Conference 1: Distorted models

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References

The conference is based mainly on the following references:

- Navarro, del Águila, Sordo and Suárez-Llorens (2013).
- Navarro, del Águila, Sordo and Suárez-Llorens (2014).
- Navarro, del Águila, Sordo and Suárez-Llorens (2016).
- Navarro (2016).
- Navarro and Gomis (2016).
- Navarro and del Águila (2017).

Outline

Notation

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- Mean, expected lifetime or mean time to failure (MTTF):

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} \bar{F}(x) dx - \int_{-\infty}^{0} F(x) dx.$$

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- Hazard rate (HR) or failure rate (FR) function $h(t) = f(t)/\bar{F}(t)$, when $\bar{F}(t) > 0$.

Distorted distributions

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- Definition

The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous *distortion function* $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is given by

$$F_q(t) = q(F(t)), \text{ for all } t \in \mathbb{R}. \quad (1.1)$$

Properties

- If q is a distortion function, then F_q is a proper distribution function for all distribution functions F.

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- If q is an strictly increasing distortion function, then F_q has the same support of F.
- From (1.1), $\bar{F} = 1 - F$ and $\bar{F}_q = 1 - F_q$ satisfy

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \text{ for all } t \in \mathbb{R}, \quad (1.2)$$

where $\bar{q}(u) := 1 - q(1-u)$ is called the *dual distortion function* in Hürlimann (2004).

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- (1.1) and (1.2) are equivalent.

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- The PDF of F_q is

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- The hazard rate of F_q is

$$h_q(t) = \frac{\bar{q}'(\bar{F}(t))}{\bar{q}(\bar{F}(t))}f(t) = \alpha(\bar{F}(t))h(t),$$

where h is the hazard rate of F and

$$\alpha(u) = \frac{u\bar{q}'(u)}{\bar{q}(u)}, \; u \in [0,1].$$

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- Definition

The **generalized distorted distribution** (GDD) associated to n distribution functions $F_1, \ldots, F_n$ and to an increasing continuous *distortion function* $Q : [0,1]^n \to [0,1]$ such that $Q(0,\ldots,0) = 0$ and $Q(1,\ldots,1) = 1$, is given by

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)), \text{ for all } t \in \mathbb{R}. \qquad (1.3)$$

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- From (1.3), $\bar{F}_i = 1 - F_i$ and $\bar{F}_Q = 1 - F_Q$ satisfy

$$\bar{F}_Q(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \text{ for all } t \in \mathbb{R}, \tag{1.4}$$

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- (1.3) and (1.4) are equivalent.

Properties

- The PDF of F_Q is

$$f_Q(t) = \sum_{i=1}^{n} f_i(t)\partial_i Q(F_1(t),\ldots,F_n(t)) = \sum_{i=1}^{n} f_i(t)\partial_i \bar{Q}(\bar{F}_1(t),\ldots,\bar{F}_n(t)).$$

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- The hazard rate of F_q is

$$h_Q(t) = \sum_{i=1}^{n} \frac{\partial_i \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t))}{\bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t))} f_i(t) = \sum_{i=1}^{n} \alpha_i(\bar{F}_1(t), \dots, \bar{F}_n(t)) h_i(t),$$

where h_i is the hazard rate of F_i and

$$\alpha_i(u) = \frac{u_i \partial_i \bar{Q}(u_1, \dots, u_n)}{\bar{Q}(u_1, \dots, u_n)}, \quad u_i \in [0,1], i = 1, \dots, n.$$

Examples of distorted distributions: PHR.

- Proportional Hazard Rate (PHR) Cox model

$$\bar{F}_\theta(t) = \bar{F}^\theta(t), t \in \mathbb{R},$$

where $\theta > 0$ is a risk (hazard) measure.

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$$h_\theta(t) = \theta \frac{\bar{F}^{\theta-1}(t)}{\bar{F}^\theta(t)} f(t) = \theta h(t),$$

that is, $\alpha_\theta(u) = \theta$ for $u \in [0,1]$.

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- Proportional Reversed Hazard Rate (PRHR) model

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$$\bar{h}_\theta(t) = \frac{f_\theta(t)}{F_\theta(t)} = \theta \bar{h}(t).$$

Examples of distorted distributions: Order statistics.

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- It is a distorted distribution with

$$\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$$

and

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- Note that both are polynomials.

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for $n = 1, \ldots, n$ which belongs to the PHR model.

