Conference 2: Applications of distortions to complex systems

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References

The conference is based mainly on the following references:

- Barlow and Proschan (1975).
- Di Crescenzo (2007)
- Navarro and Spizzichino (2010).
- Navarro, Pellerey and Di Crescenzo (2015).
- Navarro (2016, 2018).
- Navarro and del Águila (2017).
- Navarro, Durante and Fernández-Sánchez (2020).

Outline

Systems

- A (binary) **system** with (binary) components of order n is a Boolean structure function (map)

$$\phi : \{0,1\}^n \to \{0,1\},$$

where $\phi(x_1, \dots, x_n) \in \{0,1\}$ represents the system's state that is determined by the components' states $x_1, \dots, x_n \in \{0,1\}$.

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- $\phi(x_1,\dots,x_n) = 1$ means that the system works,
- $\phi(x_1,\dots,x_n) = 0$ means that the system has failed and the same for the components.

Semi-coherent systems

Definition

A **semi-coherent system** of order n is a system

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satisfying the following properties:

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- The ith component is **irrelevant** for the system ϕ if

$$\phi(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n) = \phi(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n)$$

for all $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n \in \{0,1\}$.

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(ii) ϕ is strictly increasing in each variable in at least a point.

- All the coherent systems are semi-coherent systems but the reverse is not true.
- A coherent system is a semi-coherent system without irrelevant components.

Examples

- The coherent systems $\phi_1(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$ and $\phi_2(x_1, x_2, x_3) = \min(x_2, \max(x_1, x_3))$ are different.

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- However, they have a similar "structure":

Figure: Two coherent systems of order 3 with a similar structure.

Path and cut sets

- A non-empty set $P \subseteq \{1, \ldots, n\}$ is a **path set** of a system ϕ if $\phi(x_1, \ldots, x_n) = 1$ when $x_i = 1$ for all $i \in P$.

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- A non-empty set $C \subseteq \{1, \dots, n\}$ is a **cut set** of ϕ if $\phi(x_1, \dots, x_n) = 0$ when $x_i = 0$ for all $i \in C$.

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- A path set P is a **minimal path set** if it does not contain other path sets.
- A cut set C is a **minimal cut set** if it does not contain other cut sets.
- The **dual system** of a system ϕ is the system

$$\phi^D(x_1,\ldots,x_n):=1-\phi(1-x_1,\ldots,1-x_n)$$

for all $x_1,\ldots,x_n\in\{0,1\}$.

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- The **k-out-of-n system** is defined for $k = 1, \ldots, n$ by

$$\phi_{n-k+1:n}(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_1 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + \cdots + x_n < k \end{cases} = x_{n-k+1:n}.$$

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- The **k-out-of-n:F (failed) systems** is the system that fails when k components (or more) fail. Its structure is

$$\phi_{k:n}(x_1, \dots, x_n) = x_{k:n}.$$

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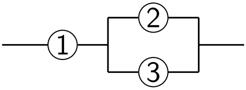
- The coherent system $\phi_1(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$ has the "structure":



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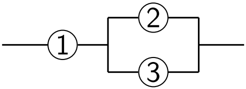

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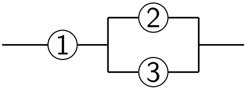

- Its minimal path sets are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.
- Its minimal cut sets are $C_1 = \{1\}$ and $C_2 = \{2, 3\}$.

Minimal path (cut) set representation

Theorem (Minimal path/cut sets representations)

Let ϕ be a coherent (or semi-coherent) system of order n and let $P_1, \dots, P_r$ and $C_1, \dots, C_s$ be its minimal path and minimal cut sets, respectively. Then

$$\phi(x_1, \dots, x_n) = \max_{i=1,\dots,r} \min_{j \in P_i} x_j \tag{1.1}$$

and

$$\phi(x_1, \dots, x_n) = \min_{i=1,\dots,s} \max_{j \in C_i} x_j \tag{1.2}$$

for all $(x_1, \dots, x_n) \in \{0, 1\}^n$.

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- Reliability or survival functions $\bar{F}_T(t) = \Pr(T > t) = 1 - F_T(t)$ and $\bar{F}_i(t) = \Pr(X_i > t)$.

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- Probability density functions (PDF) $f_T = F_T' = -\bar{F}_T'$ and $f_i = F_i' = -\bar{F}_i'$.
- Hazard rate (HR) or failure rate (FR) functions $h_T = f_T/\bar{F}_T$ and $h_i = f_i/\bar{F}_i$.

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- Probability density functions (PDF) $f_T = F'_T = -\bar{F}'_T$ and $f_i = F'_i = -\bar{F}'_i$.
- Hazard rate (HR) or failure rate (FR) functions $h_T = f_T/\bar{F}_T$ and $h_i = f_i/\bar{F}_i$.
- Identically distributed (ID) components, $F_1 = \cdots = F_n = F$.

Minimal path (cut) set representation

Theorem (Barlow and Proschan (1975))

Let ϕ be a coherent (or semi-coherent) system of order n with lifetime T and let $P_1, \ldots, P_r$ and $C_1, \ldots, C_s$ be its minimal path and minimal cut sets, respectively. Then

$$T = \max_{i=1,\ldots,r} \min_{j \in P_i} X_j \tag{1.3}$$

and

$$T = \min_{i=1,\ldots,s} \max_{j \in C_i} X_j \tag{1.4}$$

where $X_1, \ldots, X_n \geq 0$ are the component lifetimes.

Theorem (Minimal path set representation)

If T is the lifetime of a coherent (or semi-coherent) system with minimal path sets $P_1,\dots,P_r$ and component lifetimes $(X_1,\dots,X_n)$, then

$$\bar{F}_T(t)=\sum_{i=1}^{r}\bar{F}_{P_i}(t)-\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\bar{F}_{P_i\cup P_j}(t)+\dots+(-1)^{r+1}\bar{F}_{P_1\cup\dots\cup P_r}(t) \tag{1.5}$$

for all t, where $\bar{F}_P(t)=\Pr(X_P>t)$ and $X_P=\min_{j\in P}X_j$ for $P\subseteq\{1,\dots,n\}$.

Copula representation

- $(X_1, \ldots, X_n)$ random vector with joint distribution

$$\mathbf{F}(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

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- **Sklar's theorem:** There exist a copula C such that

$$\mathbf{F}(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),\ x_1, \ldots, x_n \in \mathbb{R}.$$

Moreover, if $F_1, \ldots, F_n$ are continuous, then C is unique.

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- A copula C is a multivariate distribution function with uniform marginals over the interval $(0, 1)$.

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  Moreover, if $F_1, \ldots, F_n$ are continuous, then C is unique.
- A copula C is a multivariate distribution function with uniform marginals over the interval $(0, 1)$.
- Note that we just need C in $[0, 1]^n$.

