Multivariate Distorted Distributions

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References

The conference is based on the following references:

- Navarro, Calì, Longobardi and Durante (2022). Distortion representations of multivariate distributions. Statistical Methods & Applications 31, 925–954.
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- Navarro and Buono (2022). Predicting future failure times by using quantile regression. To appear in Metrika. Published online first Sept 2022.
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Outline

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- Definitions
- Main properties
- Examples

Prediction of ordered paired data

- Representations
- Exact QR curves
- Estimated QR curves

Distorted distributions

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Definition

The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous *distortion function* $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is given by

$$F_q(t) = q(F(t)), \text{ for all } t \in \mathbb{R}. \tag{1.1}$$

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$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \text{ for all } t \in \mathbb{R}, \tag{1.2}$$

where $\bar{q}(u) := 1 - q(1 - u)$ is called the *dual distortion function*.

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  where $\bar{q}(u) := 1 - q(1-u)$ is called the *dual distortion function*.
- (1.1) and (1.2) are equivalent.

Examples of univariate distorted distributions

- Proportional Hazard Rate (PHR) Cox model $\bar{F}_\theta(t) = \bar{F}^\theta(t)$, where $\bar{q}(u) = u^\theta$ and $\theta > 0$.

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- Order statistics $X_{1:n}, \dots X_{n:n}$ from IID $X_1, \dots, X_n$:

$$\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t) = \bar{q}_{i:n}(\bar{F}(t)),$$

where $\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j}(1-u)^j u^{n-j}$.

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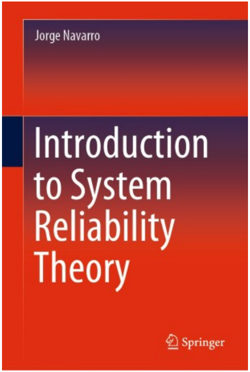
  where $\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$.
- Coherent system lifetimes T:

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)), \tag{1.3}$$

  where $\bar{Q} : [0,1]^n \to [0,1]$ is a generalized distortion function, see e.g. Navarro (2022a).

Navarro (2022a)

- This is my new book on System Reliability Theory.



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$$\mathsf{F}(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)),$$

where $F_1, \dots, F_n$ are the marginals, $F_i(x_i) = \Pr(X_i \le x_i)$.

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where $F_1, \ldots, F_n$ are the marginals, $F_i(x_i) = \Pr(X_i \leq x_i)$.

- A similar representation holds for the joint survival function

$$\overline{\mathrm{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = \widehat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)).$$

Definition

Definition (Navarro, Calì, Longobardi and Durante (2022))

A multivariate distribution function F is said to be a *multivariate distorted distribution* (MDD) of the univariate distribution functions $G_1, \ldots, G_n$ if there exists a *distortion* function D such that

$$\mathrm{F}(x_1, \ldots, x_n) = D(G_1(x_1), \ldots, G_n(x_n)), \ \forall x_1, \ldots, x_n \in \mathbb{R}. \quad (1.4)$$

We write $\mathrm{F} \equiv MDD(G_1, \ldots, G_n)$, when F is a MDD of $G_1, \ldots, G_n$.

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A continuous function $D : [0,1]^n \to [0,1]$ is called *(n–dimensional) distortion function* (shortly written as $D \in \mathcal{D}_n$) if:

(i) $D(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n) = 0$ for all $u_1, \dots, u_n \in [0,1]$.

(ii) $D(1, \dots, 1) = 1$.

(iii) D is n–increasing, i.e. for all $\mathrm{x} = (x_1, \dots, x_n)$ and $\mathrm{y} = (y_1, \dots, y_n)$ with $x_i \le y_i$, it holds $\triangle_{\mathrm{x}}^{\mathrm{y}} D \ge 0$, where

$$\triangle_{(x_1,\dots,x_n)}^{(y_1,\dots,y_n)} D := \sum_{z_i \in \{x_i, y_i\}} (-1)^{1(z_1,\dots,z_n)} D(z_1, \dots, z_n),$$

with $1(z_1, \dots, z_n) = \sum_{i=1}^{n} 1(z_i = x_i)$ and $1(A) = 1$ (respectively, 0) if A is true (respectively, false).

