Dependence Models and Copulas in Coherent Systems

Jorge Navarro[1]
Universidad de Murcia, Spain.
E-mail: jorgenav@um.es,
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Notation

- $X_1, \ldots, X_n$ component lifetimes with RF

$$\overline{F}_i(t) = \Pr(X_i > t).$$

- $T = \phi(X_1, \ldots, X_n)$ system (network) lifetime with RF

$$\overline{F}_T(t) = \Pr(T > t).$$

- We assume $\overline{F}_i(t) = \Pr(X_i > t) > 0$ and $\overline{F}_T(t) > 0$ for $t \geq 0$.
- Component residual lifetimes $X_{i,t} = (X_i - t | X_i > t)$ with RF:

$$\overline{F}_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x | X_i > t) = \frac{\overline{F}_i(t + x)}{\overline{F}_i(t)}.$$

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System residual lifetimes

- We have two main options to define the system residual lifetime at time $t > 0$:
- The usual **residual lifetime** $T_t = (T - t | T > t)$ with RF

$$\overline{F}_t(x) = \Pr(T - t > x | T > t) = \frac{\overline{F}_T(t + x)}{\overline{F}_T(t)}.$$

- The **residual lifetime at the system level** $T_t^* = (T - t | X_1 > t, \ldots, X_n > t)$ with RF

$$\overline{F}_t^*(x) = \Pr(T_t^* > x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}$$

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System residual lifetimes

- Which one is the best system?
- Intuitively, it seems that T_t^* should be always better than T_t.
- It should be better to know that all the components are working at time t!
- For $T = \min(X_1, \dots, X_n)$, $T_t =_{ST} T_t^*$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$.

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System residual lifetimes

- If $X_1, \dots, X_n$ are independent, then

$$T_t = (T - t | T > t) \leq_{ST} T_t^* = (T - t | X_1 > t, \dots, X_n > t); \quad (1)$$

  see Pellerey and Petakos (IEEE Tr Rel, 2002) and Li and Lu (PEIS, 2003).
- Conditions on $(X_1, \dots, X_n)$ to have (1) were given in Li, Pellerey and You (2013).
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Outline

- 1. Representations as distorted distributions for these systems are obtained.
- 2. These representations are used to compare these systems under different orderings.
- Conditions to get (1) for some orders are given when the dependence (copula) structure is known.
- 3. Some illustrative examples are given.
- They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.
- Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!

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Generalized distorted distribution

- The **generalized distorted distribution** (GDD) associated to n DF $F_1, \ldots, F_n$ and to an increasing continuous **multivariate distortion (aggregation) function** $Q : [0,1]^n \to [0,1]$ such that $Q(0,\ldots,0) = 0$ and $Q(1,\ldots,1) = 1$, is

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \quad (2)$$

- For the RF we have

$$\overline{F}_Q(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \quad (3)$$

where $\overline{F}_i = 1 - F_i$, $\overline{F}_Q = 1 - F_Q$ and $\overline{Q}(u_1, \ldots, u_n) = 1 - Q(1-u_1, \ldots, 1-u_n)$ is the **multivariate dual distortion function**; see Navarro et al. (MCAP 2015).

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Distorted distribution

- The **distorted distribution** (DD) associated to n DF F and to an increasing continuous distortion function $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \tag{4}$$

- They appear in Risk Theory.
- For the RF we have

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \tag{5}$$

where $\overline{F} = 1 - F$, $\overline{F}_q = 1 - F_q$ and $\overline{q}(u) = 1 - q(1-u)$ is called the **dual distortion function**.

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Coherent systems-GENERAL case

- A **path set** of T is a set $P \subseteq \{1, \ldots, n\}$ such that if all the components in P work, then the system works.
- A **minimal path set** of T is a path set which does not contain other path sets.
- If $P_1, \ldots, P_m$ are the minimal path sets of T, then $T = \max_{j=1,\ldots,m} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

$$\bar{F}_T(t) = \Pr\left(\max_{j=1,\ldots,m} X_{P_j} > t\right) = \Pr\left(\cup_{j=1}^m \{X_{P_j} > t\}\right)$$

$$= \sum_{i=1}^m \bar{F}_{P_i}(t) - \sum_{i \neq j} \bar{F}_{P_i \cup P_j}(t) + \cdots \pm \bar{F}_{P_1 \cup \cdots \cup P_m}(t)$$

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Coherent system representation

- The copula representation for the RF of $(X_1, \dots, X_n)$ is

$$\overline{\mathbf{F}}(x_1, \dots, x_n) = \Pr(X_1 > x_1, \dots, X_n > x_n) = K(\overline{F}_1(x_1), \dots, \overline{F}_n(x_n)),$$

where $\overline{F}_i(t) = \Pr(X_i > t)$ and K is the survival copula. Hence

$$\overline{F}_{1:k}(t) = \Pr(X_1 > t, \dots, X_k > t) = K(\overline{F}_1(t), \dots, \overline{F}_r(t), 1, \dots, 1).$$

- Analogously, for X_P, we have

$$\overline{F}_P(t) = K_P(\overline{F}_1(t), \dots, \overline{F}_n(t)),$$

where $K_P(u_1, \dots, u_n) = K(u_1^P, \dots, u_n^P)$ and $u_i^P = u_i$ if $i \in P$ or $u_i^P = 1$ if $i \notin P$.

