2013).


ORBIS

Example 2

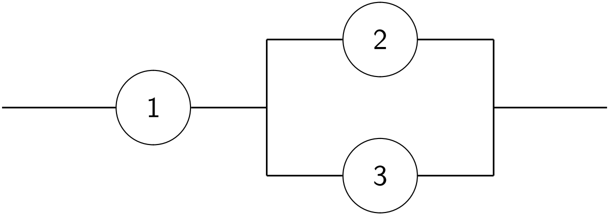

Figure: System with lifetime $T = \min(X_1, \max(X_2, X_3))$.

Example 2- ID case

- $T = \max(\min(X_1, X_2), \min(X_1, X_3))$ with ID components

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$$\begin{aligned}
\bar{F}_T(t) &= \Pr(\{\min(X_1, X_2) > t\} \cup \{\min(X_1, X_3) > t\}) \\
&= \Pr(X_{\{1,2\}} > t) + \Pr(X_{\{1,3\}} > t) - \Pr(X_{\{1,2,3\}} > t) \\
&= K(\bar{F}(t), \bar{F}(t), 1) + K(\bar{F}(t), 1, \bar{F}(t)) - K(\bar{F}(t), \bar{F}(t), \bar{F}(t)) \\
&= \bar{q}(\bar{F}(t))
\end{aligned}$$

where $\bar{q}(u) = K(u, u, 1) + K(u, 1, u) - K(u, u, u)$.

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where $\bar{q}(u) = K(u, u, 1) + K(u, 1, u) - K(u, u, u)$.

- If the components are IID, then $\bar{q}(u) = 2u^2 - u^3$.

Example 2

- If $K(u_1, u_2, u_3) = u_1 u_2 u_3 \left(1 + \alpha(2 - u_1 - u_2)(1 - u_3)\right)$, for $\alpha \in [-0.5, 0.5]$, then

$$\bar{q}_\alpha(u) = u^2 + u^2\left(1 + \alpha(1-u)^2\right) - u^3\left(1 + 2\alpha(1-u)^2\right).$$

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- If we want to compare $T=\min(X_1,\max(X_2,X_3))$ and X_1 in the HR order we plot $\bar{q}_\alpha(u)/u$ in $(0,1)$ for $\alpha=-0.5,-0.25,0,0.25,0.5$.

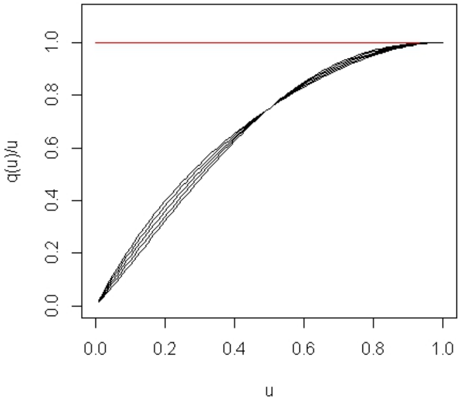

Figure: Ratio of the dual distortion functions of T and X_1 when $\alpha = -0.5, -0.25, 0, 0.25, 0.5$.

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- If we want to compare $T_\alpha = \min(X_1, \max(X_2, X_3))$ for different values of α, we should study $g = \bar{q}_\beta / \bar{q}_\alpha$ for $-0.5 \leq \alpha < \beta \leq 0.5$.

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- If we want to compare $T_\alpha = \min(X_1, \max(X_2, X_3))$ for different values of α, we should study $g = \bar{q}_\beta / \bar{q}_\alpha$ for $-0.5 \leq \alpha < \beta \leq 0.5$.
- A straightforward calculation shows that g is strictly decreasing in $(0, u_0)$ and strictly increasing in $(u_0, 1)$ for

$$u_0 = \frac{13}{8} - \frac{1}{8}\sqrt{57} \cong 0.681270.$$

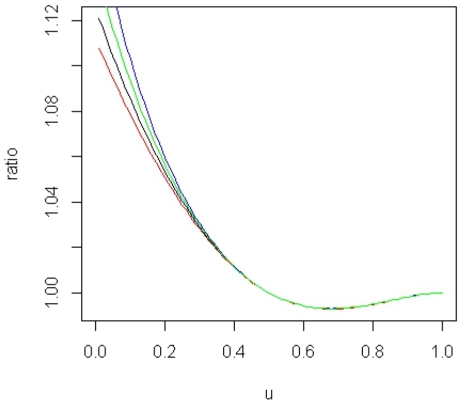

Figure: Ratio $\bar{q}_\beta/\bar{q}_\alpha$ of the dual distortion functions of T when $(\alpha,\beta)=(-0.5,-0.25)$ (blue), $(-0.25,0)$ (green), $(0,0.25)$ (black) and $(0.25,0.5)$ (red).

