Distorted Distributions

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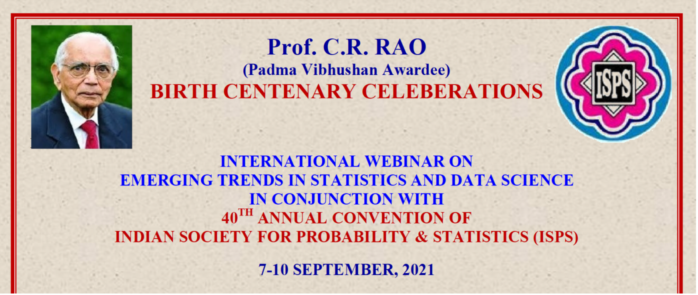

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References

The conference is based mainly on the following references:

- Navarro, del Águila, Sordo and Suárez-Llorens (2013, 2016).
- Navarro and Gomis (2016).
- Navarro and del Águila (2017).
- Navarro, Calì, Longobardi and Durante (2021).

Outline

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Notation

- X random variable (lifetime) over $(\Omega, \mathcal{S}, \Pr)$.

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- Mean, expected lifetime or mean time to failure (MTTF):

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- Hazard rate (HR) or failure rate (FR) function $h(t) = f(t)/\bar{F}(t)$, when $\bar{F}(t) > 0$.

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- Definition

The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous *distortion function* $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is given by

$$F_q(t) = q(F(t)), \text{ for all } t \in \mathbb{R}. \tag{1.1}$$

Properties

- If q is a distortion function, then F_q is a proper distribution function for all distribution functions F.

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- If q is an strictly increasing distortion function, then F_q has the same support of F.
- From (1.1), $\bar{F} = 1 - F$ and $\bar{F}_q = 1 - F_q$ satisfy

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \text{ for all } t \in \mathbb{R}, \quad (1.2)$$

where $\bar{q}(u) := 1 - q(1 - u)$ is called the *dual distortion function*.

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- (1.1) and (1.2) are equivalent.

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- The hazard rate of F_q is

$$h_q(t) = \frac{\bar{q}'(\bar{F}(t))}{\bar{q}(\bar{F}(t))}f(t) = \alpha(\bar{F}(t))h(t),$$

where h is the hazard rate of F and

$$\alpha(u) = \frac{u\bar{q}'(u)}{\bar{q}(u)}, \ u \in [0,1].$$

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- Definition

The **generalized distorted distribution** (GDD) associated to n distribution functions $F_1, \dots, F_n$ and to an increasing continuous *distortion function* $Q : [0,1]^n \to [0,1]$ such that $Q(0,\dots,0) = 0$ and $Q(1,\dots,1) = 1$, is given by

$$F_Q(t) = Q(F_1(t), \dots, F_n(t)), \text{ for all } t \in \mathbb{R}. \tag{1.3}$$

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- From (1.3), $\bar{F}_i = 1 - F_i$ and $\bar{F}_Q = 1 - F_Q$ satisfy

$$\bar{F}_Q(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)), \text{ for all } t \in \mathbb{R}, \qquad (1.4)$$

where $\bar{Q}(u_1, \dots, u_n) := 1 - Q(1 - u_1, \dots, 1 - u_n)$ is called the *dual distortion function*.

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Properties

- The PDF of F_Q is

$$f_Q(t)=\sum_{i=1}^{n} f_i(t)\partial_i Q(F_1(t),\dots,F_n(t))=\sum_{i=1}^{n} f_i(t)\partial_i \bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t)).$$

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- The hazard rate of F_q is

$$h_Q(t)=\sum_{i=1}^{n}\frac{\partial_i \bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t))}{\bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t))} f_i(t)=\sum_{i=1}^{n}\alpha_i(\bar{F}_1(t),\dots,\bar{F}_n(t))h_i(t),$$

where h_i is the hazard rate of F_i and

$$\alpha_i(u)=\frac{u_i\partial_i \bar{Q}(u_1,\dots,u_n)}{\bar{Q}(u_1,\dots,u_n)},\ \ u_i\in[0,1], i=1,\dots,n.$$

Examples of distorted distributions: PHR.

- Proportional Hazard Rate (PHR) Cox model

$$\bar{F}_\theta(t) = \bar{F}^\theta(t), t \in \mathbb{R},$$

where $\theta > 0$ is a risk (hazard) measure.

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- Its hazard rate is

$$h_\theta(t) = \theta \frac{\bar{F}^{\theta-1}(t)}{\bar{F}^\theta(t)} f(t) = \theta h(t),$$

that is, $\alpha_\theta(u) = \theta$ for $u \in [0,1]$.

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- Proportional Reversed Hazard Rate (PRHR) model

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- Its hazard rate is

$$h_\theta(t) = \frac{\theta F^{\theta-1}(t)}{1-(1-\bar{F}(t))^\theta} f(t) = \alpha_\theta(\bar{F}(t)) h(t),$$

that is, $\alpha_\theta(u) = \frac{\theta u (1-u)^{\theta-1}}{1-(1-u)^\theta}$ for $u \in [0,1]$.

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that is, $\alpha_\theta(u) = \frac{\theta u (1-u)^{\theta-1}}{1-(1-u)^\theta}$ for $u \in [0,1]$.
- Its reversed hazard rate is

$$\bar{h}_\theta(t) = \frac{f_\theta(t)}{F_\theta(t)} = \theta \bar{h}(t).$$

Examples of distorted distributions: Order statistics.

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- It is a distorted distribution with

$$\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$$

and

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- Note that both are polynomials.

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- $X_{n:n} = \max(X_1, \ldots, X_n)$ with

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for $n = 1, \ldots, n$ which belongs to the PRHR model.

Examples of generalized distorted distributions: Mixtures.

- The mixture distribution

$$F_{\mathbf{p}}(t) = p_1 F_1(t) + \cdots + p_n F_n(t), t \in \mathbb{R},$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, $p_i \geq 0$ and $p_1 + \cdots + p_n = 1$.