Survival copula representation

- $(X_1,\dots,X_n)$ with joint reliability (survival) function

$$\overline{\mathbf{F}}(x_1,\dots,x_n)=\Pr(X_1>x_1,\dots,X_n>x_n).$$

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- $(X_1, \ldots, X_n)$ with joint reliability (survival) function

$$\mathbf{\overline{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n).$$

- Marginal reliability functions

$$\bar{F}_i(x_i) = \Pr(X_i > x_i) = \lim_{x_j \to -\infty,\ \forall j \neq i} \mathbf{\overline{F}}(x_1, \ldots, x_n).$$

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- **Sklar's theorem:** There exist a copula $\widehat{C}$ (called survival copula) such that

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \widehat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),\ x_1, \ldots, x_n \in \mathbb{R}.$$

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  Moreover, if $\bar{F}_1, \ldots, \bar{F}_n$ are continuous, then $\widehat{C}$ is unique.
- $\widehat{C}$ is a copula (distribution), not a survival function.

Series systems

- The reliability function of $X_{1:n}$ is

$$\bar{F}_{1:n}(t) = \Pr(X_1 > t, \ldots, X_n > t) = \widehat{C}(\bar{F}_1(t), \ldots, \bar{F}_n(t)).$$

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- The reliability function of $X_{1:k}$ $(k < n)$ is

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- The reliability function of $X_P = \min_{j \in P} X_j$ is

$$\bar{F}_P(t) = \widehat{C}_P(\bar{F}_1(t), \dots, \bar{F}_n(t)),$$

where

$$\widehat{C}_P(u_1, \dots, u_n) = \widehat{C}(u_1^P, \dots, u_n^P)$$

with $u_j^P = u_j$ for $j \in P$ and $u_j^P = 1$ for $j \notin P$.

Distortion representation

Theorem (Distortion representation, general case)

If T is the lifetime of a semi-coherent system and the component lifetimes $(X_1, \ldots, X_n)$ have the survival copula $\widehat{C}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \tag{1.6}$$

for all t, where $\bar{Q}$ is a distortion function which depends on ϕ (that is, on $P_1, \ldots, P_r$) and $\widehat{C}$.

Distortion representation, IND case

Theorem (Distortion representation, IND case)

If T is the lifetime of a semi-coherent system with independent component lifetimes $X_1, \ldots, X_n$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

for all t, where $\bar{Q}$ is a multinomial (called reliability structure function in Barlow and Proschan (1975)) which only depends on ϕ (structure).

Distortion representation, ID case

Theorem (Distortion representation, ID case)

If T is the lifetime of a semi-coherent system and the component lifetimes $(X_1, \ldots, X_n)$ have the survival copula $\widehat{C}$ and a common reliability $\bar{F}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

for all t, where $\bar{q}$ is a distortion function which only depends on ϕ and on $\widehat{C}$.

Distortion representation, IID case

Theorem (Distortion representation, IID case)

If T is the lifetime of a semi-coherent system with IID component lifetimes $X_1, \dots, X_n$ having a common reliability $\bar{F}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

for all t, where $\bar{q}(u) = \sum_{i=1}^{n} a_i u^i$ is a distortion function and $a = (a_1, \dots, a_n)$ is the minimal signature which only depends on ϕ.

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If T is the lifetime of a semi-coherent system with IID component lifetimes $X_1, \ldots, X_n$ having a common reliability $\bar{F}$, then the reliability function of T can be written as

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- $F_T(t) = q(F(t))$, where $q(u) = \sum_{i=1}^{n} b_i u^i$ is a distortion function and $b = (b_1, \ldots, b_n)$ is the maximal signature.

Example 1

- $T = \min(X_1, \max(X_2, X_3))$.

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$$\bar{F}_T(t) = \widehat{C}(\bar{F}_1(t), \bar{F}_2(t), 1) + \widehat{C}(\bar{F}_1(t), 1, \bar{F}_3(t)) - \widehat{C}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t)).$$

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$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t))$$

with $\bar{Q}(u_1, u_2, u_3) = \widehat{C}(u_1, u_2, 1) + \widehat{C}(u_1, 1, u_3) - \widehat{C}(u_1, u_2, u_3)$.

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- IID case

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

with $\bar{q}(u) = \bar{Q}(u, u, u) = 2u^2 - u^3$ and $a = (0, 2, -1)$.

Example 1

- If we choose the FGM copula:

$$\widehat{C}(u_1, u_2, u_3) = u_1 u_2 u_3 (1 + \theta(1 - u_1)(1 - u_2)(1 - u_3))$$

for $\theta \in [-1, 1]$, then

$$\bar{Q}(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 - u_1 u_2 u_3 (1 + \theta(1 - u_1)(1 - u_2)(1 - u_3)).$$

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- ID case $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ with $\bar{q}(u) = 2u^2 - u^3 - \theta u^3 (1 - u)^3$.

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for $\theta \in [-1, 1]$, then

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- IND case $\bar{Q}(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 - u_1 u_2 u_3$.
- ID case $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ with $\bar{q}(u) = 2u^2 - u^3 - \theta u^3 (1 - u)^3$.
- IID case $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ with $\bar{q}(u) = 2u^2 - u^3$.

Example 1: Reliability and hazard rate functions

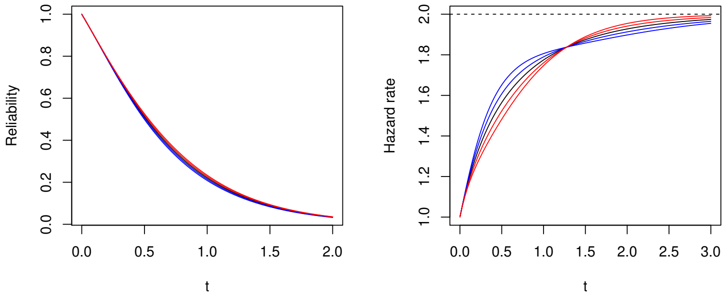

Figure: Reliability (left) and hazard rate (right) functions of T for a standard exponential distribution, a FGM survival copula and $\theta = -1, -0.5$ (red), 0 (black) and $\theta = 0.5, 1$ (blue).

R code

```
# Reliability functions
C<-function(u1,u2,u3,z)u1*u2*u3*(1+z*(1-u1)*(1-u2)*(1-u3))
bQ<-function(u1,u2,u3,z)
C(u1,u2,1,z)+C(u1,1,u3,z)-C(u1,u2,u3,z)
bq<-function(u,z) bQ(u,u,u,z)
R<-function(t) exp(-t)
RT<-function(t,z) bq(R(t),z)
curve(RT(x,0),0,2,xlab='t',ylab='Reliability')
curve(RT(x,0.5),add=T,col='blue')
curve(RT(x,1),add=T,col='blue')
curve(RT(x,-0.5),add=T,col='red')
curve(RT(x,-1),add=T,col='red')
```

R code

```
#Hazard rate functions
f<-function(t) exp(-t)
bqp<-function(u,z)
4*u-3*u^ 2-3*z*u^ 2*(1-u)^ 3+3*z*u^ 3*(1-u)^ 2
fT<-function(t,z) bqp(R(t),z)*f(t)
hT<-function(t,z) fT(t,z)/RT(t,z)
curve(hT(x,0),0,3,xlab='t',ylab='Hazard
rate',ylim=c(1,2))
curve(hT(x,0.5),add=T,col='blue')
curve(hT(x,1),add=T,col='blue')
curve(hT(x,-0.5),add=T,col='red')
curve(hT(x,-1),add=T,col='red')
abline(h=2,lty=2)
```