- Hence the system reliability can be written as

$$\overline{F}_T(t) = \overline{Q}_{\phi,K}(\overline{F}_1(t), \dots, \overline{F}_n(t)).$$

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Coherent system representations

- Particular cases:
- If the components are ID, then $\overline{F}_T(t) = \overline{q}_{\phi,K}(\overline{F}(t))$ where

$$\overline{q}_{\phi,K}(u) = \overline{Q}_{\phi,K}(u,\ldots,u).$$

- If the components are IND, then $\overline{Q}_{\phi,K}$ is a multinomial.
- If the components are IID, then $\overline{q}_{\phi,K}(u) = \sum_{i=1}^{n} a_i u^i$, where $(a_1,\ldots,a_n)$ is the minimal signature.

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Representations for the system residual lifetimes

- The RF of $T_t = (T - t | T > t)$ is

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- Then

$$\overline{F}_t(x) = \frac{\overline{Q}(\overline{F}_1(t)\overline{F}_{1,t}(x),\dots,\overline{F}_n(t)\overline{F}_{n,t}(x))}{\overline{Q}(\overline{F}_1(t),\dots,\overline{F}_n(t))},$$

where $\overline{F}_{i,t}(x) = \overline{F}_i(t+x)/\overline{F}_i(t)$.

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Representations for the system residual lifetimes

- The RF of $T_t^* = (T - t | X_1 > t, \dots, X_n > t)$ is

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Parallel system with two components

- $T = \max(X_1, X_2)$.
- Minimal path sets $P_1 = \{1\}$ and $P_2 = \{2\}$.
- System reliability function:

$$\overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).$$

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$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \overline{F}_2(t)),$$

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Coherent system representation

- Particular cases:
- If the components are ID, then $\overline{F}_T(t) = \overline{q}(\overline{F}(t))$ where

$$\overline{q}(u) = \overline{Q}(u, u) = 2u - K(u, u).$$

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Comparison results-DD

- If q_1 and q_2 are two DF,

$$q_1(F) \leq_{ord} q_2(F) \text{ for all } F?$$

- If q is a DF,

$$F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)?$$

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Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X-t|X>t) \leq_{ST} (Y-t|Y>t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X-t|X>t) \leq E(Y-t|Y>t)$ for all t.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
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- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all t.
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$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

Main stochastic orderings

- $X \le_{ST} Y \Leftrightarrow \overline{F}_X(t) \le \overline{F}_Y(t)$, stochastic order.
- $X \le_{HR} Y \Leftrightarrow h_X(t) \ge h_Y(t)$, hazard rate order.
- $X \le_{HR} Y \Leftrightarrow (X - t|X > t) \le_{ST} (Y - t|Y > t)$ for all t.
- $X \le_{MRL} Y \Leftrightarrow E(X - t|X > t) \le E(Y - t|Y > t)$ for all t.
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Comparison results-DD

- If T_i has the RF $\overline{q}_i(\overline{F}(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all F if and only if $\overline{q}_2/\overline{q}_1 \geq 1$ in $(0, 1)$.
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- $T_1 \leq_{RHR} T_2$ for all F if and only if q_2/q_1 increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F if and only if $\overline{q}'_2/\overline{q}'_1$ decreases.
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Comparison results-System residual lifetimes

- These results can be applied to compare T_t and T_t^*. For example:
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Example 1: Parallel system with two ID components

- $T = \max(X_1, X_2)$ where X_1 and X_2 have DF F.
- Then $\overline{F}_T(t) = \overline{q}(\overline{F}(t))$ where

$$\overline{q}(u) = \overline{Q}(u,u) = 2u - K(u,u).$$

- The RF of $T_t = (T - t | T > t)$ is $\overline{F}_t(x) = \overline{q}_t(\overline{F}_t(x))$ where

$$\overline{q}_t(u) = \overline{Q}_t(u,u) = \frac{\overline{q}(cu)}{\overline{q}(c)} = \frac{2cu - K(cu,cu)}{2c - K(c,c)},$$