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- $X_{n:n} = \max(X_1, \dots, X_n)$ with

$$F_{n:n}(t) = \binom{n}{n} F^n(t)\bar{F}^{n-n}(t) = F^n(t)$$

for $n = 1, \dots, n$ which belongs to the PRHR model.

Examples of generalized distorted distributions: Average.

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- It is a generalized distorted distribution with

$$Q(u_1, \dots, u_n) = \bar{Q}(u_1, \dots, u_n) = \frac{u_1 + \cdots + u_n}{n}, \ u_1, \dots, u_n \in [0, 1].$$

- Its PDF is

$$f_a(t) = \frac{f_1(t) + \cdots + f_n(t)}{n}, t \in \mathbb{R}.$$

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$$Q_{gmd}(u_1, \dots, u_n) = (u_1 \dots u_n)^{1/n}, \ u_1, \dots, u_n \in [0,1]$$

and

$$\bar{Q}_{gmr}(u_1, \dots, u_n) = (u_1 \dots u_n)^{1/n}, \ u_1, \dots, u_n \in [0,1],$$

respectively.

Examples of generalized distorted distributions: Mixtures.

- The mixture distribution

$$F_{\mathbf{p}}(t) = p_1 F_1(t) + \cdots + p_n F_n(t), t \in \mathbb{R},$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, $p_i \geq 0$ and $p_1 + \cdots + p_n = 1$.

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- Its PDF is $f_{\mathbf{p}}(t) = p_1 f_1(t) + \cdots + p_n f_n(t), t \in \mathbb{R}$.
- Its HR is

$$h_{\mathbf{p}}(t) = w_1(t) h_1(t) + \cdots + w_n(t) h_n(t), \ w_i(t) = \frac{p_i \bar{F}_i(t)}{\bar{F}_{\mathbf{p}}(t)} \geq 0.$$

Examples of generalized distorted distributions: GPHR.

- The generalized proportional hazard rate (GPHR) model

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Copula representation

- $(X_1, \dots, X_n)$ random vector with joint distribution

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- Then

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

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- $T_1 \leq_{MRL} T_2$ for all F such that $E(T_1)\leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0,1)$.

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- $X_{1:2} \leq_{LR} X_i \leq_{LR} X_{2:2}$ holds for all abs. cont. F.

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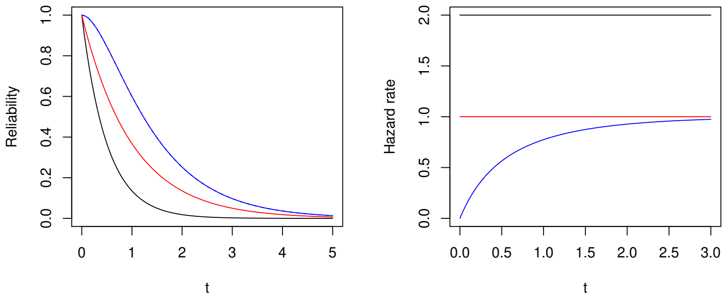

Figure: Reliability (left) and hazard rate (right) functions of $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for a standard exponential distribution.

R code

```
# Reliability functions
R<-function(t) exp(-t)
qIID<-function(u) u^ 2
G12<-function(t) qIID(R(t))
G22<-function(t) 2*R(t)-G12(t)
curve(G12(x),xlab='t',ylab='Reliability',0,5)
curve(G22(x),add=T,col='blue')
curve(R(x),add=T,col='red')
```

R code

```
#Hazard rate functions
f<-function(t) exp(-t)
qpIID<-function(u) 2*u
g12<-function(t) f(t)*qpIID(R(t))
g22<-function(t) 2*f(t)-g12(t)
curve(g12(x)/G12(x),xlab='t',ylab='Hazard rate',0,3,
ylim=c(0,2))
curve(g22(x)/G22(x),add=T,col='blue')
curve(f(x)/R(x),add=T,col='red')
```

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- $T_1 \leq_{HR} T_2$ for all $F_1 \geq_{HR} \cdots \geq_{HR} F_n$ iff the function

$$\bar{H}(v_1, \dots, v_n) = \frac{\bar{Q}_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{\bar{Q}_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \qquad (2.1)$$

is decreasing in $(0,1)^n$;

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If T_i has DF $F_{T_i} = Q_i(F_1, \dots, F_n)$, $i = 1, 2$, then:

- $T_1 \le_{ST} T_2$ for all $F_1 \ge_{ST} \cdots \ge_{ST} F_n$ iff $\bar{Q}_1 \le \bar{Q}_2$ in $D = \{(u_1, \dots, u_n) \in [0,1]^n : u_1 \ge \cdots \ge u_n\}$;
- $T_1 \le_{HR} T_2$ for all $F_1 \ge_{HR} \cdots \ge_{HR} F_n$ iff the function

$$\bar{H}(v_1, \dots, v_n) = \frac{\bar{Q}_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{\bar{Q}_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \qquad (2.1)$$

  is decreasing in $(0,1)^n$;
- $T_1 \le_{RHR} T_2$ for all $F_1 \le_{RHR} \cdots \le_{RHR} F_n$ iff the function

$$H(v_1, \dots, v_n) = \frac{Q_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{Q_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \qquad (2.2)$$

  is increasing in $(0,1)^n$.

Comparisons of GDD. Example 2.

- X_1, X_2 independent (IND) $\sim F_1, F_2$.

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- Yes for $X_{1:2} \leq_{HR} X_i$ and $X_{1:2} \leq_{HR} X_{2:2}$ for all F_1, F_2.
- No for $X_i \leq_{HR} X_{2:2}$ and for all F_1, F_2.

Comparisons of GDD. Example 2.

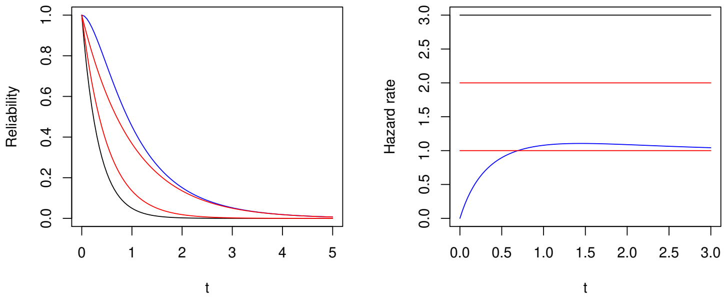

Figure: Reliability (left) and hazard rate (right) functions of $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for two exponential distributions.

R code

```
# Reliability functions
R1<-function(t) exp(-t)
R2<-function(t) exp(-2*t)
QIND<-function(u1,u2) u1*u2
G12<-function(t) QIND(R1(t),R2(t))
G22<-function(t) R1(t)+R2(t)-G12(t)
curve(G12(x),xlab='t',ylab='Reliability',0,5)
curve(G22(x),add=T,col='blue')
curve(R1(x),add=T,col='red')
curve(R2(x),add=T,col='red')
```

R code

```
 #Hazard rate
f1<-function(t) exp(-t)
f2<-function(t) 2*exp(-2*t)
Q1<-function(u,v) v
Q2<-function(u,v) u
g12<-function(t)f1(t)*Q1(R1(t),R2(t))+f2(t)*Q2(R1(t),R2(t))
g22<-function(t) f1(t)+f2(t)-g12(t)
curve(g12(x)/G12(x),xlab='t',ylab='Hazard rate',0,3,
ylim=c(0,3))
curve(g22(x)/G22(x),add=T,col='blue')
curve(f1(x)/R1(x),add=T,col='red')
curve(f2(x)/R2(x),add=T,col='red')
```

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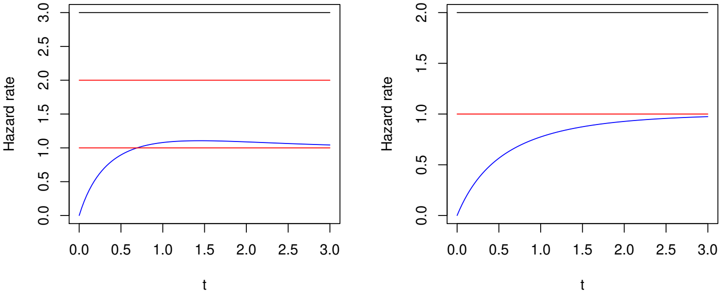

Figure: Hazard rate functions of $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for two exponential distributions with hazard rates $1, 2$ (left) and $1, 1$ (right).

Main aging classes

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- X is Increasing (Decreasing) Failure Rate Average, IFRA (DFRA), if $A(t) = \frac{1}{t}\int_0^t h(x)dx = -\frac{1}{t}\ln \bar{F}(t)$ is increasing (decreasing) (or $\bar{F}(ct) \geq \bar{F}^c(t)$, $0 < c < 1$) for all $t \geq 0$.