Minimal and maximal signatures

Table: Minimal **a** and maximal **b** signatures of all the coherent systems with 1-4 IID components.

i	T_i	**a**	**b**
1	$X_{1:1} = X_1$	(1)	(1)
2	$X_{1:2} = \min(X_1, X_2)$ (2-series)	$(0, 1)$	$(2, -1)$
3	$X_{2:2} = \max(X_1, X_2)$ (2-parallel)	$(2, -1)$	$(0, 1)$
4	$X_{1:3} = \min(X_1, X_2, X_3)$ (3-series)	$(0, 0, 1)$	$(3, -3, 1)$
5	$\min(X_1, \max(X_2, X_3))$	$(0, 2, -1)$	$(1, 1, -1)$
6	$X_{2:3}$ (2-out-of-3)	$(0, 3, -2)$	$(0, 3, -2)$
7	$\max(X_1, \min(X_2, X_3))$	$(1, 1, -1)$	$(0, 2, -1)$
8	$X_{3:3} = \max(X_1, X_2, X_3)$ (3-parallel)	$(3, -3, 1)$	$(0, 0, 1)$
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$ (series)	$(0, 0, 0, 1)$	$(4, -6, 4, -1)$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(0, 0, 2, -1)$	$(2, 0, -2, 1)$
11	$\min(X_{2:3}, X_4)$	$(0, 0, 3, -2)$	$(1, 3, -5, 2)$

Minimal and maximal signatures

i	T_i	**a**	**b**
12	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$(0,1,1,-1)$	$(1,2,-3,1)$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$(0,3,-3,1)$	$(1,0,1,-1)$
14	$X_{2:4}$ (3-out-of-4)	$(0,0,4,-3)$	$(0,6,-8,3)$
15	$\max(\min(X_1, X_2), \min(X_1, X_3, X_4),$ $\min(X_2, X_3, X_4))$	$(0,1,2,-2)$	$(0,5,-6,2)$
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	$(0,2,0,-1)$	$(0,4,-4,1)$
17	$\max(\min(X_1, X_2), \min(X_1, X_3),$ $\min(X_2, X_3, X_4))$	$(0,2,0,-1)$	$(0,4,-4,1)$
18	$\max(\min(X_1, X_2), \min(X_2, X_3),$ $\min(X_3, X_4))$	$(0,3,-2,0)$	$(0,3,-2,0)$
19	$\max(\min(X_1, \max(X_2, X_3, X_4)),$ $\min(X_2, X_3, X_4))$	$(0,3,-2,0)$	$(0,3,-2,0)$
20	$\min(\max(X_1, X_2), \max(X_1, X_3),$ $\max(X_2, X_3, X_4))$	$(0,4,-4,1)$	$(0,2,0,-1)$
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	$(0,4,-4,1)$	$(0,2,0,-1)$

Minimal and maximal signatures

i	T_i	**a**	**b**
22	$\min(\max(X_1, X_2), \max(X_1, X_3, X_4), \max(X_2, X_3, X_4))$	$(0, 5, -6, 2)$	$(0, 1, 2, -2)$
23	$X_{3:4}$ (2-out-of-4)	$(0, 6, -8, 3)$	$(0, 0, 4, -3)$
24	$\max(X_1, \min(X_2, X_3, X_4))$	$(1, 0, 1, -1)$	$(0, 3, -3, 1)$
25	$\max(X_1, \min(X_2, X_3), \min(X_3, X_4))$	$(1, 2, -3, 1)$	$(0, 1, 1, -1)$
26	$\max(X_{2:3}, X_4)$	$(1, 3, -5, 2)$	$(0, 0, 3, -2)$
27	$\min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4))$	$(2, 0, -2, 1)$	$(0, 0, 2, -1)$
28	$X_{4:4} = \max(X_1, X_2, X_3, X_4)$ (4-parallel)	$(4, -6, 4, -1)$	$(0, 0, 0, 1)$

Comparisons of systems with ID components

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- $T_1 \le_{LR} T_2$ for all F iff $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.

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- $T_1 \leq_{RHR} T_2$ for all F iff q_2/q_1 increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F iff $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.
- $T_1 \leq_{MRL} T_2$ for all F such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.

Comparisons of systems with ID. Example 1.

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- $X_{1:2} \leq_{ST} X_i \leq_{ST} X_{2:2}$ holds for all F and all $\widehat{C}$.

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- $X_{1:2} \leq_{HR} X_i$ holds for all F iff the ratio

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is increasing in $(0, 1)$.

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$$\frac{\bar{q}_{2:2}(u)}{\bar{q}_i(u)} = \frac{2u - \widehat{C}(u,u)}{u}$$

is decreasing in $(0,1)$.

Comparisons of systems with ID. Example 1.

- If the components are IID, that is, $\widehat{C}(u,v)=uv$, then $\widehat{C}(u,u)/u=u$ is increasing and so

$$X_{1:2} \le_{HR} X_i \le_{HR} X_{2:2}\ \forall F. \tag{2.1}$$

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$$X_{1:2}\le_{HR} X_i \le_{HR} X_{2:2}\ \forall F. \quad (2.1)$$

- If the components are ID with the Clayton copula

$$\widehat{C}(u,v)=\frac{uv}{u+v-uv}$$

(positive dependence), then

$$\frac{\widehat{C}(u,u)}{u}=\frac{u^2}{2u^2-u^3}=\frac{1}{2-u}$$

which is increasing in $(0,1)$. So (2.1) holds for all F.

Comparisons of DD. Example 1.

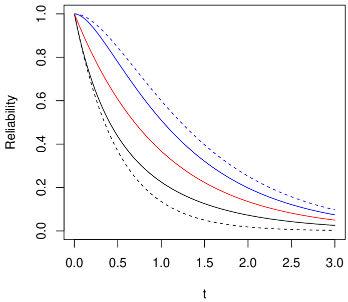

Figure: Reliability (left) and hazard rate (right) functions for $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for the case of IID (dashed lines) or ID components with a Clayton survival copula (continuous lines).

Comparisons of systems with ID. Example 1.

- Note that $X_{1:2}^{IID} \leq_{HR} X_{1:2}^{C}$ $(\geq_{HR})$ holds for all F iff

$$\frac{\widehat{C}(u,u)}{u^2} = \frac{u^2}{2u^2 - u^3} = \frac{1}{u(2-u)}$$

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- As it is decreasing, $X_{1:2}^{IID} \leq_{HR} X_{1:2}^{C}$ holds for all F.
- Analogously $X_{2:2}^{IID} \geq_{HR} X_{2:2}^{C}$ holds for all F since

$$\frac{2u - \widehat{C}(u,u)}{2u - u^2} = \frac{2u - \frac{u}{2-u}}{2u - u^2} = \frac{3-u}{(2-u)^2}$$

is increasing in $(0,1)$.

Comparisons of systems with ID. Example 1.