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$$\bar{F}_{\mathbf{p}}(t) = p_1 \bar{F}_1(t) + \cdots + p_n \bar{F}_n(t), t \in \mathbb{R}.$$

- It is a generalized distorted distribution with

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- Its PDF is $f_{\mathbf{p}}(t) = p_1 f_1(t) + \cdots + p_n f_n(t), t \in \mathbb{R}$.
- Its HR is

$$h_{\mathbf{p}}(t) = w_1(t) h_1(t) + \cdots + w_n(t) h_n(t), \; w_i(t) = \frac{p_i \bar{F}_i(t)}{\bar{F}_{\mathbf{p}}(t)} \geq 0.$$

Systems

- $(X_1, \ldots, X_n)$ component lifetimes of a system with joint distribution

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- Marginal distributions $F_i(x_i) = \Pr(X_i \leq x_i)$, $i = 1, \ldots, n$.
- **Sklar's theorem:** There exist a copula C such that

$$\mathbf{F}(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \ x_1, \ldots, x_n \in \mathbb{R}.$$

Moreover, if $F_1, \ldots, F_n$ are continuous, then C is unique.

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- A copula C is a multivariate distribution function with uniform marginals over the interval $(0, 1)$ (see Nelsen (2006)).

Systems

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  Moreover, if $F_1, \dots, F_n$ are continuous, then C is unique.
- A copula C is a multivariate distribution function with uniform marginals over the interval $(0, 1)$ (see Nelsen (2006)).
- Note that we just need C in $[0, 1]^n$.

Survival copula representation

- $(X_1, \ldots, X_n)$ with joint reliability (survival) function

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- Marginal reliability (survival) functions $\bar{F}_i(x_i)=\Pr(X_i>x_i)$, $i=1,\dots,n$.
- **Sklar's theorem:** There exist a copula $\widehat{C}$ (called survival copula) such that
$$\overline{\mathbf{F}}(x_1,\dots,x_n)=\widehat{C}(\bar{F}_1(x_1),\dots,\bar{F}_n(x_n)),\ x_1,\dots,x_n\in\mathbb{R}.$$
Moreover, if $\bar{F}_1,\dots,\bar{F}_n$ are continuous, then $\widehat{C}$ is unique.

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  Moreover, if $\bar{F}_1, \ldots, \bar{F}_n$ are continuous, then $\widehat{C}$ is unique.
- $\widehat{C}$ is a copula (distribution function), not a survival function.

Parallel systems

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- Its distribution function is

$$F_{n:n}(t) = \Pr(X_{n:n} \leq t) = \Pr(X_1 \leq t, \dots, X_n \leq t) = \mathbf{F}(t, \dots, t).$$

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- Its distribution function is

$$F_{n:n}(t) = \Pr(X_{n:n} \le t) = \Pr(X_1 \le t, \dots, X_n \le t) = \mathbf{F}(t, \dots, t).$$

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- The reverse is not true.

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- It is a generalized distorted distribution from $F_1, \ldots, F_n$ with $\bar{Q}_{1:n} = \widehat{C}$.

Systems

Theorem (Distortion representation, general case)

If T is the lifetime of a semi-coherent system and its component lifetimes $(X_1, \dots, X_n)$ have the survival copula $\widehat{C}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)) \tag{1.5}$$

for all t, where $\bar{Q}$ is a distortion function which depends on the structure ϕ of the sytem and on $\widehat{C}$.

(see, e.g., Navarro, del Águila, Sordo and Suárez-Llorens, 2016)

Distortion representation, ID case

Theorem (Distortion representation, ID case)

If T is the lifetime of a semi-coherent system and the component lifetimes $(X_1, \dots, X_n)$ have the survival copula $\widehat{C}$ and a common reliability $\bar{F}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

for all t, where $\bar{q}$ is a distortion function which only depends on ϕ and on $\widehat{C}$.

Proof. Take $\bar{q}(u) = \bar{Q}(u, \dots, u)$.

Distortion representation, IID case

Theorem (Distortion representation, IID case)

If T is the lifetime of a semi-coherent system with IID component lifetimes $X_1, \dots, X_n$ having a common reliability $\bar{F}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

for all t, where $\bar{q}(u) = \sum_{i=1}^n a_i u^i$ is a distortion function and $a = (a_1, \dots, a_n)$ is the minimal signature which only depends on ϕ. Moreover, $q(u) = \sum_{i=1}^n b_i u^i$ where $b = (b_1, \dots, b_n)$ is the maximal signature.

Table 1: Minimal and maximal signatures

Table: Minimal **a** and maximal **b** signatures of all the coherent systems with 1-4 IID components.

i	T_i	**a**	**b**
1	$X_{1:1} = X_1$	(1)	(1)
2	$X_{1:2} = \min(X_1, X_2)$ (2-series)	$(0, 1)$	$(2, -1)$
3	$X_{2:2} = \max(X_1, X_2)$ (2-parallel)	$(2, -1)$	$(0, 1)$
4	$X_{1:3} = \min(X_1, X_2, X_3)$ (3-series)	$(0, 0, 1)$	$(3, -3, 1)$
5	$\min(X_1, \max(X_2, X_3))$	$(0, 2, -1)$	$(1, 1, -1)$
6	$X_{2:3}$ (2-out-of-3)	$(0, 3, -2)$	$(0, 3, -2)$
7	$\max(X_1, \min(X_2, X_3))$	$(1, 1, -1)$	$(0, 2, -1)$
8	$X_{3:3} = \max(X_1, X_2, X_3)$ (3-parallel)	$(3, -3, 1)$	$(0, 0, 1)$
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$ (series)	$(0, 0, 0, 1)$	$(4, -6, 4, -1)$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(0, 0, 2, -1)$	$(2, 0, -2, 1)$
11	$\min(X_{2:3}, X_4)$	$(0, 0, 3, -2)$	$(1, 3, -5, 2)$

Table 1: Minimal and maximal signatures

i	T_i	**a**	**b**
12	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$(0, 1, 1, -1)$	$(1, 2, -3, 1)$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$(0, 3, -3, 1)$	$(1, 0, 1, -1)$
14	$X_{2:4}$ (3-out-of-4)	$(0, 0, 4, -3)$	$(0, 6, -8, 3)$
15	$\max(\min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_2, X_3, X_4))$	$(0, 1, 2, -2)$	$(0, 5, -6, 2)$
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	$(0, 2, 0, -1)$	$(0, 4, -4, 1)$
17	$\max(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_3, X_4))$	$(0, 2, 0, -1)$	$(0, 4, -4, 1)$
18	$\max(\min(X_1, X_2), \min(X_2, X_3), \min(X_3, X_4))$	$(0, 3, -2, 0)$	$(0, 3, -2, 0)$
19	$\max(\min(X_1, \max(X_2, X_3, X_4)), \min(X_2, X_3, X_4))$	$(0, 3, -2, 0)$	$(0, 3, -2, 0)$
20	$\min(\max(X_1, X_2), \max(X_1, X_3), \max(X_2, X_3, X_4))$	$(0, 4, -4, 1)$	$(0, 2, 0, -1)$
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	$(0, 4, -4, 1)$	$(0, 2, 0, -1)$