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- X is Increasing (Decreasing) Likelihood Ratio, ILR (DLR), if $X_s \geq_{LR} X_t$ ($\leq_{LR}$) for all $0 \leq s \leq t$ (or f is logconcave).

Main among the main aging classes

$$\begin{array}{ccccccc} \text{ILR} & \Rightarrow & \text{IFR} & \Rightarrow & \text{IFRA} & \Rightarrow & \text{NBU} \\ \text{DLR} & \Rightarrow^{*} & \text{DFR} & \Rightarrow & \text{DFRA} & \Rightarrow & \text{NWU} \end{array}$$

Table: Relationships among the main aging classes (* when the support is (a, ∞).).

Distorted distributions

Theorem

Let $F_q = q(F)$ *and* $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$. *Then:*

- *The IFR (DFR) class is preserved by* q *iff* α *is decreasing (increasing) for* $u \in (0,1)$.

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- *The IFRA (DFRA) class is preserved by q iff $\bar{q}$ satisfies*

$$\bar{q}(u^c) \geq (\bar{q}(u))^c, \ (\leq) \text{ for all } u, c \in [0,1]. \tag{3.2}$$

Distorted distributions

- If F is absolutely continuous and ILR and there exists $u_0 \in [0,1]$ such that $\beta(u) = u\bar{q}''(u)/\bar{q}'(u)$ is non-negative and decreasing in $[0,u_0]$ and $\bar{\beta}(u) = (1-u)\bar{q}''(u)/\bar{q}'(u)$ is non-positive and decreasing in $[u_0,1]$, then F_q is ILR.

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- Reverse results: If F_q is IFR and α is increasing, then F is also IFR.

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- If α is increasing (or decreasing) but it is not constant, then just the DFR (IFR) class is preserved.
- If α is not monotone, then neither of them are preserved.

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- The conditions for the IFR/DFR classes are quite strong, while the conditions for the NBU/NWU and IFRA/DFRA classes a mild.

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- X_1, X_2 IID$\sim F$, $X_{1:2} = \min(X_1, X_2)$ and $X_{2:2} = \max(X_1, X_2)$.

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- As $\alpha_{2:2}$ is strictly decreasing, then IFR, NBU and IFRA classes are preserved and DFR, NWU and DFRA are not.

Preservation of IFR/DFR. Example 1.

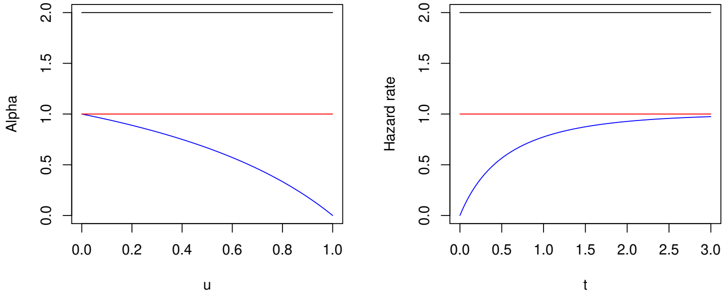

Figure: Alpha function (left) and hazard rate (right) functions of $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for a standard exponential distribution.

Generalized distorted distributions

Theorem

Let $F_Q = Q(F_1, \ldots, F_n)$, $\mathbf{u} = (u_1, \ldots, u_n)$ *and* $\alpha_i(\mathbf{u}) = u_i \partial_i \bar{Q}(\mathbf{u}) / \bar{Q}(\mathbf{u})$. *Then:*

- *If* $\alpha_1, \ldots, \alpha_n$ *are decreasing (increasing) for* $u_1, \ldots, u_n \in (0, 1)$ *and* $i = 1, \ldots, n$, *then the IFR (DFR) class is preserved.*

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- *The IFRA (DFRA) class is preserved by* Q *if* $\bar{Q}$ *satisfies*

$$\bar{Q}(u_1^c,\dots,u_n^c) \ge (\bar{Q}(u_1,\dots,u_n))^c,\ (\le)\ u_i,c \in [0,1].$$

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$$\alpha_1(u_1, u_2) = u_1 \frac{\partial_1 \bar{Q}_{1:2}(u_1, u_2)}{\bar{Q}_{1:2}(u_1, u_2)} = 1$$

and $\alpha_2(u_1, u_2) = 1$.