- Note that $X_{1:2} \leq_{LR} X_i$ holds for all abs. cont. F iff $1/\bar{q}'_{1:2}(u)$ is decreasing in $(0,1)$, that is, $\bar{q}_{1:2}$ is convex.

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- Note that $X_{1:2} \leq_{LR} X_i$ holds for all abs. cont. F iff $1/\bar{q}'_{1:2}(u)$ is decreasing in $(0,1)$, that is, $\bar{q}_{1:2}$ is convex.
- In the IID case $\bar{q}(u) = u^2$ is convex, and so this order holds for all abs. cont. F.

Comparisons of systems with ID. Example 1.

- Note that $X_{1:2} \leq_{LR} X_i$ holds for all abs. cont. F iff $1/\bar{q}'_{1:2}(u)$ is decreasing in $(0,1)$, that is, $\bar{q}_{1:2}$ is convex.
- In the IID case $\bar{q}(u) = u^2$ is convex, and so this order holds for all abs. cont. F.
- In the ID case with this Clayton copula $\bar{q}(u) = u/(2-u)$ is convex, and so this order holds for all abs. cont. F and this copula.

Comparisons of systems with ID.

Proposition

Let X_1 and X_2 be the lifetimes of two components having a common distribution function F and copula and survival copulas C and $\widehat{C}$, respectively. Then the following properties are equivalent:

(i) $X_{1:2} \leq_{HR} X_1$ *for all* F*;*

(ii) $X_1 \leq_{HR} X_{2.2}$ *for all* F*;*

(iii) $X_{1:2} \leq_{HR} X_{2:2}$ *for all* F*;*

(iv) $\widehat{C}(u,u)/u$ *is increasing in* $(0,1)$*;*

(v) $(1 - C(u,u))/(1-u)$ *is increasing in* $(0,1)$*.*

Comparisons of systems with ID.

Proposition

Let X_1 and X_2 be the lifetimes of two components having a common absolutely continuous distribution function F and copula and survival copulas C and $\widehat{C}$, respectively. Then the following properties are equivalent:

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(ii) $X_1 \leq_{LR} X_{2.2}$ *for all* F*;*

(iii) $X_{1:2} \leq_{LR} X_{2:2}$ *for all* F*;*

(iv) $\widehat{C}(u,u)$ *is convex in* $(0,1)$.

(v) $C(u,u)$ *is convex in* $(0,1)$.

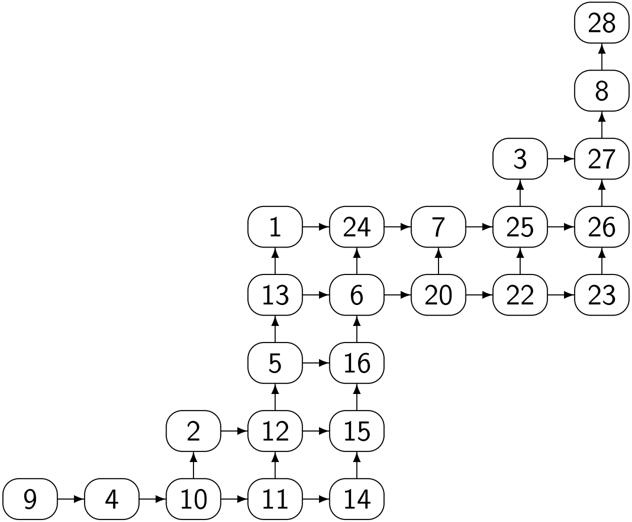


Figure: All ST orderings for the systems in Table 1 (IID case).

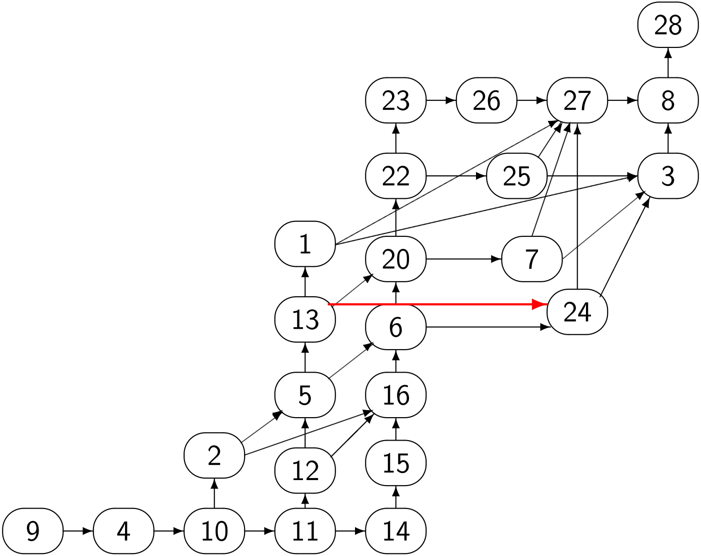
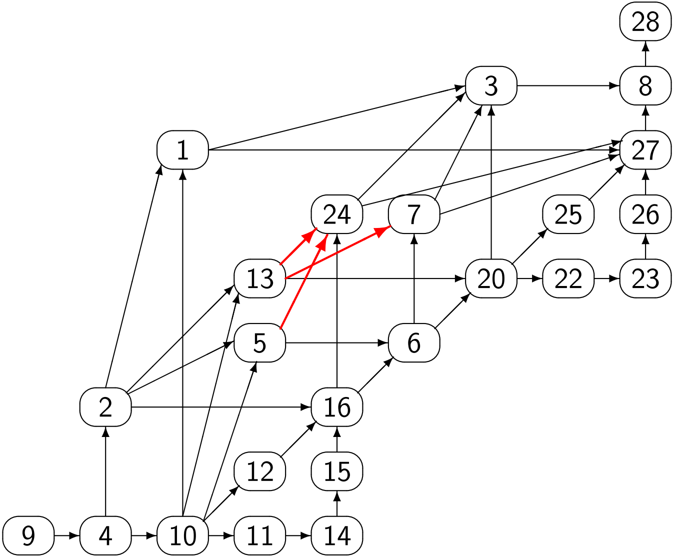


Figure: All the HR orders for the systems in Table 1 (IID case).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Comparisons for systems with non-ID components

Theorem (Navarro and del Águila (2017))

If T_i *has DF* $F_{T_i} = Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, *then:*

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- $T_1 \leq_{HR} T_2$ for all $F_1, \ldots, F_n$ iff $\bar{Q}_2 / \bar{Q}_1$ is decreasing in $(0, 1)^n$.

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- $T_1 \leq_{HR} T_2$ for all $F_1, \dots, F_n$ iff $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^n$.
- $T_1 \leq_{RHR} T_2$ for all $F_1, \dots, F_n$ iff Q_2/Q_1 is increasing in $(0,1)^n$.

Comparisons for systems with ordered components

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- $T_1 \leq_{HR} T_2$ for all $F_1 \geq_{HR} \cdots \geq_{HR} F_n$ iff the function

$$\bar{H}(v_1, \dots, v_n) = \frac{\bar{Q}_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{\bar{Q}_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \qquad (2.2)$$

is decreasing in $(0,1)^n$;

Comparisons for systems with ordered components

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is decreasing in $(0,1)^n$;
- $T_1 \leq_{RHR} T_2$ for all $F_1 \leq_{RHR} \cdots \leq_{RHR} F_n$ iff the function

$$H(v_1, \dots, v_n) = \frac{Q_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{Q_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \qquad (2.3)$$

is increasing in $(0,1)^n$.