Table 1: Minimal and maximal signatures

i	T_i	**a**	**b**
22	$\min(\max(X_1,X_2),\max(X_1,X_3,X_4),$ $\max(X_2,X_3,X_4))$	$(0,5,-6,2)$	$(0,1,2,-2)$
23	$X_{3:4}$ (2-out-of-4)	$(0,6,-8,3)$	$(0,0,4,-3)$
24	$\max(X_1,\min(X_2,X_3,X_4))$	$(1,0,1,-1)$	$(0,3,-3,1)$
25	$\max(X_1,\min(X_2,X_3),\min(X_3,X_4))$	$(1,2,-3,1)$	$(0,1,1,-1)$
26	$\max(X_{2:3},X_4)$	$(1,3,-5,2)$	$(0,0,3,-2)$
27	$\min(\max(X_1,X_2,X_3),\max(X_2,X_3,X_4))$	$(2,0,-2,1)$	$(0,0,2,-1)$
28	$X_{4:4}=\max(X_1,X_2,X_3,X_4)$ (4-parallel)	$(4,-6,4,-1)$	$(0,0,0,1)$

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- Stochastic order: $X \leq_{ST} Y \Leftrightarrow \bar{F}_X \leq \bar{F}_Y$.

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- Reversed hazard rate order: $X \leq_{RHR} Y \Leftrightarrow F_Y/F_X$ increases (or $\bar{h}_X \leq \bar{h}_Y$).
- Then

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

Comparisons of distorted distributions

Theorem (Navarro, del Águila, Sordo and Suárez-Llorens (2013); Navarro and Gomis (2016))

If T_i *has the DF* $F_i(t) = q_i(F(t))$, $i = 1, 2$, *then:*

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- $T_1 \leq_{RHR} T_2$ for all F iff q_2/q_1 increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F iff $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.

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- $T_1 \leq_{RHR} T_2$ for all F iff q_2/q_1 increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all F iff $\bar{q}_2'/\bar{q}_1'$ decreases in $(0, 1)$.
- $T_1 \leq_{MRL} T_2$ for all F such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.

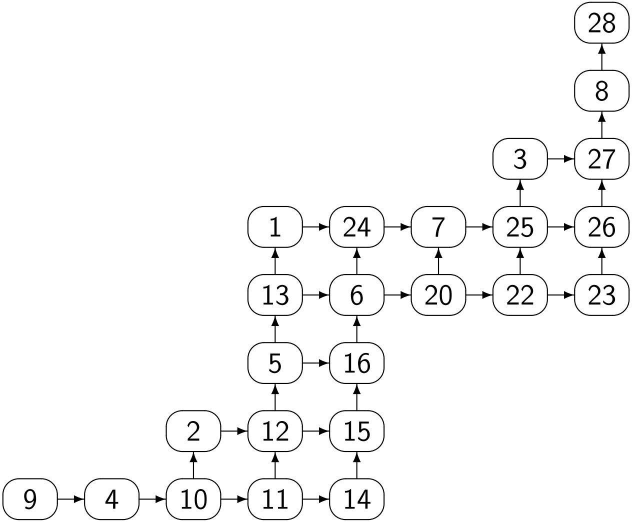

Figure: All ST orderings for the systems in Table 1 (IID case).

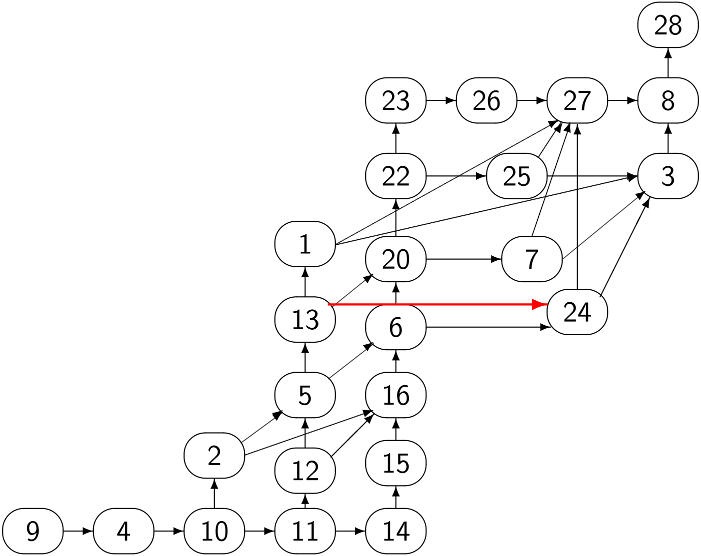

Figure: All the HR orders for the systems in Table 1 (IID case).

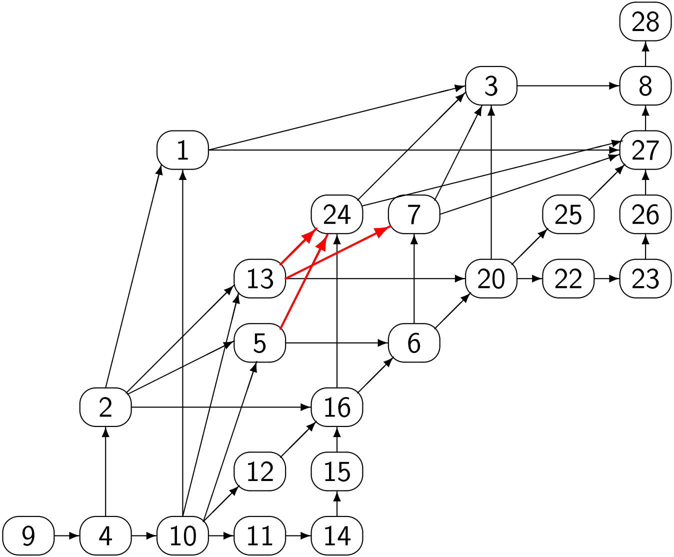

Figure: All the LR orders for the systems in Table 1 (IID case).