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is a multivariate distribution function for all DF $G_1, \dots, G_n$.
- If $\mathrm{F} \equiv MDD(G_1, \dots, G_n)$, then

$$\overline{\mathrm{F}}(x_1, \dots, x_n) = \widehat{D}(\bar{G}_1(x_1), \dots, \bar{G}_n(x_n)), \qquad (1.5)$$

where $\bar{G}_i = 1 - G_i$ and $\widehat{D} \in \mathcal{D}_n$.

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- In particular, the ith marginal is

$$F_i(x_i) = D(1, \ldots, 1, G_i(x_i), 1, \ldots, 1) = D_i(G_i(x_i)), \qquad (1.6)$$

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  where $D_i(u) := D(1, \dots, 1, u, 1, \dots, 1)$ and the value u is placed at the ith position.
- Clearly, we have $G_i = F_i$ for a fixed $i \in \{1, \dots, n\}$ when $D_i(u) = u$ for all $u \in [0, 1]$.

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Proposition (Navarro, Calì, Longobardi and Durante (2022))

Let (X_1, X_2) *with* $\mathrm{F} \equiv MDD(G_1, G_2)$ *for* $D \in \mathcal{D}_2$*, then*

$$F_{2|1}(x_2|x_1) = D_{2|1}(G_2(x_2)|G_1(x_1)) \tag{1.7}$$

whenever $\lim_{v\to 0^+} \partial_1 D(G_1(x_1), v) = 0$*, where*

$$D_{2|1}(v|G_1(x_1)) = \frac{\partial_1 D(G_1(x_1), v)}{\partial_1 D(G_1(x_1), 1)}$$

for $0 < v < 1$ *and* x_1 *such that* $\partial_1 D(G_1(x_1), 1) > 0$*.*

Theoretical Quantile Regression

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- This quantile function $F_{2|1}^{-1}$ can be computed from (1.7) as

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$$F_{2|1}^{-1}(q|x_1) = G_2^{-1}(D_{2|1}^{-1}(q|G_1(x_1))),\ 0 < q < 1.$$

- Moreover, it can be used to obtain α-prediction bands for X_2

$$\left[F_{2|1}^{-1}(\beta_1|x_1), F_{2|1}^{-1}(\beta_2|x_1)\right]$$

taking $0 \le \beta_1 < \beta_2 \le 1$ such that $\beta_2 - \beta_1 = \alpha \in (0, 1)$.

Examples of MDD

- **Multivariate residual lifetimes**
  $\mathrm{X}_t = (X_1 - t, \dots, X_n - t | X_1 > t, \dots, X_n > t)$, see Navarro, Calì, Longobardi and Durante (2022).

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- **Coherent systems with ID components**, see Navarro, Calì, Longobardi and Durante (2022); Navarro et al. (2023).

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- **Order statistics** (k-out-of-n systems with ID components), see Navarro and Buono (2022).
- **Record values**, see Navarro (2022b).

Prediction of ordered paired data

Representations of ordered paired data

- Let us assume that X and Y are ID, that is, they have a common absolutely continuous distribution function F. Then

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- We want to predict $U = \max(X, Y)$ from $L = \min(X, Y)$.
- To this purpose we need the conditional distribution function

$$G_{2|1}(s|t) := \Pr(U \leq s|L = t), \ s \geq t.$$

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- For other individuals, we just know $L = \min(X, Y)$ and we want to estimate $U = \max(X, Y)$.
- So, we want to obtain a MDD representation for the random vector (L, U) in terms of F and C.
- It can be used to compute $G^{-1}_{2|1}(q|t)$, the median regression curve $m(t)$ and the associated prediction bands.

Representations of ordered paired data

- The joint distribution function $\mathrm{G}(x, y) = \Pr(L \leq x, U \leq y)$ of (L, U) can be written as

$$\mathrm{G}(x, y) = \begin{cases} C(F(y), F(y)) & \text{for} \quad y \leq x; \\ 2C(F(x), F(y)) - C(F(x), F(x)) & \text{for} \quad y > x. \end{cases}$$

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- Therefore, $\mathrm{G}\equiv MDD(F,F)$, that is,

$$\mathrm{G}(x,y)=D(F(x),F(y)) \tag{2.1}$$

with the following distortion function

$$D(u,v)=\begin{cases} C(v,v) & \text{for} \quad v\leq u; \\ 2C(u,v)-C(u,u) & \text{for} \quad u<v. \end{cases} \tag{2.2}$$

Representations of ordered paired data

- Then the marginal distributions of (L, U) can be written as

$$G_1(x) := \Pr(L \leq x) = D(F(x), 1) = D_1(F(x)),$$

$$G_2(y) := \Pr(U \leq y) = D(1, F(y)) = D_2(F(y)),$$

where

$$D_1(u) = D(u, 1) = 2u - C(u, u)$$

and

$$D_2(v) = D(1, v) = C(v, v)$$

for all $u, v \in [0, 1]$.