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- X_1, X_2 IND$\sim F_1, F_2$, $X_{1:2} = \min(X_1, X_2)$.
- Then $\bar{Q}_{1:2}(u_1, u_2) = u_1 u_2$ and

$$\alpha_1(u_1, u_2) = u_1 \frac{\partial_1 \bar{Q}_{1:2}(u_1, u_2)}{\bar{Q}_{1:2}(u_1, u_2)} = 1$$

  and $\alpha_2(u_1, u_2) = 1$.
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- As α_i are constant, all the aging classes IFR, NBU, IFRA, DFR, NWU, and DFRA are preserved.
- If $X_{2:2} = \max(X_1, X_2)$, then $\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$ and

$$\alpha_1(u_1, u_2) = u_1 \frac{\partial_1 \bar{Q}_{2:2}(u_1, u_2)}{\bar{Q}_{2:2}(u_1, u_2)} = \frac{u_1(1 - u_2)}{u_1 + u_2 - u_1 u_2}$$

  is not monotone. IFR and DFR are not preserved because

Preservation of IFR/DFR. Example 2.

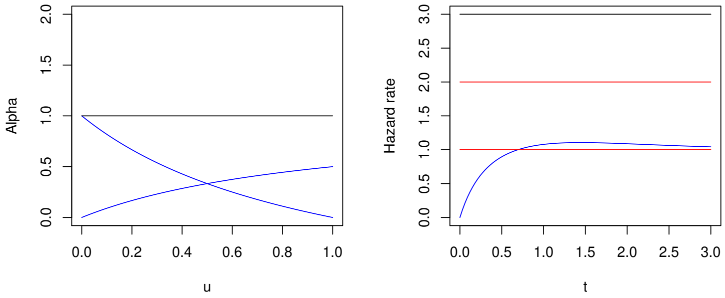

Figure: Alpha function $\alpha_1(0.5, u)$ and $\alpha_1(u, 0.5)$ (left) and hazard rate (right) functions of $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for two exponential distributions.

Mixtures

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- Therefore, some preservations are not detected by the above conditions.
- For example, if $\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t)$, with $\bar{F}_i(t) = \exp(-it)$ for $t \geq 0$ and $i = 1, 2$, then

$$h(t) = \frac{pe^{-t} + 2(1-p)e^{-2t}}{pe^{-t} + (1-p)e^{-2t}}, \ t \geq 0.$$

Preservation of IFR/DFR in Mixtures

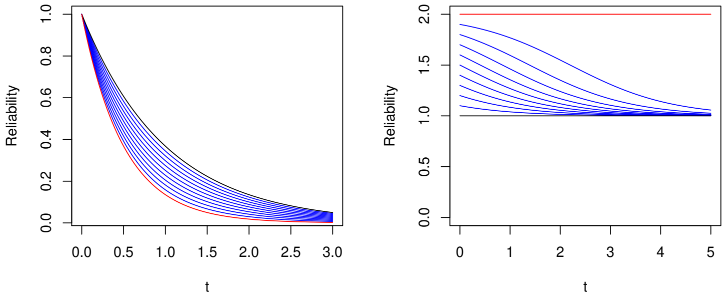

Figure: Reliability functions (left) and hazard rate (right) functions of a mixture of two exponential distributions with hazard rates 1 (black) and 2 (red) and $p = 0.1, \dots, 0.9$ (blue).

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The slides and more references can be seen in my webpage:

https://webs.um.es/jorgenav/miwiki/doku.php

Exercises

1. Prove that if q is a distortion function, then F_q is a proper distribution function for all F.
2. Provide a valid distortion function of dimension 1.
3. Prove that if Q is a distortion function, then F_Q is a proper distribution function for all F.
4. Provide a valid distortion function of dimension $n > 1$.
5. Compute the distortion functions of the median $X_{2:3}$.
6. Compute the distortion function of a fifty-fifty mixture of $\bar{F}$ and $\bar{F}^2$.
7. Compute the distortion functions of the parallel system $X_{2:2} = \max(X_1, X_2)$ for a copula C. What happen if X_1 and X_2 are ID?

8. Find a distortion function that is not a copula.
9. Compare the order statistics $X_{2:3}$ and $X_{3:3}$ (IID case).
10. Study which aging classes are preserved by the median $X_{2:3}$ (IID case).
11. Study which aging classes are preserved by a fifty-fifty mixture of $\bar{F}$ and $\bar{F}^2$.
12. Prove that the NBU class is preserved by $X_{2:2}$ in the IID case.
13. Prove that the NBU class is preserved by $X_{2:2}$ in the IND case.
14. Study which classes are preserved by $X_{2:2}$ in the ID case for a copula C.

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- Questions?