Comparisons for systems with non-ID components

Theorem (Navarro and del Águila (2017))

If T_i *has DF* $F_{T_i} = Q_i(F_1, \dots, F_n)$, $i = 1, 2$, *then:*

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If T_i has DF $F_{T_i} = Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, then:

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- $T_1 \leq_{HR} T_2$ for all $F_1 \geq_{HR} \cdots \geq_{HR} F_n$ iff the function

$$\bar{H}(v_1, \ldots, v_n) = \frac{\bar{Q}_2(v_1, v_1 v_2, \ldots, v_1 \ldots v_n)}{\bar{Q}_1(v_1, v_1 v_2, \ldots, v_1 \ldots v_n)} \qquad (1.6)$$

is decreasing in $(0,1)^n$;

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- $T_1 \leq_{HR} T_2$ for all $F_1 \geq_{HR} \cdots \geq_{HR} F_n$ iff the function
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is decreasing in $(0,1)^n$;
- $T_1 \leq_{RHR} T_2$ for all $F_1 \leq_{RHR} \cdots \leq_{RHR} F_n$ iff the function
$$H(v_1, \dots, v_n) = \frac{Q_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{Q_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \qquad (1.7)$$
is increasing in $(0,1)^n$.

Table 2: Dual distortions of systems with IND components

Table: Dual distortions functions of 5 systems with 1-3 IND components.

N	$T=\psi(X_1,X_2,X_3)$	$\overline{Q}(u_1,u_2,u_3)$
1	$X_{1:3}=\min(X_1,X_2,X_3)$	$u_1u_2u_3$
2	$\min(X_2,X_3)$	u_2u_3
3	$\min(X_1,X_3)$	u_1u_3
4	$\min(X_1,X_2)$	u_1u_2
5	$\min(X_3,\max(X_1,X_2))$	$u_1u_3+u_2u_3-u_1u_2u_3$
6	$\min(X_2,\max(X_1,X_3))$	$u_1u_2+u_2u_3-u_1u_2u_3$
7	$\min(X_1,\max(X_2,X_3))$	$u_1u_2+u_1u_3-u_1u_2u_3$
8	X_3	u_3

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N	$T=\psi(X_1,X_2,X_3)$	$\overline{Q}(u_1,u_2,u_3)$
9	X_2	u_2
10	X_1	u_1
11	$X_{2:3}$	$u_1u_2+u_1u_3+u_2u_3-2u_1u_2u_3$
12	$\max(X_3,\min(X_1,X_2))$	$u_3+u_1u_2-u_1u_2u_3$
13	$\max(X_2,\min(X_1,X_3))$	$u_2+u_1u_3-u_1u_2u_3$
14	$\max(X_1,\min(X_2,X_3))$	$u_1+u_2u_3-u_1u_2u_3$
15	$\max(X_2,X_3)$	$u_2+u_3-u_2u_3$
16	$\max(X_1,X_3)$	$u_1+u_3-u_1u_3$
17	$\max(X_1,X_2)$	$u_1+u_2-u_1u_2$
18	$X_{3:3}=\max(X_1,X_2,X_3)$	$u_1+u_2+u_3-u_1u_2-u_1u_3-u_2u_3$ $+u_1u_2u_3$

Systems with IND components

Table: Relationships for the ST order between the coherent systems with independent components given in Table 2. The value 2 indicates that $T_i \leq_{ST} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column). The value 1 indicates that $T_i \leq_{ST} T_j$ holds for all $F_1 \geq_{ST} F_2 \geq_{ST} F_3$. It also indicates that $T_i \leq_{ST} T_j$ does not hold for all F_1, F_2, F_3. The value 0 indicates that $T_i \leq_{ST} T_j$ does not hold for all $F_1 \geq_{ST} F_2 \geq_{ST} F_3$.

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	2	2	1	2	2	1	2	2	2	2	2	2	2	2
3	0	2	1	2	1	2	2	1	2	2	2	2	2	2	2	2	2
4	0	0	2	0	2	2	0	2	2	2	2	2	2	2	2	2	2
5	0	0	0	2	1	1	2	1	1	2	2	2	2	2	2	2	2
6	0	0	0	0	2	1	0	2	1	2	2	2	2	2	2	2	2
7	0	0	0	0	0	2	0	0	2	2	2	2	2	2	2	2	2

Systems with IND components

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	0	0	0	0	0	0	2	1	1	0	2	1	1	2	2	1	2
9	0	0	0	0	0	0	0	2	1	0	0	2	1	2	1	2	2
10	0	0	0	0	0	0	0	0	2	0	0	0	2	0	2	2	2
11	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
12	0	0	0	0	0	0	0	0	0	0	2	1	1	2	2	1	2
13	0	0	0	0	0	0	0	0	0	0	0	2	1	2	1	2	2
14	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2	2	2
15	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	2
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	2
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