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- If K is the Clayton copula

$$K(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta > 0,$$

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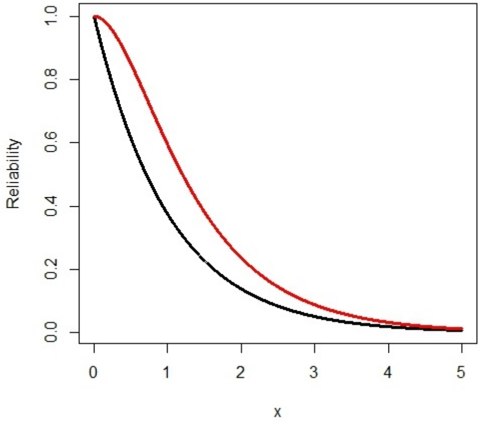

Figure: Reliability functions of T_t (black) and T_t^* (red) when $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 2$.

Example 1: Gumbel-Barnett Archimedean copula

- If K is the Gumbel-Barnett Archimedean copula

$$K(u,v) = uv \exp\left[-\theta(\ln u)(\ln v)\right], \quad \theta \in (0,1], \qquad (8)$$

then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0,1]$ when $\theta = 1$.

- Therefore T_t and T_t^* are not ST ordered (for all F and t).
- These conditions lead to a Gumbel bivariate exponential with a negative correlation.
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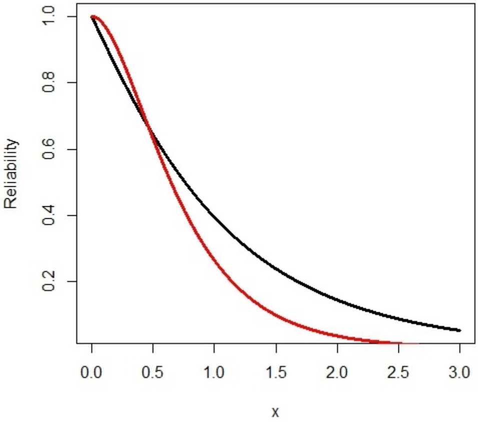

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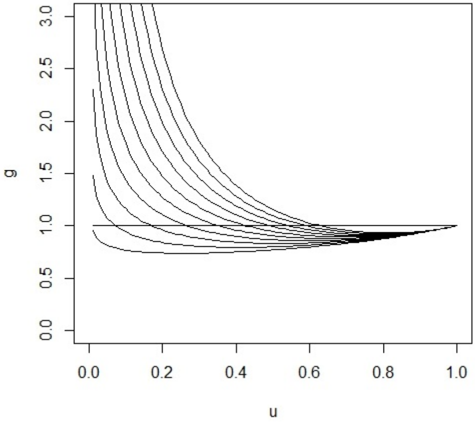

Figure: Ratio $g(u) = \overline{q}_t(u)/\overline{q}^*_t(u)$ for $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 0.1, 0.2, \ldots, 1$ (from the bottom to the top).

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Example 2: Parallel system with two INID components

- If $T = X_{2:2} = \max(X_1, X_2)$, X_1, X_2 IND, then

$$\overline{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2.$$

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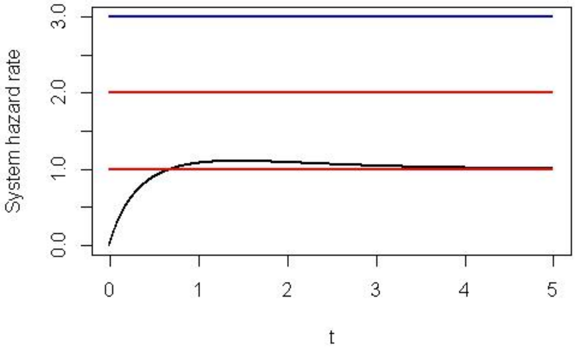

Figure: Hazard rate functions of X_i (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when $X_i \equiv Exp(\mu = 1/i)$, $i = 1, 2$.

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- If X_1, X_2 are IID with DF F, then $X_1 \leq_{HR} X_{2:2}$ holds for all F since

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is a decreasing in u in the set $[0,1]$.

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where $c_1=\overline{F}_1(t)$ and $c_2=\overline{F}_2(t)$.