Comparisons of systems. Example 2.

- $X_1, X_2 \sim C, \widehat{C}, F_1, F_2.$

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- Does $X_{1:2} \leq_{HR} X_1$ hold for all F_1, F_2?
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- Does $X_{1:2} \leq_{HR} X_1$ hold for all F_1, F_2?
- It holds iff $\widehat{C}(u, v)/u$ is increasing in $(0, 1)^2$.
- For the Clayton survival copula

$$\frac{\widehat{C}(u, v)}{u} = \frac{v}{u + v - uv}$$

is decreasing in u and increasing in v.

Comparisons of systems. Example 2.

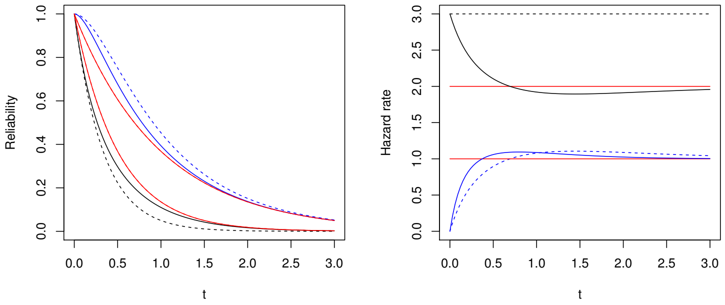

Figure: Reliability (left) and hazard rate functions (right) for $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for IND (dashed lines) and dependent (continuous lines) components with a Clayton survival copula.

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- It holds for the Clayton copula since

$$\bar{H}_1(v_1, v_2) = \frac{v_1(v_1 + v_1 v_2 - v_1^2 v_2)}{v_1^2 v_2} = \frac{1 + v_2 - v_1 v_2}{v_2},$$

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Comparisons of systems. Example 2.

- Does $X_{1:2} \leq_{HR} X_2$ hold for all $F_1 \geq_{HR} F_2$?
- It holds iff the function

$$\bar{H}_2(v_1, v_2) = \frac{\bar{Q}_2(v_1, v_1 v_2)}{\bar{Q}_{1:2}(v_1, v_1 v_2)} = \frac{v_1 v_2}{\widehat{C}(v_1, v_1 v_2)}$$

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- Does $X_{1:2} \leq_{HR} X_2$ hold for all $F_1 \geq_{HR} F_2$?
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is decreasing in $(0,1)^2$.

- It does not hold for the Clayton copula since

$$\bar{H}_2(v_1, v_2) = \frac{v_1 v_2 (v_1 + v_1 v_2 - v_1^2 v_2)}{v_1^2 v_2} = 1 + v_2 - v_1 v_2,$$

is decreasing in v_1 and increasing in v_2 in $(0,1)^2$.

Comparisons of systems. Example 2.

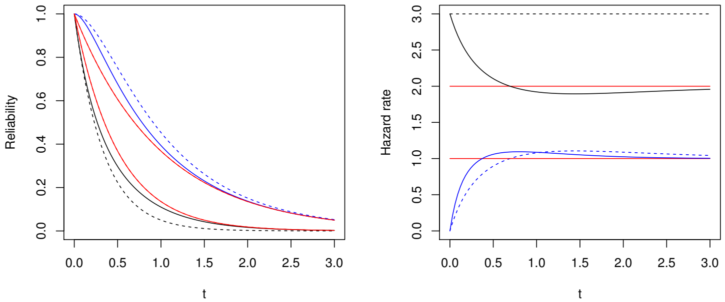


Figure: Reliability (left) and hazard rate functions (right) for $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for the case of IND components (dashed lines) and dependent (continuous lines) components a Clayton survival copula.

Systems with IND components

Table: Dual distortions functions of coherent systems with 1-3 IND components.

N	$T = \psi(X_1, X_2, X_3)$	$\overline{Q}(u_1, u_2, u_3)$
1	$X_{1:3} = \min(X_1, X_2, X_3)$	$u_1 u_2 u_3$
2	$\min(X_2, X_3)$	$u_2 u_3$
3	$\min(X_1, X_3)$	$u_1 u_3$
4	$\min(X_1, X_2)$	$u_1 u_2$
5	$\min(X_3, \max(X_1, X_2))$	$u_1 u_3 + u_2 u_3 - u_1 u_2 u_3$
6	$\min(X_2, \max(X_1, X_3))$	$u_1 u_2 + u_2 u_3 - u_1 u_2 u_3$
7	$\min(X_1, \max(X_2, X_3))$	$u_1 u_2 + u_1 u_3 - u_1 u_2 u_3$
8	X_3	u_3

Systems with IND components

N	$T=\psi(X_1,X_2,X_3)$	$\overline{Q}(u_1,u_2,u_3)$
9	X_2	u_2
10	X_1	u_1
11	$X_{2:3}$	$u_1u_2+u_1u_3+u_2u_3-2u_1u_2u_3$
12	$\max(X_3,\min(X_1,X_2))$	$u_3+u_1u_2-u_1u_2u_3$
13	$\max(X_2,\min(X_1,X_3))$	$u_2+u_1u_3-u_1u_2u_3$
14	$\max(X_1,\min(X_2,X_3))$	$u_1+u_2u_3-u_1u_2u_3$
15	$\max(X_2,X_3)$	$u_2+u_3-u_2u_3$
16	$\max(X_1,X_3)$	$u_1+u_3-u_1u_3$
17	$\max(X_1,X_2)$	$u_1+u_2-u_1u_2$
18	$X_{3:3}=\max(X_1,X_2,X_3)$	$u_1+u_2+u_3-u_1u_2-u_1u_3-u_2u_3$ $+u_1u_2u_3$

Systems with IND components

Table: Relationships for the ST order between the coherent systems with independent components given in Table 2. The value 2 indicates that $T_i \leq_{ST} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column). The value 1 indicates that $T_i \leq_{ST} T_j$ holds for all $F_1 \geq_{ST} F_2 \geq_{ST} F_3$. It also indicates that $T_i \leq_{ST} T_j$ does not hold for all F_1, F_2, F_3. The value 0 indicates that $T_i \leq_{ST} T_j$ does not hold for all $F_1 \geq_{ST} F_2 \geq_{ST} F_3$.