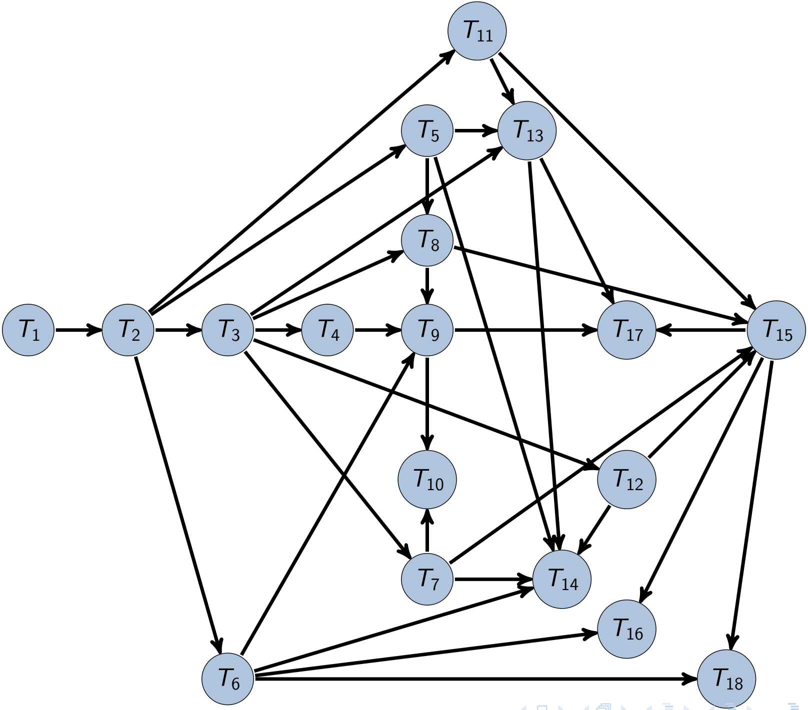
Systems with IND components

Table: Relationships for the HR order between the coherent systems with independent components given in Table 2. The value 2 indicates that $T_i \leq_{HR} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column). The value 1 indicates that $T_i \leq_{HR} T_j$ holds for all $F_1 \geq_{HR} F_2 \geq_{HR} F_3$. It also indicates that $T_i \leq_{HR} T_j$ does not hold for all F_1, F_2, F_3. The value 0 means that $T_i \leq_{HR} T_j$ does not hold for all $F_1 \geq_{HR} F_2 \geq_{HR} F_3$.

HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2	1	1	1	1	1	2	1	1	1
3	0	2	1	0	0	1	2	1	2	0	1	1	1	1	2	1	1
4	0	0	2	0	0	0	0	2	2	0	0	0	0	0	0	2	0
5	0	0	0	2	0	0	2	1	1	0	0	1	1	1	1	2	2
6	0	0	0	0	2	0	0	2	1	0	0	0	1	0	2	1	2
7	0	0	0	0	0	2	0	0	2	0	0	0	1	2	1	1	2

Systems with IND components

HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	0	0	0	0	0	0	2	1	1	0	0	0	0	1	1	1	1
9	0	0	0	0	0	0	0	2	1	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	2	0	1	1	2	2	2	2
12	0	0	0	0	0	0	0	0	0	0	2	0	1	1	1	1	1
13	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	1	0
14	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2



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Multivariate distorted distributions

Notation

- $(X_1, \dots, X_n)$ random vector over $(\Omega, \mathcal{S}, \Pr)$.

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$$\mathbf{F}(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

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- A similar representation holds for the joint survival function

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n).$$

Definition

Definition (Navarro, Calì, Longobardi and Durante (2021))

A multivariate distribution function $\mathbf{F}$ is said to be a *multivariate distorted distribution* (MDD) of the univariate distribution functions $G_1, \ldots, G_n$ if there exists a *distortion* function D such that

$$\mathbf{F}(x_1, \ldots, x_n) = D(G_1(x_1), \ldots, G_n(x_n)), \ \forall x_1, \ldots, x_n \in \mathbb{R}. \quad (2.1)$$

We write $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$, when $\mathbf{F}$ is a MDD of $G_1, \ldots, G_n$.

Definition

Definition

A continuous function $D : [0,1]^n \to [0,1]$ is called *(n–dimensional) distortion function* (shortly written as $D \in \mathcal{D}_n$) if:

(i) $D(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n) = 0$ for all $u_1, \dots, u_n \in [0,1]$.

(ii) $D(1, \dots, 1) = 1$.

(iii) D is n–increasing, i.e. for all $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ with $x_i \le y_i$, it holds $\triangle_{\mathbf{x}}^{\mathbf{y}} D \ge 0$, where

$$\triangle_{(x_1,\dots,x_n)}^{(y_1,\dots,y_n)} D := \sum_{z_i \in \{x_i, y_i\}} (-1)^{\mathbf{1}(z_1,\dots,z_n)} D(z_1, \dots, z_n),$$

with $\mathbf{1}(z_1, \dots, z_n) = \sum_{i=1}^n 1(z_i = x_i)$ and $1(A) = 1$ (respectively, 0) if A is true (respectively, false).

Main properties

- According to Sklar's theorem, any multivariate distribution function can be expressed in terms of its univariate marginal distributions via a copula representation.

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- If the marginals are continuous then this representation (copula) is unique.
- In the following result, we state a similar Sklar-type theorem for MDD under mild conditions.

Sklar-type theorem

Proposition

Let $(X_1, \ldots, X_n)$ be a random vector with joint continuous distribution function $\mathbf{F}$. Let $G_1, \ldots, G_n$ be arbitrary continuous distribution functions and let us assume that G_i is strictly increasing in the support of X_i for $i = 1, \ldots, n$. Then there exists a unique distortion $D \in \mathcal{D}_n$ such that

$$\mathbf{F}(x_1, \ldots, x_n) = D(G_1(x_1), \ldots, G_n(x_n))$$

holds for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Construction of new multivariate models

The converse of the preceding proposition can be stated as follows.

Proposition

If $D \in \mathcal{D}_n$, then

$$D(G_1(x_1), \ldots, G_n(x_n))$$

is a multivariate distribution function for all univariate distribution functions $G_1, \ldots, G_n$.

Relationship with the copula

Proposition

Let $(X_1, \ldots, X_n)$ be a random vector with joint continuous distribution function $\mathbf{F}$. Let $G_1, \ldots, G_n$ be arbitrary continuous distribution functions. Suppose that $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$ with distortion D. Then,

$$D(u_1, \ldots, u_n) = C(F_1(G_1^{-1}(u_1)), \ldots, F_n(G_n^{-1}(u_n)))$$

for all $(u_1, \ldots, u_n) \in [0,1]^n$, where G_i^{-1} is the quasi-inverse of G_i and F_i is the ith marginal of $\mathbf{F}$ for $i = 1, \ldots, n$.

Joint survival function.

Proposition

Let $(X_1, \ldots, X_n)$ be a random vector with distribution function $\mathbf{F}$*. If (2.1) holds for $G_1, \ldots, G_n$ and $D \in \mathcal{D}_n$, then the joint survival function of $(X_1, \ldots, X_n)$ can be written as*

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \hat{D}(\bar{G}_1(x_1), \ldots, \bar{G}_n(x_n)) \quad (2.2)$$

for all $x_1, \ldots, x_n$, where $\bar{G}_i = 1 - G_i$ is the survival function associated to G_i for $i = 1, \ldots, n$ and $\hat{D} \in \mathcal{D}_n$.

Marginal distributions

- A relevant property of the MDD representation $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$ is that all the multivariate marginal distributions of $\mathbf{F}$ are also MDD from $G_1, \ldots, G_n$.