Representations of ordered paired data, IID case

- If X and Y are independent, then the distortion function is

$$D(u,v)=\begin{cases} v^2 & \text{for} \quad v\leq u; \\ 2uv-u^2 & \text{for} \quad u<v. \end{cases} \tag{2.3}$$

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- Note that D is not a copula and that the marginals G_1 and G_2 of G do not appear in (2.1) (we use F instead).

Representations of ordered paired data

- From (1.7) and (2.1), the distribution function of $(U|L = x)$ is

$$G_{2|1}(y|x) = D_{2|1}(F(y)|F(x)) \tag{2.4}$$

for $y \geq x$, where

$$D_{2|1}(v|F(x)) := \frac{\partial_1 D(F(x), v)}{\partial_1 D(F(x), 1)}$$

and $\partial_1 D(u, v) = 2\partial_1 C(u, v) - 2\partial_1 C(u, u)$, for $0 < u \leq v \leq 1$.

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and $\partial_1 D(u,v) = 2\partial_1 C(u,v) - 2\partial_1 C(u,u)$, for $0 < u \leq v \leq 1$.

- In the IID case, $\partial_1 D(u,v) = 2(v-u)$ for $u \leq v \leq 1$ and

$$D_{2|1}(v|F(x)) = \frac{v - F(x)}{\bar{F}(x)}$$

for $F(x) \leq v \leq 1$.

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- Note that L_i and U_i are dependent.
- From (2.4), the distribution function of $(U|L = x)$ is

$$G_{2|1}(y|x) = D_{2|1}(F(y)|F(x)) \tag{2.5}$$

for $y \geq x$, where

$$D_{2|1}(v|F(x)) = \frac{v - F(x)}{\bar{F}(x)}$$

for $F(x) \leq v \leq 1$.

Example 1: IID exponential case

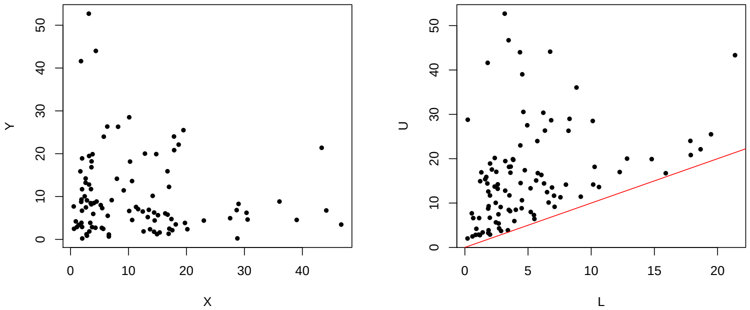

Figure: Independent data from two exponential distributions with mean $\mu = 10$ (left) and the associated ordered paired data (right).

Example 1: IID exponential case

- The quantile function $F_{2|1}^{-1}$ can be computed as

$$F_{2|1}^{-1}(q|x) = F^{-1}(D_{2|1}^{-1}(q|F(x)))$$

for $0 < v < 1$, where $D_{2|1}^{-1}(q|F(x)) = F(x) + q\bar{F}(x)$, when $\bar{F}(x) = \exp(-x/\mu)$ and $F^{-1}(y) = -\mu \log(1-y)$. Then

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- Therefore, the exact QR curve is

$$m(x) = x - \mu \log(0.5).$$

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- The exact QR lower 90% prediction band is

$$[x, x - \mu \log(0.10)] .$$

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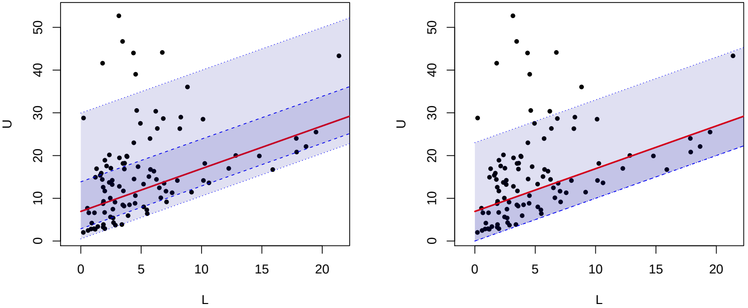

Figure: QR for the ordered paired data (L, U) associated to independent data (X, Y) from two exponential distributions with mean $\mu = 10$ jointly with 50% and 90% centered (left) or bottom (right) prediction bands.