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	2	2	1	2	2	1	2	2	2	2	2	2	2	2
3	0	2	1	2	1	2	2	1	2	2	2	2	2	2	2	2	2
4	0	0	2	0	2	2	0	2	2	2	2	2	2	2	2	2	2
5	0	0	0	2	1	1	2	1	1	2	2	2	2	2	2	2	2
6	0	0	0	0	2	1	0	2	1	2	2	2	2	2	2	2	2
7	0	0	0	0	0	2	0	0	2	2	2	2	2	2	2	2	2

Systems with IND components

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	0	0	0	0	0	0	2	1	1	0	2	1	1	2	2	1	2
9	0	0	0	0	0	0	0	2	1	0	0	2	1	2	1	2	2
10	0	0	0	0	0	0	0	0	2	0	0	0	2	0	2	2	2
11	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
12	0	0	0	0	0	0	0	0	0	0	2	1	1	2	2	1	2
13	0	0	0	0	0	0	0	0	0	0	0	2	1	2	1	2	2
14	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2	2	2
15	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	2
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	2
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

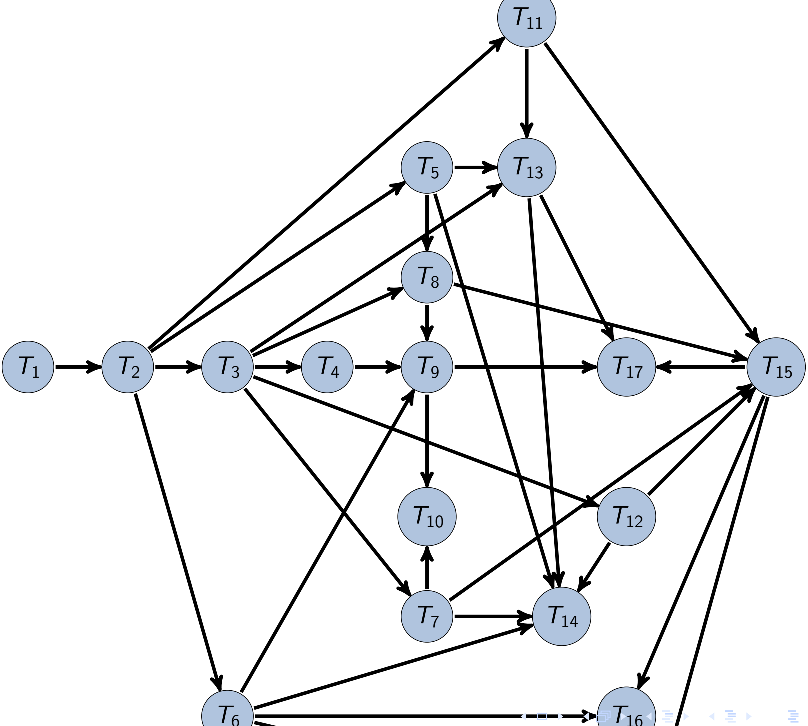
Systems with IND components

Table: Relationships for the HR order between the coherent systems with independent components given in Table 2. The value 2 indicates that $T_i \leq_{HR} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column). The value 1 indicates that $T_i \leq_{HR} T_j$ holds for all $F_1 \geq_{HR} F_2 \geq_{HR} F_3$. It also indicates that $T_i \leq_{HR} T_j$ does not hold for all F_1, F_2, F_3. The value 0 means that $T_i \leq_{HR} T_j$ does not hold for all $F_1 \geq_{HR} F_2 \geq_{HR} F_3$.

HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2	1	1	1	1	1	2	1	1	1
3	0	2	1	0	0	1	2	1	2	0	1	1	1	1	2	1	1
4	0	0	2	0	0	0	0	2	2	0	0	0	0	0	0	2	0
5	0	0	0	2	0	0	2	1	1	0	0	1	1	1	1	2	2
6	0	0	0	0	2	0	0	2	1	0	0	0	1	0	2	1	2
7	0	0	0	0	0	2	0	0	2	0	0	0	1	2	1	1	2

Systems with IND components

HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	0	0	0	0	0	0	2	1	1	0	0	0	0	1	1	1	1
9	0	0	0	0	0	0	0	2	1	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	2	0	1	1	2	2	2	2
12	0	0	0	0	0	0	0	0	0	0	2	0	1	1	1	1	1
13	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	1	0
14	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2


T_1
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T_{17}

A Parrondo paradox in reliability

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A Parrondo paradox in Reliability

- The respective reliability functions are

$$\bar{F}_T(t) = \Pr(X_1 > t, X_2 > t) = \bar{F}_1(t)\bar{F}_2(t) = \bar{Q}_T(\bar{F}_1(t), \bar{F}_2(t)),$$

$$\bar{F}_S(t) = \Pr(Y_1 > t, Y_2 > t) = \left(\frac{1}{2}\bar{F}_1(t) + \frac{1}{2}\bar{F}_2(t)\right)^2 = \bar{Q}_S(\bar{F}_1(t), \bar{F}_2(t)),$$

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- Hence $T \leq_{ST} S$ for all $\bar{F}_1, \bar{F}_2$.

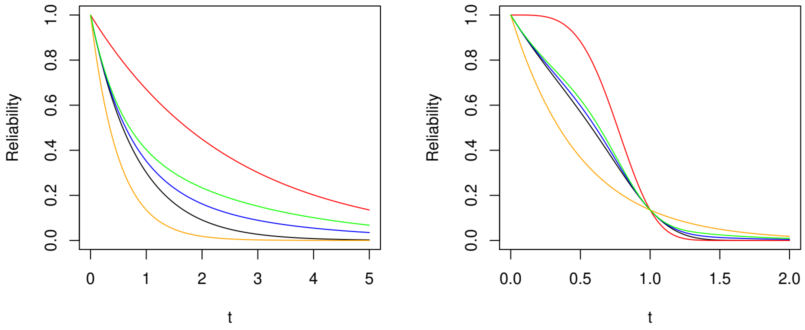

Figure: Reliability functions for the series systems T (black) and S (blue) in Parrondo's paradox for exponential (left) and Weibull (right) distributions.

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- What happen in other system structures?
- Do these properties hold when the components are dependent?
- The answers to these questions were obtained in Navarro and Spizzichino (2010) and they are based on the notions of Schur-concave and weakly Schur-concave functions.

A Parrondo paradox in Reliability

Definition (Durante and Papini (2007))

A function $g : \mathbb{R}^n \to \mathbb{R}$ is weakly Schur-concave (convex) if

$$g(u_1, \dots, u_n) \leq g(\bar{u}, \dots, \bar{u}) \ (\geq)$$

for all $(u_1, \dots, u_n)$, where $\bar{u} = (u_1 + \cdots + u_n)/n$.

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for all $u_1, \dots, u_n, v_1, \dots, v_n$ such that $u_1 + \cdots + u_n = v_1 + \cdots + v_n$ and such that $\sum_{i=1}^{j} u_{i:n} \leq \sum_{i=1}^{j} v_{i:n}$ where $u_{i:n}$ and $v_{i:n}$ are the ordered values obtained from the respective vectors.

A Parrondo paradox in Reliability

Theorem (Navarro and Spizzichino (2010))

Let $\bar{Q}$ be the dual distortion function of a system. The Parrondo paradox holds (is reverted) for this system if and only if $\bar{Q}$ is weakly Schur-concave (convex).

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- For this survival copula, the Parrondo paradox is reverted in this series system with dependent components.