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- Let $F_{1,\ldots,m}$ be the distribution function of $(X_1, \ldots, X_m)$.

- Proposition

If $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$ *and* $1 \leq m \leq n$, *then*

$$F_{1,\ldots,m}(x_1, \ldots, x_m) = D_{1,\ldots,m}(G_1(x_1), \ldots, G_m(x_m)) \tag{2.3}$$

for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, *where*

$$D_{1,\ldots,m}(u_1, \ldots, u_m) := D(u_1, \ldots, u_m, 1, \ldots, 1)$$

for all $(u_1, \ldots, u_m) \in [0, 1]^m$ *and* $D_{1,\ldots,m} \in \mathcal{D}_m$.

Univariate marginal distributions.

- In particular, the ith marginal distribution function of X_i can be written as

$$F_i(x_i) = D(1, \dots, 1, G_i(x_i), 1, \dots, 1) = D_i(G_i(x_i)) \qquad (2.4)$$

for all $x_i \in \mathbb{R}$, where

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- Clearly, we have $G_i = F_i$ for a fixed $i \in \{1, \ldots, n\}$ when $D_i(u) = u$ for all $u \in [0, 1]$.

Probability density function

Let us assume that **F** is absolutely continuous with joint probability density function (PDF) **f**, where

$$\mathbf{f}(x_1,\dots,x_n)=\partial_{1,\dots,n}\mathbf{F}(x_1,\dots,x_n)\ (a.e.).$$

Proposition

If $\mathbf{F}\equiv MDD(G_1,\dots,G_n)$ *for absolutely continuous distribution functions* $G_1,\dots,G_n$ *with PDFs* $g_1,\dots,g_n$*, respectively, and a distortion function* D *that admits continuous mixed derivatives of order* n*, then*

$$\mathbf{f}(x_1,\dots,x_n)=g_1(x_1)\dots g_n(x_n)\ \partial_{1,\dots,n}D(G_1(x_1),\dots,G_n(x_n)). \quad (2.5)$$

Conditional distributions

- All the conditional distributions of $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$ have MDD representations.

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- Proposition

Let (X_1, X_2) *with* $\mathbf{F} \equiv MDD(G_1, G_2)$ *for a distortion function* $D \in \mathcal{D}_2$ *that admits continuous mixed derivatives of order* 2, *then*

$$F_{2|1}(x_2|x_1) = D_{2|1}(G_2(x_2)|G_1(x_1)) \tag{2.6}$$

whenever $\lim_{v\to 0^+} \partial_1 D(G_1(x_1), v) = 0$, *where*

$$D_{2|1}(v|G_1(x_1)) = \frac{\partial_1 D(G_1(x_1), v)}{\partial_1 D(G_1(x_1), 1)}$$

for $0 < v < 1$ *and* x_1 *such that* $\partial_1 D(G_1(x_1), 1) > 0$.

Theoretical Quantile Regression

- The (mean) regression curve to predict X_2 from X_1 is

$$m_{2|1}(x_1) = E(X_2|X_1 = x_1)$$

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- Another option is the *conditional median regression curve*

$$\widetilde{m}_{2|1}(x_1) := F_{2|1}^{-1}(0.5|x_1)$$

(see Koenker (2005) or Nelsen (2006), p. 217).

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- This quantile function $F_{2|1}^{-1}$ can be computed from (2.6) as

$$F_{2|1}^{-1}(q|x_1) = G_2^{-1}(D_{2|1}^{-1}(q|G_1(x_1))),\ 0 < q < 1.$$

Confidence bands

- Moreover, we can obtain α-confidence bands in a similar way with
$$\left[F_{2|1}^{-1}(\beta_1|x_1), F_{2|1}^{-1}(\beta_2|x_1)\right]$$
taking $0 \leq \beta_1 < \beta_2 \leq 1$ such that $\beta_2 - \beta_1 = \alpha$.

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taking $0 \leq \beta_1 < \beta_2 \leq 1$ such that $\beta_2 - \beta_1 = \alpha$.
- For example, the centered 50% and 90% quantile-confidence bands for $(X_2|X_1 = x_1)$ are determined, respectively, by
$$\left[F_{2|1}^{-1}(0.25|x_1), F_{2|1}^{-1}(0.75|x_1)\right]$$
and
$$\left[F_{2|1}^{-1}(0.05|x_1), F_{2|1}^{-1}(0.95|x_1)\right].$$

Example 1: Residual lifetimes

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- $(X_i - t | X_i > t)$ denotes the univariate residual lifetimes at time $t > 0$ with

$$\bar{F}_{i,t}(x) := \Pr(X_i - t > x | X_i > t) = \frac{\bar{F}_i(t+x)}{\bar{F}_i(t)}$$

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for every $x \geq 0$, whenever $\bar{F}_i(t) > 0$.
- The mean residual lifetime is $m_i(t) = E(X_i - t | X_i > t)$.
- From $\mathbf{X} = (X_1, \dots, X_n)$, we can consider

$$\mathbf{X}_t = (X_1 - t, \dots, X_n - t | X_1 > t, \dots, X_n > t)$$

whose survival function for $x_1, \dots, x_n \geq$ is

$$\bar{F}_t(x_1, \dots, x_n) := \Pr(X_1 > x_1 + t, \dots, X_n > x_n + t | X_1 > t, \dots, X_n > t).$$

Example 1: Residual lifetimes

Proposition

If $\bar{F}(t,\dots,t)>0$ for some $t\geq 0$, then

$$\bar{F}_t(x_1,\dots,x_n)=\widehat{D}_t(\bar{F}_{1,t}(x_1),\dots,\bar{F}_{n,t}(x_n)) \quad (2.7)$$

for all $x_1,\dots,x_n\geq t$ and distortion function

$$\widehat{D}_t(u_1,\dots,u_n):=\frac{\widehat{C}(\bar{F}_1(t)u_1,\dots,\bar{F}_n(t)u_n)}{\widehat{C}(\bar{F}_1(t),\dots,\bar{F}_n(t))},\ u_1,\dots,u_n\in[0,1], \quad (2.8)$$

which depends on $\bar{F}_1(t),\dots,\bar{F}_n(t)$.

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- Hence (2.7) is not a copula representation and $\widehat{D}_t$ is not always a copula.
- If $X_1, \ldots, X_n$ are exponential, then $\bar{F}_{i,t} = \bar{F}_i \neq \bar{H}_{i,t}$.