Example 2

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- These systems are not ST ordered since g takes values greater and smaller than 1.

Comparisons IID case-Navarro (TEST, 2016)

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- $T_1 \leq_{ST} T_2$ holds for all F if and only if

$$\sum_{i=1}^{n} (q_i - p_i) x^i \geq 0 \text{ for all } x \in (0,1).$$

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$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j-i)(p_j q_i - p_i q_j) x^{i+j-2} \geq 0 \text{ for all } x \in (0,1).$$

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- $T_1 \leq_{LR} T_2$ holds for all F if and only if

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} ij(j-i)(p_j q_i - p_i q_j)x^{i+j-2} \geq 0 \text{ for all } x \in (0,1).$$

Table: Minimal signatures for all the systems with 1-4 components.

N	$T_N = \phi(X_1, X_2, X_3, X_4)$	**a**
1	$X_{1:1} = X_1$	$(1, 0, 0, 0)$
2	$X_{1:2} = \min(X_1, X_2)$	$(0, 1, 0, 0)$
3	$X_{2:2} = \max(X_1, X_2)$	$(2, -1, 0, 0)$
4	$X_{1:3} = \min(X_1, X_2, X_3)$	$(0, 0, 1, 0)$
5	$\min(X_1, \max(X_2, X_3))$	$(0, 2, -1, 0)$
6	$X_{2:3}$	$(0, 3, -2, 0)$
7	$\max(X_1, \min(X_2, X_3))$	$(1, 1, -1, 0)$
8	$X_{3:3} = \max(X_1, X_2, X_3)$	$(3, -3, 1, 0)$
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$	$(0, 0, 0, 1)$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(0, 0, 2, -1)$
11	$\min(X_{2:3}, X_4)$	$(0, 0, 3, -2)$
12	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$(0, 1, 1, -1)$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$(0, 3, -3, 1)$

14	$X_{2:4}$	$(0,0,4,-3)$
15	$\max(\min(X_1,X_2),\min(X_1,X_3,X_4),\min(X_2,X_3,X_4))$	$(0,1,2,-2)$
16	$\max(\min(X_1,X_2),\min(X_3,X_4))$	$(0,2,0,-1)$
17	$\max(\min(X_1,X_2),\min(X_1,X_3),\min(X_2,X_3,X_4))$	$(0,2,0,-1)$
18	$\max(\min(X_1,X_2),\min(X_2,X_3),\min(X_3,X_4))$	$(0,3,-2,0)$
19	$\min(\max(X_1,X_2),\max(X_2,X_3),\max(X_3,X_4))$	$(0,3,-2,0)$
20	$\min(\max(X_1,X_2),\max(X_1,X_3),\max(X_2,X_3,X_4))$	$(0,4,-4,1)$
21	$\min(\max(X_1,X_2),\max(X_3,X_4))$	$(0,4,-4,1)$
22	$\min(\max(X_1,X_2),\max(X_1,X_3,X_4),\max(X_2,X_3,X_4))$	$(0,5,-6,2)$
23	$X_{3:4}$	$(0,6,-8,3)$
24	$\max(X_1,\min(X_2,X_3,X_4))$	$(1,0,1,-1)$
25	$\max(X_1,\min(X_2,X_3),\min(X_3,X_4))$	$(1,2,-3,1)$
26	$\max(X_{2:3},X_4)$	$(1,3,-5,2)$
27	$\min(\max(X_1,X_2,X_3),\max(X_2,X_3,X_4))$	$(2,0,-2,1)$
28	$X_{4:4}=\max(X_1,X_2,X_3,X_4)$	$(4,-6,4,-1$

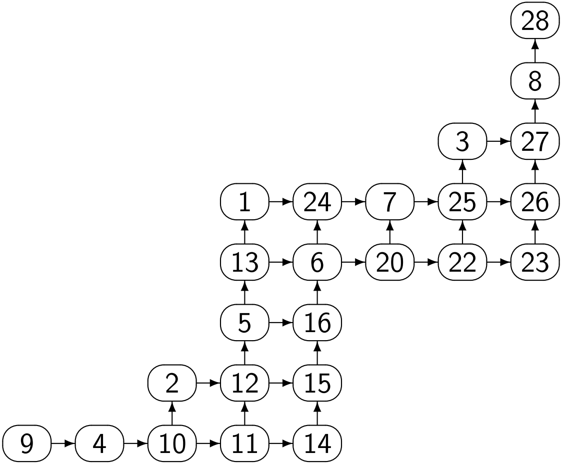

Figure: All the ST orderings for IID components.