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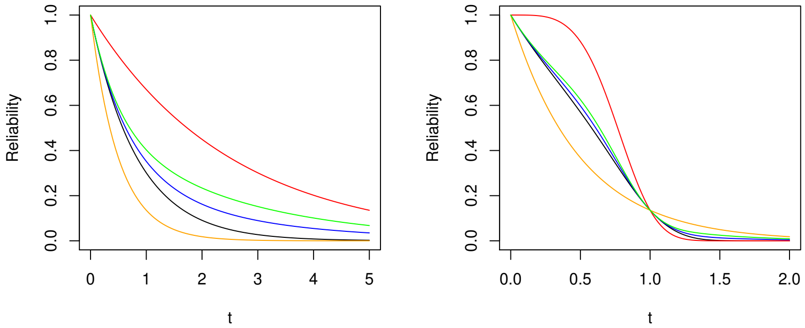

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- What to do in practice?

Main aging classes

- $X \geq 0$ (lifetime).
- $X_t = (X - t|X > t)$ (residual lifetime) for $t \geq 0$.
- X is Increasing (Decreasing) Failure Rate, IFR (DFR), if $X_s \geq_{ST} X_t$ ($\leq_{ST}$) for all $0 \leq s \leq t$ (or h_X increases).
- X is New Better (Worse) than Used, NBU (NWU), if $X \geq_{ST} X_t$ ($\leq_{ST}$) for all $t \geq 0$.
- X is Increasing (Decreasing) Failure Rate Average, IFRA (DFRA), if $A(t) = \frac{1}{t}\int_0^t h(x)dx = -\frac{1}{t}\ln \bar{F}(t)$ is increasing (decreasing) (or $\bar{F}(ct) \geq \bar{F}^c(t)$, $0 < c < 1$) for all $t \geq 0$.
- X is Increasing (Decreasing) Likelihood Ratio, ILR (DLR), if $X_s \geq_{LR} X_t$ ($\leq_{ST}$) for all $0 \leq s \leq t$ (or f is logconcave).

Main among the main aging classes

ILR	$\Rightarrow$	IFR	$\Rightarrow$	IFRA	$\Rightarrow$	NBU
DLR	$\Rightarrow^*$	DFR	$\Rightarrow$	DFRA	$\Rightarrow$	NWU

Table: Relationships among the main aging classes (* when the support is (a, ∞).).

Distorted distributions

Theorem

Let $F_q = q(F)$ and $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$. Then:

- *The IFR (DFR) class is preserved by q iff α is decreasing (increasing) for $u \in (0,1)$.*
- *The NBU (NWU) class is preserved by q iff $\bar{q}$ is submultiplicative (supermultiplicative), that is,*

$$\bar{q}(uv) \leq \bar{q}(u)\bar{q}(v), \ (\geq) \text{ for all } u, v \in [0,1]. \tag{3.1}$$

- *The IFRA (DFRA) class is preserved by q iff $\bar{q}$ satisfies*

$$\bar{q}(u^c) \geq (\bar{q}(u))^c, \ (\leq) \text{ for all } u, c \in [0,1]. \tag{3.2}$$

Preservation in systems with IID

- If the IFR class is preserved, then the NBU and IFRA classes are also preserved.
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- The IFR class in preserved in k-out-of-n systems with IID components (Barlow and Proschan (1975)) and the DFR not.
- The NBU class is preserved in all the coherent systems with IID components (Barlow and Proschan (1975)).

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$$\alpha_1(u) = \frac{u\bar{q}_1'(u)}{\bar{q}_1(u)} = \frac{4-3u}{2-u}.$$

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- As α_1 is strictly decreasing, then IFR, NBU and IFRA classes are preserved and DFR, NWU and DFRA are not.

Preservation of IFR/DFR. Example 1.

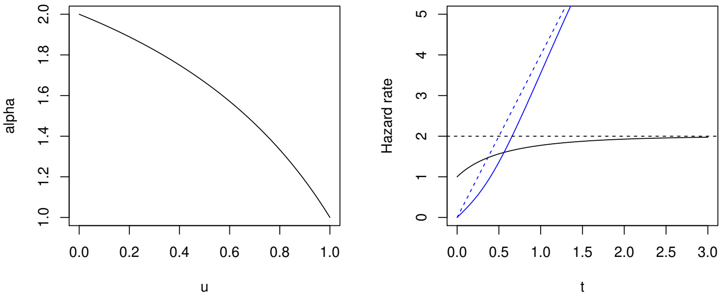

Figure: Alpha (left) and hazard rate (right) functions of T_1 (continuous lines) for an exponential distribution with $h(t) = 1$ (black) and a Weibull distribution with $h(t) = 2t$ for $t \geq 0$. The dashed lines are $2h(t)$.

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- X_1, X_2, X_3 IID$\sim F$ and $T_2 = \max(X_1, \min(X_2, X_3))$.
- Then $\bar{q}_2(u) = u + u^2 - u^3$ for $u \in [0,1]$.
- Is the IFR (DFR) class preserved?
- For this system

$$\alpha_2(u) = \frac{u\bar{q}_2'(u)}{\bar{q}_2(u)} = \frac{1 + 2u - 3u^2}{1 + u - u^2}.$$

- As α_2 is strictly increasing and then decreasing, the IFR and DFR, NWU are not preserved. The NBU and IFRA are preserved.

Preservation of IFR/DFR. Example 2.

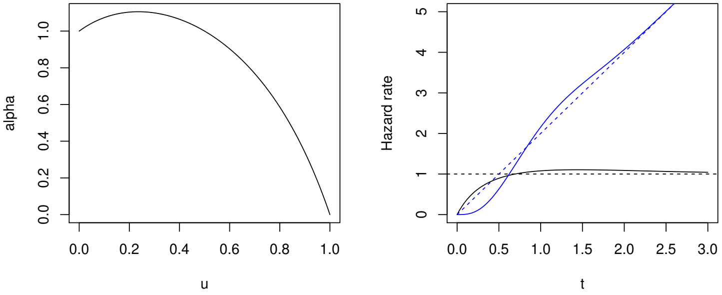

Figure: Alpha function (left) and hazard rate (right) functions of T_2 (continuous lines) for an exponential distribution with $h(t) = 1$ (black) and a Weibull distribution with $h(t) = 2t$ (blue) for $t \geq 0$.

Example 3.

- We consider the series and parallel systems with dependent ID components (X_1, X_2) having the Clayton survival copula

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- $\alpha_{1:2}$ is increasing and $\alpha_{2:2}$ is decreasing.
- The IFR class is preserved in $X_{2:2}$ but it is not preserved in $X_{1:2}$ and the opposite for the DFR class.

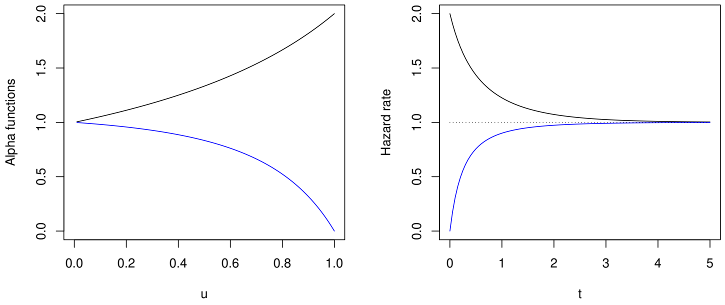

Figure: Alpha functions (left) and hazard rate functions for the series (black) and parallel (blue) systems with a Clayton survival copula. The dotted line represents the hazard rate of the components.