Example 2: Ordered paired data

- Let us assume that X and Y have a common absolutely continuous distribution function F. Then

$$F_{X,Y}(x,y) = C(F(x), F(y)).$$

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- It can be used to compute the median regression curve and the confidence bands.

Example 2: Ordered paired data

- We assume that we have a training sample $(X_1, Y_1), \ldots, (X_m, Y_m)$ from (X, Y).

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- Note that both F and C can be estimated from the training sample by using parametric models or empirical or kernel type estimators.
- So, we want to obtain a MDD representation for the random vector (L, U) in terms of F and C.

Example 2: Ordered paired data

- The joint distribution function $\mathbf{G}(x,y)=\Pr(L\le x, U\le y)$ of (L,U) is

$$\mathbf{G}(x,y)=\begin{cases} C(F(y),F(y)) & \text{for } \quad y\le x;\\ C(F(x),F(y))+C(F(y),F(x))-C(F(x),F(x)) & \text{for } \quad y>x.\end{cases}$$

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- Therefore, $\mathbf{G} \equiv MDD(F, F)$, i.e.

$$\mathbf{G}(x, y) = D(F(x), F(y)) \tag{2.9}$$

with the following distortion function

$$D(u,v) = \begin{cases} C(v,v) & \text{for} \quad v \leq u; \\ C(u,v) + C(v,u) - C(u,u) & \text{for} \quad u < v. \end{cases} \tag{2.10}$$

Example 2: Ordered paired data

- Then the marginal distributions of (L, U) can be written as

$$G_1(x) := \Pr(L \leq x) = D(F(x), 1) = D_1(F(x)),$$

$$G_2(y) := \Pr(U \leq y) = D(1, F(y)) = D_2(F(y)),$$

where

$$D_1(u) = D(u, 1) = 2u - C(u, u)$$

and

$$D_2(v) = D(1, v) = C(v, v)$$

for all $u, v \in [0, 1]$.

Example 2: Ordered paired data, IID case

- For example, if X and Y are independent, then

$$D_1(u) = D(u,1) = 2u - u^2 \neq u$$

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- The distortion function is

$$D(u,v) = \begin{cases} v^2 & \text{for} \quad v \leq u; \\ 2uv - u^2 & \text{for} \quad u < v. \end{cases} \tag{2.11}$$

- Note that it is not a copula and that the marginals G_1 and G_2 of $\mathbf{G}$ do not appear in (2.9) (we use F instead).

Example 2: Ordered paired data

- From (2.6) and (2.9), the distribution function of $(U|L = x)$ is

$$G_{2|1}(y|x) = D_{2|1}(F(y)|F(x)) \tag{2.12}$$

for $y \geq x$, where

$$D_{2|1}(v|F(x)) := \frac{\partial_1 D(F(x), v)}{\partial_1 D(F(x), 1)},$$

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- In the EXC case, we have

$$\partial_1 D(u,v) = 2\partial_1 C(u,v) - 2\partial_1 C(u,u), \; u \leq v \leq 1.$$

Example 3: Coherent systems

Theorem

If T_1 and T_2 are two coherent systems with ID$\sim F$ components $(X_1, \ldots, X_n)$, then its joint distribution is MDD(F,F).

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- In particular, it can be applied to the k-out-of-n systems (order statistics).
- In a more particular case, for $X_{1:2}$ and $X_{2:2}$ we obtain the distortion D of the preceding subsection.
- Other examples: Sequential order statistics, record values, ...

Exact QR curves for paired ordered data. IID case.

- Let (X_i, Y_i) be a sample from (X, Y) where X, Y are IID$\sim F$.

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- Let $L_i = \min(X_i, Y_i)$ and $U_i = \max(X_i, Y_i)$.
- Note that L_i and U_i are dependent.
- From (2.12), the distribution function of $(U|L = x)$ is

$$G_{2|1}(y|x) = D_{2|1}(F(y)|F(x)) \qquad (2.13)$$

for $y \geq x$, where

$$D_{2|1}(v|F(x)) = \frac{v - F(x)}{\bar{F}(x)}$$

for $F(x) \leq v \leq 1$.

Paired ordered data. IID case.

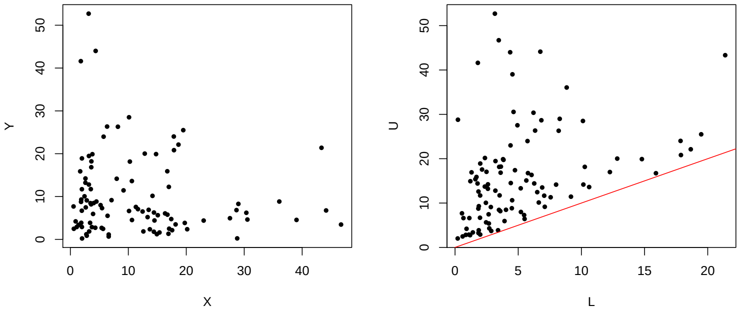

Figure: Independent data from two exponential distributions with mean $\mu = 10$ (left) and the associated paired ordered data (right).

Exact QR curves for paired ordered data. IID case.

- The quantile function $F_{2|1}^{-1}$ can be computed as

$$F_{2|1}^{-1}(q|x) = F^{-1}(D_{2|1}^{-1}(q|F(x)))$$

for $0 < v < 1$, where $D_{2|1}^{-1}(q|F(x)) = F(x) + q\bar{F}(x)$, when $\bar{F}(x) = \exp(-x/\mu)$ and $F^{-1}(y) = -\mu\log(1-y)$. Then

$$F_{2|1}^{-1}(q|x) = -\mu\log\left((1-q)e^{-x/\mu}\right) = x - \mu\log(1-q).$$

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$$F_{2|1}^{-1}(q|x) = -\mu \log\left((1-q)e^{-x/\mu}\right) = x - \mu \log(1-q).$$

- Therefore, the exact QR curve is

$$m(x) = x - \mu \log(0.5).$$

Exact QR confidence bands for paired ordered data

- Analogously, the exact QR centered 90% confidence band is

$$[x - \mu \log(0.05), x - \mu \log(0.95)] .$$

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- The exact QR lower 90% confidence band is

$$[x, x - \mu \log(0.90)] .$$

QR for paired ordered data. IID case.