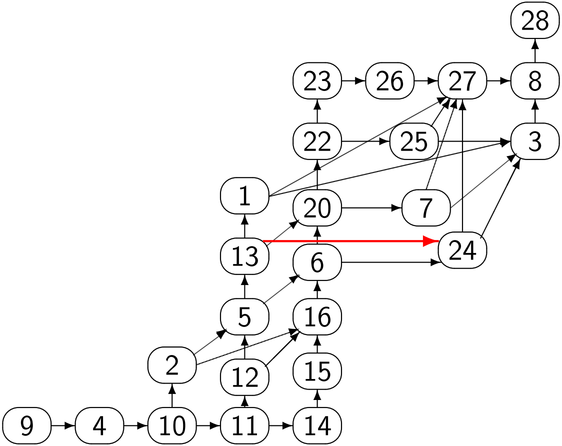

Figure: All the HR orderings for IID components.

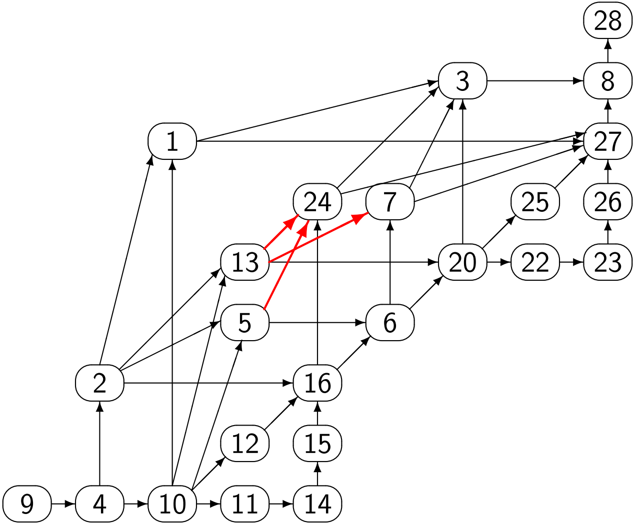

Figure: All the LR orderings for IID components.

Example 3

- $T = X_{2:2} = \max(X_1, X_2)$, X_1, X_2 ID DEP$\sim K$.

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- Then the system reliability is

$$\bar{F}_{2:2}(t) = 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t)) = \bar{q}_{2:2}(\bar{F}(t)),$$

where $\bar{q}_{2:2}(u) = 2u - K(u, u)$.

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- The reliability of the series system $X_{1:2} = \min(X_1, X_2)$ is

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- Then $X_{1:2} \leq_{HR} X_{2:2}$ holds for all $\bar{F}$ if and only if

$$\frac{\bar{q}_{2:2}(u)}{\bar{q}_{1:2}(u)} = \frac{2u - K(u,u)}{K(u,u)} = \frac{2u}{K(u,u)} - 1$$

  is decreasing in $(0, 1)$.

Example 3

- If X_1, X_2 are ID$\sim \bar{F}$ and DEP$\sim K$, the following conditions are equivalent:
  (i) $X_{1:2} \leq_{HR} X_{2:2}$ for all $\bar{F}$.
  (ii) $X_{1:2} \leq_{HR} X_i$ for all $\bar{F}$.
  (iii) $X_i \leq_{HR} X_{2:2}$ for all $\bar{F}$.
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  - (iii) $X_i \leq_{HR} X_{2:2}$ for all $\bar{F}$.
  - (iv) $K(u,u)/u$ is increasing in $(0,1)$.
- These properties are not necessarily true for all K (see Navarro, Torrado and del Águila, Methodology and Computing in Applied Probability, 2017).

Coherent systems- NID case

- If $T = \phi(X_1, \ldots, X_n)$, then

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

where $\bar{Q}$ is a dual distortion function.

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where $\bar{Q}$ is a dual distortion function.
- Therefore we can use the results obtained in Navarro et al. (Methodology and Computing in Applied Probability, 2016) and in Navarro and del Águila (Metrika, 2017) for generalized distorted distributions.

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where $\bar{Q}_{1:2}(u_1, u_2) = u_1 u_2$.
- Then $X_{1:2} \leq_{HR} X_{2:2}$ holds for all $\bar{F}_1, \bar{F}_2$ since

$$\frac{\bar{Q}_{2:2}(u_1, u_2)}{\bar{Q}_{1:2}(u_1, u_2)} = \frac{u_1 + u_2 - u_1 u_2}{u_1 u_2} = \frac{1}{u_1} + \frac{1}{u_2} - 1$$

is decreasing in $(0, 1)^2$.