- $T_t \leq_{HR} T_t^*$ holds for all F_1, F_2 if and only if

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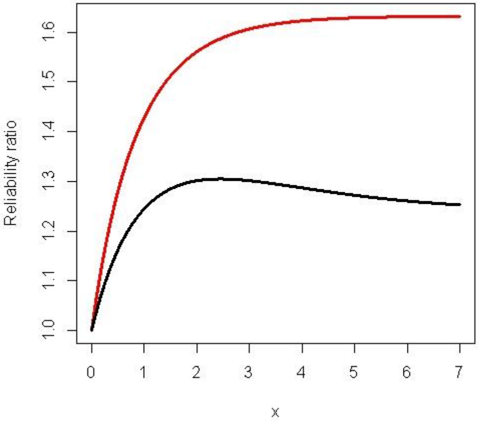

Figure: Ratio $\overline{F}^*_t/\overline{F}_t$ for $t=1$, $\overline{F}_1(x)=e^{-x}$ and $\overline{F}_2(x)=e^{-x/2}$ (black) or $\overline{F}_2(x)=e^{-x}$ (red).

Example 3: Coherent system with DID components

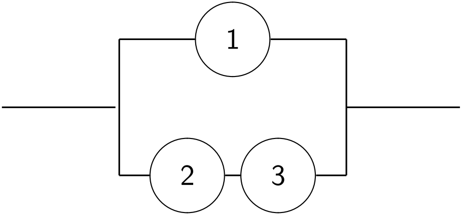

Figure: System in Example 3.

Example 3: Coherent system with DID components

- $T = \max(X_1, \min(X_2, X_3))$, X_1, X_2, X_3 DID with DF F.
- Then $P_1 = \{1\}$, $P_2 = \{2, 3\}$ and

$$\overline{q}(u) = u + K(1, u, u) - K(u, u, u).$$

- Therefore $\overline{q}_t(u) = \overline{q}(cu)/\overline{q}(c)$ and

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where $c = \overline{F}(t)$.

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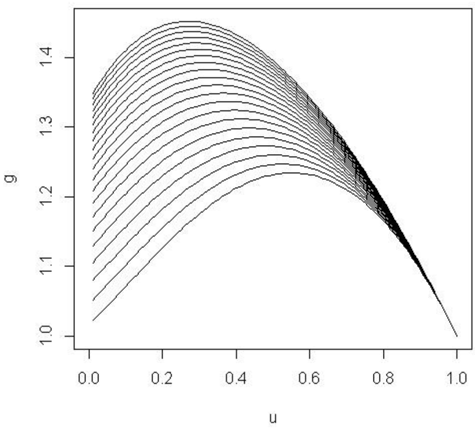

Figure: Ratio $g(u) = \overline{q}_t^*(u)/\overline{q}_t(u)$ for $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = -1, -0.9, \ldots, 1$ (from the bottom to the top).

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Further results

- Navarro and Durante (2016):
- Case 3: $T^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t} = \left(T - t | H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}\right)$ where the (past) history of the system can be represented as

$$H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t} = \{X_{i_1} = t_1, \dots, X_{i_r} = t_r, X_j > t \text{ for } j \notin \{i_1,\dots,i_r\}\},$$

where $0 < r < n$, $0 < t_1 < \cdots < t_r < t$, $\Pr\left(H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}\right) > 0$ and the event $H^{(i_1,\dots,i_r)}_{t_1,\dots,t_r,t}$ implies $T > t$.

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- Case 1: At time t we know that the system has failed. The inactivity time is

$$_tT = (t - T | T \leq t).$$

- Case 2: At time t we know which components W are working. The other W^c have failed, that is, $A_t = \{X_W > t, X^{W^c} \leq t\}$, where $X_W = \min_{i \in W} X_i$ and $X^{W^c} = \max_{i \in W^c} X_i$, for $W \subset \{1, \dots, n\}$. If A_t implies $T < t$, the inactivity time is

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- Case 2: At time t we know which components W are working. The other W^c have failed, that is, $A_t = \{X_W > t, X^{W^c} \leq t\}$, where $X_W = \min_{i \in W} X_i$ and $X^{W^c} = \max_{i \in W^c} X_i$, for $W \subset \{1, \dots, n\}$. If A_t implies $T < t$, the inactivity time is

$$_tT^W = (t - T | X_W > t, X^{W^c} \leq t).$$

- These cases can also be represented as DD.

Further results

- Navarro, Pellerey and Longobardi (2016), Inactivity times:
- Case 1: At time t we know that the system has failed. The inactivity time is

$$ {}_tT = (t - T | T \leq t). $$

- Case 2: At time t we know which components W are working. The other W^c have failed, that is, $A_t = \{X_W > t, X^{W^c} \leq t\}$, where $X_W = \min_{i \in W} X_i$ and $X^{W^c} = \max_{i \in W^c} X_i$, for $W \subset \{1, \ldots, n\}$. If A_t implies $T < t$, the inactivity time is

$$ {}_tT^W = (t - T | X_W > t, X^{W^c} \leq t). $$

- These cases can also be represented as DD.

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References

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