Predictions

- The first ordered pair in our simulated sample is $L_1 = 10.15771$ and $U_1 = 14.17195$.

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- Let us consider now that (X, Y) are DID with a copula C and a common marginal distribution F.
- We consider again the exponential model

$$\bar{F}(t) = \exp(-t/\mu), \ \ t \geq 0$$

and the Clayton EXC copula

$$C(u, v) = \frac{uv}{u + v - uv}, \ \ (u, v) \in [0, 1]^2. \qquad (2.6)$$

Example 2: Dependent EXC exponential case

- To get the QR curves we need the distribution $G_{2|1}(y|x)$ of $(U|L = x)$. From (2.4) we need

$$\partial_1 D(u, v) = 2\partial_1 C(u, v) - 2\partial_1 C(u, u) = \frac{2v^2}{(u + v - uv)^2} - \frac{2}{(2 - u)^2}$$

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- To compute the inverse, we need to solve in y the equation $G_{2|1}(y|x) = q$ for $q \in (0,1)$.

Example 2: Dependent EXC exponential case

- This equation leads to

$$G_{2|1}^{-1}(q|x) = F^{-1}\left(\frac{F(x)}{F(x) - 1 + \frac{2-F(x)}{\sqrt{1-q+q(2-F(x))^2}}}\right).$$

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- For an exponential distribution with $\mu = 10$ we get

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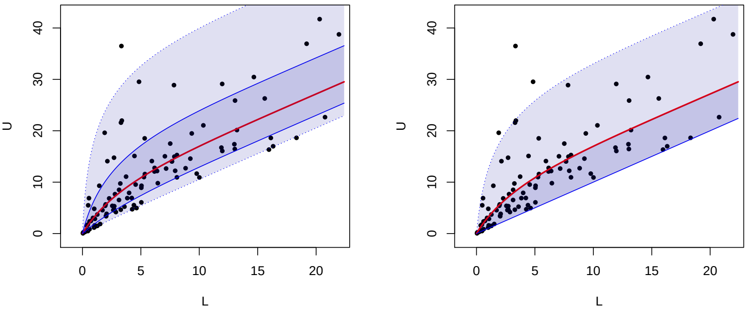

Figure: QR curves for paired ordered data (L, U) associated to dependent data (X, Y) from two exponential distributions with centered (left) and bottom (right) prediction bands.

Parametric QR curves

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- They can be both in F and/or in C.
- We can use the training sample (X_i, Y_i) from (X, Y) to estimate the unknown parameters.
- Then we can use the MDD representation with the estimated parameters to get the estimated QR curves.

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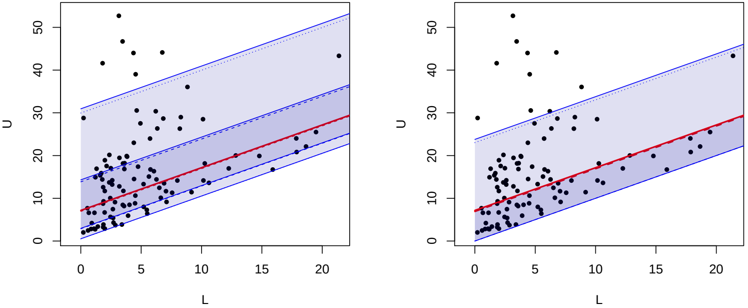
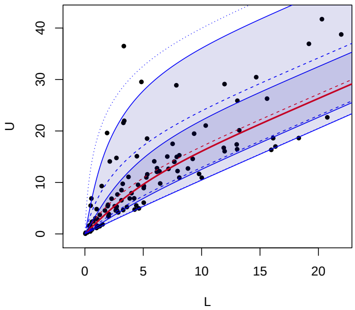


Figure: Parametric QR curves for (L, U) associated to IID data (X, Y) from an exponential distribution jointly with centered (left) and bottom (right) prediction bands. The dashed lines are the exact curves.