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- If we want to compare $T = X_{2:2} = \max(X_1, X_2)$ with X_1, we should study

$$\frac{\bar{Q}_{2:2}(u_1, u_2)}{u_1} = \frac{u_1 + u_2 - u_1 u_2}{u_1} = 1 + \frac{u_2}{u_1} - u_2 = 1 + u_2 \left(\frac{1}{u_1} - 1 \right).$$

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- As it is is decreasing in u_1 and increasing in u_2 in $(0,1)^2$, $X_{2:2}$ and X_1 are not HR ordered. The same happen for X_2.

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- As it is is decreasing in u_1 and increasing in u_2 in $(0,1)^2$, $X_{2:2}$ and X_1 are not HR ordered. The same happen for X_2.
- If we assume $X_1 \geq_{HR} X_2$, we should study

$$H(v_1,v_2) = \frac{\bar{Q}_2(v_1,v_1v_2)}{\bar{Q}_1(v_1,v_1v_2)} = \frac{v_1+v_1v_2-v_1^2v_2}{v_1} = 1+v_2-v_1v_2.$$

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- If we assume $X_1 \geq_{HR} X_2$, we should study

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- As it is is decreasing in v_1 and increasing in v_2 in $(0,1)^2$, $X_{2:2}$ and X_1 are not HR ordered.

Example 4

- However, to compare $X_{2:2}$ and X_2, we should study

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$$H(v_1, v_2) = \frac{\bar{Q}_2(v_1, v_1 v_2)}{\bar{Q}_1(v_1, v_1 v_2)} = \frac{v_1 + v_1 v_2 - v_1^2 v_2}{v_1 v_2} = 1 + \frac{1}{v_2} - v_1.$$

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- In particular, in the IID case, we have

$$X_{1:2} \leq_{HR} X_1 =_{HR} X_2 \leq_{HR} X_{2:2}$$

(a well known property).

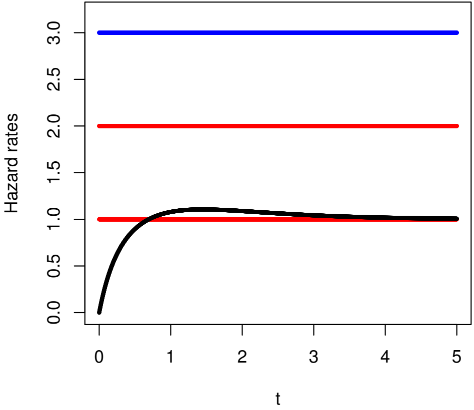

Figure: Hazard rates of X_i (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when $X_i \sim Exp(\mu = 1/i)$, $i = 1, 2$.

Table: Dual distortions functions of systems with 1-3 INID components.

N	$T = \psi(X_1, X_2, X_3)$	$\bar{Q}(u_1, u_2, u_3)$
1	$X_{1:3} = \min(X_1, X_2, X_3)$	$u_1 u_2 u_3$
2	$\min(X_2, X_3)$	$u_2 u_3$
3	$\min(X_1, X_3)$	$u_1 u_3$
4	$\min(X_1, X_2)$	$u_1 u_2$
5	$\min(X_3, \max(X_1, X_2))$	$u_1 u_3 + u_2 u_3 - u_1 u_2 u_3$
6	$\min(X_2, \max(X_1, X_3))$	$u_1 u_2 + u_2 u_3 - u_1 u_2 u_3$
7	$\min(X_1, \max(X_2, X_3))$	$u_1 u_2 + u_1 u_3 - u_1 u_2 u_3$
8	X_3	u_3
9	X_2	u_2
10	X_1	u_1

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11	$X_{2:3}$	$u_1u_2 + u_1u_3 + u_2u_3 - 2u_1u_2u_3$
12	$\max(X_3, \min(X_1, X_2))$	$u_3 + u_1u_2 - u_1u_2u_3$
13	$\max(X_2, \min(X_1, X_3))$	$u_2 + u_1u_3 - u_1u_2u_3$
14	$\max(X_1, \min(X_2, X_3))$	$u_1 + u_2u_3 - u_1u_2u_3$
15	$\max(X_2, X_3)$	$u_2 + u_3 - u_2u_3$
16	$\max(X_1, X_3)$	$u_1 + u_3 - u_1u_3$
17	$\max(X_1, X_2)$	$u_1 + u_2 - u_1u_2$
18	$X_{3:3} = \max(X_1, X_2, X_3)$	$u_1 + u_2 + u_3 - u_1u_2 - u_1u_3$ $-u_2u_3 + u_1u_2u_3$