Generalized distorted distributions

Theorem

Let $F_Q = Q(F_1, \dots, F_n)$, $\mathbf{u} = (u_1, \dots, u_n)$ *and* $\alpha_i(\mathbf{u}) = u_i \partial_i \bar{Q}(\mathbf{u})/\bar{Q}(\mathbf{u})$. *Then:*

- *If* $\alpha_1, \dots, \alpha_n$ *are decreasing (increasing) for* $u_1, \dots, u_n \in (0,1)$ *and* $i = 1, \dots, n$, *then the IFR (DFR) class is preserved.*
- *The NBU (NWU) class is preserved by* Q *if* $\bar{Q}$ *is submultiplicative (supermultiplicative), that is,*

$$\bar{Q}(u_1 v_1, \dots, u_n v_n) \le \bar{Q}(u_1, \dots, u_n)\bar{Q}(v_1, \dots, v_n),\ (\ge)\ u_i, v_i \in [0,1].$$

- *The IFRA (DFRA) class is preserved by* Q *if* $\bar{Q}$ *satisfies*

$$\bar{Q}(u_1^c, \dots, u_n^c) \ge (\bar{Q}(u_1, \dots, u_n))^c,\ (\le)\ u_i, c \in [0,1].$$

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- If the IFR class is preserved, then the NBU and IFRA classes are also preserved.

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- The IFR and DFR classes are not always preserved.

Example 4.

- X_1, X_2 IND$\sim F_1, F_2$, $X_{1:2} = \min(X_1, X_2)$.

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- Then $\bar{Q}_{1:2}(u_1, u_2) = u_1 u_2$ and

$$\alpha_1(u_1, u_2) = u_1 \frac{\partial_1 \bar{Q}_{1:2}(u_1, u_2)}{\bar{Q}_{1:2}(u_1, u_2)} = 1$$

and $\alpha_2(u_1, u_2) = 1$.

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  and $\alpha_2(u_1, u_2) = 1$.
- As α_i are constant, all the aging classes IFR, NBU, IFRA, DFR, NWU, and DFRA are preserved.
- If $X_{2:2} = \max(X_1, X_2)$, then $\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$ and

$$\alpha_1(u_1, u_2) = u_1 \frac{\partial_1 \bar{Q}_{2:2}(u_1, u_2)}{\bar{Q}_{2:2}(u_1, u_2)} = \frac{u_1(1 - u_2)}{u_1 + u_2 - u_1 u_2}$$

  is not monotone. IFR and DFR are not preserved because

Preservation of IFR/DFR. Example 4.

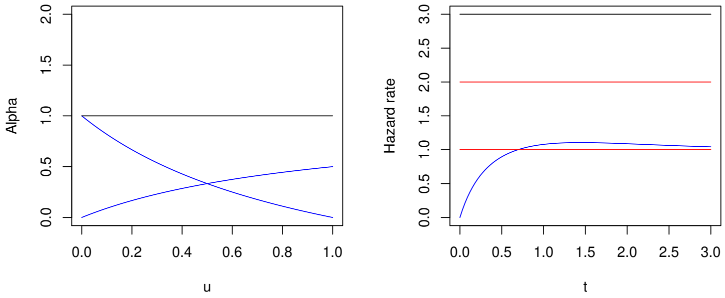

Figure: Alpha function $\alpha_1(0.5, u)$ and $\alpha_1(u, 0.5)$ (left) and hazard rate (right) functions of $X_{1:2}$ (black), X_i (red) and $X_{2:2}$ (blue) for two exponential distributions.

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- A straightforward calculation show that $\bar{Q}$ is submultiplicative.

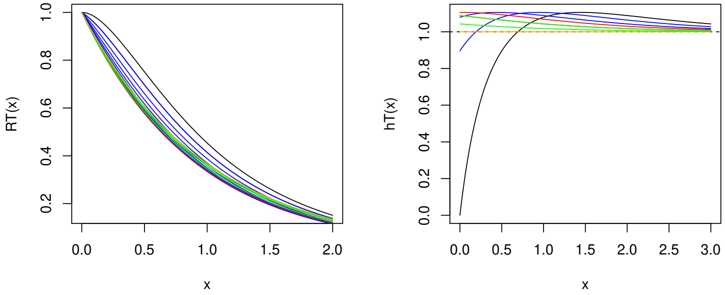

Figure: Reliability $\bar{F}_{T_t}$ (left) and hazard rate h_T (right) functions of T_2 (continuous lines) for an exponential distribution with $h(t) = 1$ (black) and $t = 0.1, 0.2, \ldots, 1$ (blue), 1.4 (red), $2, 3, 4, 5$ (green) and 10 (orange).

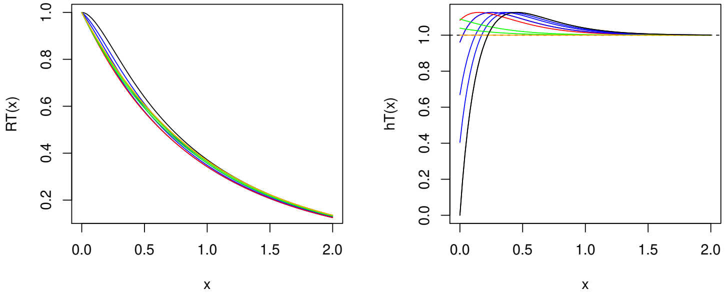

Figure: Reliability $\bar{F}_{T_t}$ (left) and hazard rate h_T (right) functions of T_2 (continuous lines) for three exponential distribution with $h(t) = 1, 2, 3$ (black) and $t = 0.05, 0.1, 0.2$ (blue), 0.3 (red), $0.7, 1$ (green) and 2 (orange).

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The slides and more references can be seen in my webpage:

https : //webs.um.es/jorgenav/miwiki/doku.php

Exercises

1. Determine the minimal path and minimal cut sets of a coherent system with four components.
2. Compute the reliability of a coherent system with four components in the general case.
3. Compute the reliability of a coherent system with four components in the IND case.
4. Compute the reliability of a coherent system with four components in the ID case.
5. Compute the reliability of a coherent system with four components in the IID case.
6. Compute the reliability of a plane with four engines, two in each wing, that can fly if at least one engine works in each wing.

7. Compute the minimal and maximal signatures of a system with four components.
8. Check an arrow in the figures for the ST, HR and LR orders of systems with IID components
9. Check a no arrow in the figures for the ST, HR and LR orders of systems with IID components
10. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND components.
11. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND HR-ordered components.
12. Check if $X_i \leq_{HR} X_{2:2}$ holds for dependent components with the Clayton copula in the slides.
13. Check if $X_i \leq_{HR} X_{2:2}$ holds for HR-ordered components with the Clayton copula in the slides.
14. Check an arrow in the tables and figure for the ST and HR orders of systems with IND components.

15. Check a no arrow in the figures for the ST, HR and LR orders of systems with IID components
16. Check if the IFR class is preserved in a system with four IID components.
17. Check if the Parrondo paradox holds in a system with three IND components.

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- Questions?