Example 2: Dependent EXC exponential case



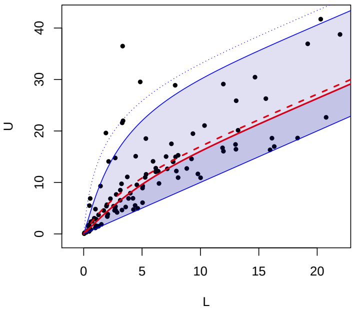


Figure: Parametric QR curves for (L, U) associated to data (X, Y) from an exponential distribution with unknown mean μ and a Clayton copula with unknown parameter θ. The dashed lines are the exact curves.

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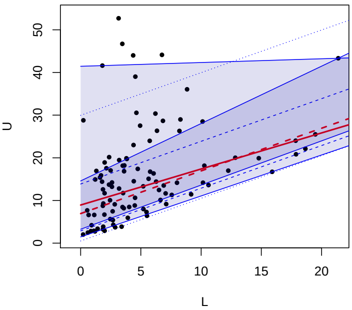
- We can also use the statistical program R with `library('quantreg')` to estimate the exact curves from the training sample (see Koenker, 2005).
- Here k denotes the degree of the polynomial used to estimate the QR curves.

Example 1: Non-parametric QR, IID exponential case



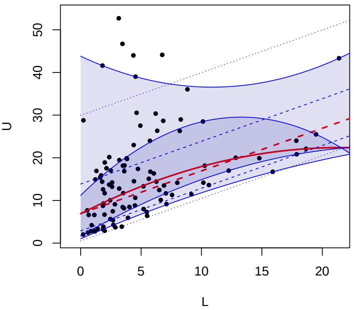


Figure: Non-parametric QR curves for paired ordered data (L, U) associated to IID data (X, Y) from an exponential distribution with $\mu = 10$ and $k = 1$ (left) or $k = 2$ (right).

Example 2: Non-parametric QR, DID exponential case

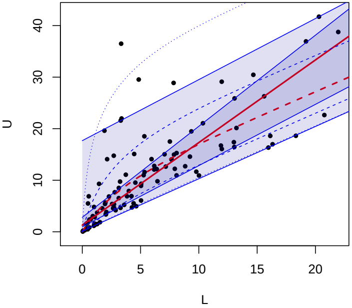


Figure: Non-parametric QR curves for (L, U) associated to data (X, Y) from an exponential distribution and a Clayton copula with $\theta = 1$ and $k = 1$ (left) or $k = 2$ (right). The dashed lines are the exact curves.

Conclusions

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- The representations based on distortions are very useful as an alternative to the classic copula representations !!

References

Awad AM, Raqab MZ (2000) Prediction intervals for the future record values from exponential distribution: comparative study. Journal of Statistical Computation and Simulation 65, 325–340.

Buono F, Navarro J (2022) Simulations and predictions of future values in the time homogeneous load-sharing model. Submitted.

Koenker R (2005). Quantile Regression. Cambridge University Press.

Navarro J (2022a). Introduction to System Reliability Theory. Springer.

Navarro, J (2022b). Prediction of record values by using quantile regression curves and distortion functions. Metrika 85, 675–706. DOI: 10.1007/s00184-021-00847-w.

Navarro J (2023). Predicting system failure times. In preparation.

Navarro J, Buono F (2022). Predicting future failure times by using quantile regression. To appear in Metrika. Published online first Sept 2022. DOI https://doi.org/10.1007/s00184-022-00884-z.

References

Navarro J, Calì C, Longobardi M, Durante F (2022). Distortion representations of multivariate distributions. Statistical Methods & Applications 31, 925–954. DOI: 10.1007/s10260-021-00613-2.

Navarro J, Pellerey F, Mulero J (2022). On sums of dependent random lifetimes under the time-transformed exponential model. To appear in Test. Published online first March 2022. DOI: 10.1016/j.physa.2022.127214.

Nelsen RB (2006). An introduction to copulas. Second edition. Springer, New York.

Nevzorov VB (2001) Records: Mathematical Theory. American Mathematical Society, Providence, RI.

Wang S (1996). Premium calculation by transforming the layer premium density. Astin Bulletin 26, 71–92.

Yaari ME (1987). The dual theory of choice under risk. Econometrica 55, 95–115.

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- Questions?