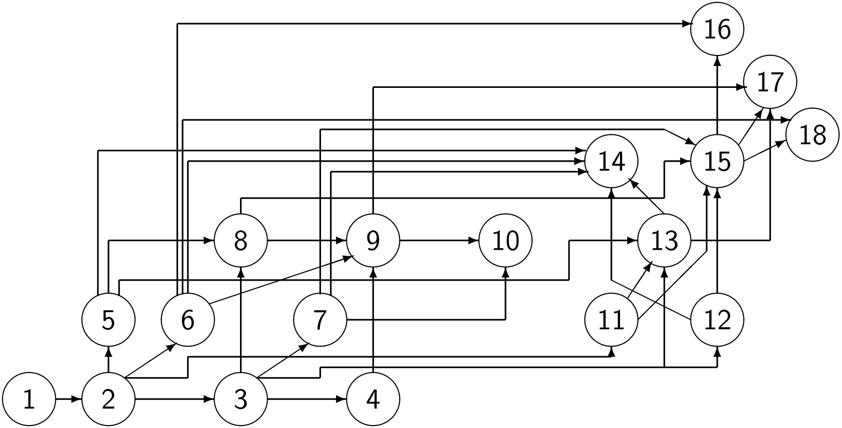

Figure: Hazard rate ordering relationships between the coherent systems with 1-3 independent components such that $X_1 \geq_{HR} X_2 \geq_{HR} X_3$.

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- Then $X_{1:2} \leq_{HR} X_{2:2}$ holds for all $\bar{F}_1, \bar{F}_2$ if and only if

$$\frac{\bar{Q}_{2:2}(u_1, u_2)}{\bar{Q}_{1:2}(u_1, u_2)} = \frac{u_1 + u_2 - K(u_1, u_2)}{K(u_1, u_2)} = \frac{u_1 + u_2}{K(u_1, u_2)} - 1$$

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- This property is not necessarily true.

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- Then $X_{1:2} \leq_{HR} X_{2:2}$ does not hold for all $\bar{F}_1, \bar{F}_2$.
- Thus, if $\bar{F}_1(t) = \exp(-t)$ (Exponential) and $\bar{F}_2(t) = 1/(1 + 5t)$ (Pareto) we obtain the following hazard rate functions.

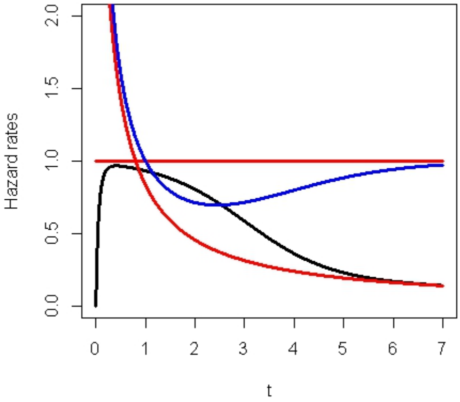

Figure: Hazard rates of X_i (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when X_1 is Exponential, X_2 is Pareto and K is the above Clayton-Oakes copula.

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- Even more, it can be proved that $X_{1:2} \leq_{LR} X_i \leq_{LR} X_{2:2}$ holds for all $\bar{F}$.

Bounds-ID case

- The system lifetime T has the RF $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$. Then:

$$\bar{F}^{\alpha}(t) \inf_{u\in(0,1]} \frac{\bar{q}(u)}{u^{\alpha}} \leq \bar{F}_T(t) \leq \bar{F}^{\alpha}(t) \sup_{u\in(0,1]} \frac{\bar{q}(u)}{u^{\alpha}}.$$

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- More examples can be seen in Navarro and Rychlik (European Journal of Operational Research, 2010).

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- If we compare $\bar{F}_T$ with $\bar{G}_1(t) = (\bar{F}_1(t) + \cdots + \bar{F}_n(t))/n$, then

$$c\,\bar{G}_1(t) \leq \bar{F}_T(t) \leq \frac{n}{|P_1|}\bar{G}_1(t)$$

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- These bounds are sharp.
- More bounds and examples can be seen in Miziula and Navarro (NRL, 2017 and ASMBI, 2017).

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- $(t - T | T \leq t)$, $(t - T | X_1 \leq t, \ldots, X_n \leq t)$ and $(t - T | X_i > t \ \forall i \in W, X_j \leq t \ \forall j \notin W)$ (see Navarro, Pellerey and Longobardi Test, 2017).

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References

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- Thank you for your attention!!