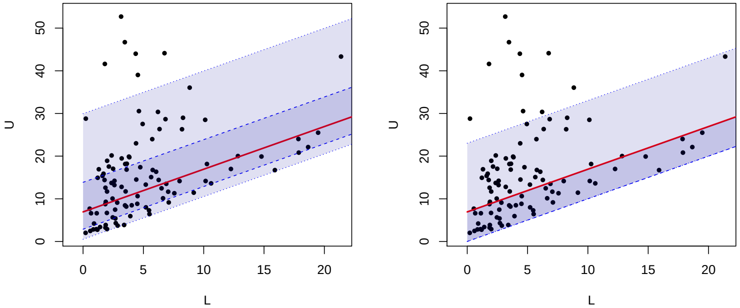

Figure: QR for the paired ordered data (L, U) associated to independent data (X, Y) from two exponential distributions with mean $\mu = 10$ jointly with 50% and 90% centered (left) or bottom (right) confidence bands.

Predictions

- The first ordered pair in our sample is $L_1 = 10.15771$ and $U_1 = 14.17195$.

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- We consider again the exponential model

$$\bar{F}(t) = \exp(-t/\mu), \ \ t \geq 0$$

and the Clayton EXC copula

$$C(u, v) = \frac{uv}{u + v - uv}, \ \ (u, v) \in [0, 1]^2. \tag{2.14}$$

Dependent EXC data

- To get the QR curves we need the distribution $G_{2|1}(y|x)$ of $(U|L=x)$. From (2.12) we need

$$\partial_1 D(u,v) = 2\partial_1 C(u,v) - 2\partial_1 C(u,u) = \frac{2v^2}{(u+v-uv)^2} - \frac{2}{(2-u)^2}$$

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- Hence, for $v \geq u$, we get

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- To compute the inverse, we need to solve in y the equation $G_{2|1}(y|x) = q$ for $q \in (0,1)$.

Dependent EXC data

- This leads to

$$G_{2|1}^{-1}(q|x) = y = F^{-1}\left(\frac{F(x)}{F(x) - 1 + \frac{2-F(x)}{\sqrt{1-q+q(2-F(x))^2}}}\right)$$

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- The other confidence bands can be obtained in a similar way.
- For an exponential distribution with $\mu = 10$ we get

QR for paired ordered data. ID case.

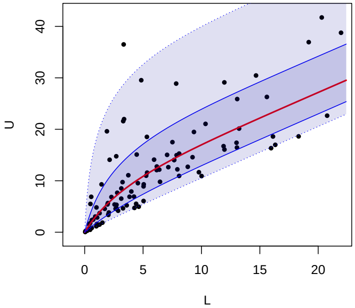

Figure: QR curves for paired ordered data (L, U) associated to dependent data (X, Y) from two exponential distributions with centered (left) and bottom (right) confidence bands.

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- They can be both in F or in C.
- We can use the training sample (X_i, Y_i) from (X, Y) to estimate the unknown parameters.
- Then we can use the MDD representation with the estimated parameters to get the estimated QR curves.

Parametric QR for paired ordered data IID case

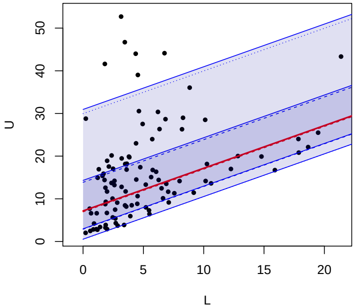

Figure: Parametric QR curves for (L, U) associated to IID data (X, Y) from an exponential distribution jointly with centered (left) and bottom (right) confidence bands. The dashed lines are the exact curves.

Parametric QR for paired ordered data ID case

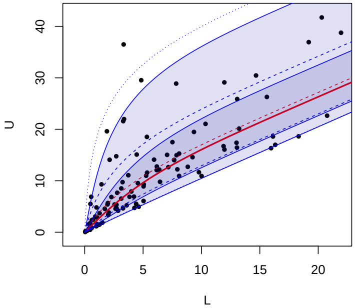

Figure: Parametric QR curves for (L, U) associated to data (X, Y) from an exponential distribution with unknown mean μ and a Clayton copula with unknown parameter θ. The dashed lines are the exact curves.

Non-parametric QR curves.

- If we do not have a parametric model, we can use non-parametric estimators for F and C.

Non-parametric QR curves.

- If we do not have a parametric model, we can use non-parametric estimators for F and C.
- We can also use the statistical program R with `library('quantreg')` to estimate the exact curves from the training sample (see Koenker, 2005; Koenker and Bassett, 1978).

Non-parametric QR for paired ordered data, IID case

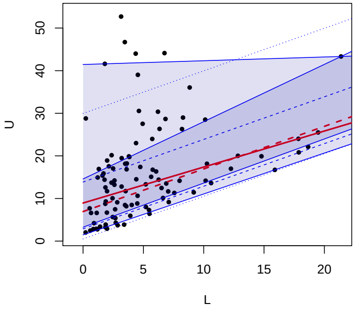

Figure: Non-parametric QR curves for paired ordered data (L, U) associated to IID data (X, Y) from an exponential distribution with $\mu = 10$ and $k = 1$ (left) or $k = 2$ (right).

Non-parametric QR for paired ordered data, ID case

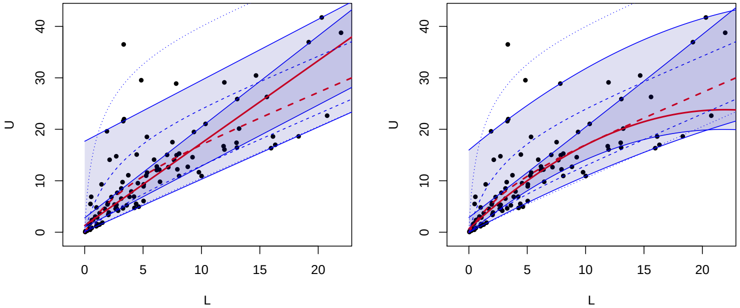

Figure: Non-parametric QR curves for (L, U) associated to data (X, Y) from an exponential distribution and a Clayton copula with $\theta = 1$ and $k = 1$ (left) or $k = 2$ (right). The dashed lines are the exact curves.

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The slides and more references can be seen in my webpage:

https://webs.um.es/jorgenav/miwiki/doku.php

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- Questions?