Distortion Representations of Multivariate Distributions

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References

The conference is based mainly on the following references:

- Navarro, del Águila, Sordo and Suárez-Llorens (2013, 2016).
- Navarro and Gomis (2016).
- Navarro and del Águila (2017).
- Navarro, Calì, Longobardi and Durante (2021).

Outline

Distorted distributions

Notation

- X random variable (lifetime) over $(\Omega, \mathcal{S}, \Pr)$.

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- Probability density function (PDF) $f(t) = F'(t) = -\bar{F}'(t)$.
- Hazard rate (HR) or failure rate (FR) function $h(t) = f(t)/\bar{F}(t)$, when $\bar{F}(t) > 0$.

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- Definition

The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous *distortion function* $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is given by

$$F_q(t) = q(F(t)), \text{ for all } t \in \mathbb{R}. \tag{1.1}$$

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- If q is a distortion function, then F_q is a proper distribution function for all distribution functions F.

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- If q is an strictly increasing distortion function, then F_q has the same support of F.
- From (1.1), $\bar{F} = 1 - F$ and $\bar{F}_q = 1 - F_q$ satisfy

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \text{ for all } t \in \mathbb{R}, \tag{1.2}$$

where $\bar{q}(u) := 1 - q(1 - u)$ is called the *dual distortion function*.

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  where $\bar{q}(u) := 1 - q(1 - u)$ is called the *dual distortion function*.
- (1.1) and (1.2) are equivalent.

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- The hazard rate of F_q is

$$h_q(t) = \frac{\bar{q}'(\bar{F}(t))}{\bar{q}(\bar{F}(t))} f(t) = \alpha(\bar{F}(t))h(t),$$

where h is the hazard rate of F and

$$\alpha(u) = \frac{u\bar{q}'(u)}{\bar{q}(u)}, \quad u \in [0,1].$$

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The **generalized distorted distribution** (GDD) associated to n distribution functions $F_1, \ldots, F_n$ and to an increasing continuous *distortion function* $Q : [0,1]^n \to [0,1]$ such that $Q(0, \ldots, 0) = 0$ and $Q(1, \ldots, 1) = 1$, is given by

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)), \text{ for all } t \in \mathbb{R}. \tag{1.3}$$

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where $\bar{Q}(u_1, \dots, u_n) := 1 - Q(1 - u_1, \dots, 1 - u_n)$ is called the *dual distortion function*.

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Properties

- The PDF of F_Q is

$$f_Q(t)=\sum_{i=1}^{n} f_i(t)\partial_i Q(F_1(t),\dots,F_n(t))=\sum_{i=1}^{n} f_i(t)\partial_i \bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t)).$$

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- The hazard rate of F_q is

$$h_Q(t)=\sum_{i=1}^{n}\frac{\partial_i \bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t))}{\bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t))} f_i(t)=\sum_{i=1}^{n}\alpha_i(\bar{F}_1(t),\dots,\bar{F}_n(t))h_i(t),$$

where h_i is the hazard rate of F_i and

$$\alpha_i(u)=\frac{u_i\partial_i \bar{Q}(u_1,\dots,u_n)}{\bar{Q}(u_1,\dots,u_n)},\ \ u_i\in[0,1], i=1,\dots,n.$$

Examples of distorted distributions: PHR.

- Proportional Hazard Rate (PHR) Cox model

$$\bar{F}_\theta(t) = \bar{F}^\theta(t), t \in \mathbb{R},$$

where $\theta > 0$ is a risk (hazard) measure.

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$$h_\theta(t) = \theta \frac{\bar{F}^{\theta-1}(t)}{\bar{F}^\theta(t)} f(t) = \theta h(t),$$

that is, $\alpha_\theta(u) = \theta$ for $u \in [0,1]$.

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- Its hazard rate is

$$h_\theta(t) = \frac{\theta F^{\theta-1}(t)}{1-(1-\bar{F}(t))^\theta} f(t) = \alpha_\theta(\bar{F}(t))h(t),$$

that is, $\alpha_\theta(u) = \frac{\theta u(1-u)^{\theta-1}}{1-(1-u)^\theta}$ for $u \in [0,1]$.

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that is, $\alpha_\theta(u) = \frac{\theta u(1-u)^{\theta-1}}{1-(1-u)^\theta}$ for $u \in [0,1]$.
- Its reversed hazard rate is

$$\bar{h}_\theta(t) = \frac{f_\theta(t)}{F_\theta(t)} = \theta \bar{h}(t).$$

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- It is a distorted distribution with

$$\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$$

and

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- Note that both are polynomials.

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- $X_{n:n} = \max(X_1, \dots, X_n)$ with

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Examples of generalized distorted distributions: Mixtures.

- The mixture distribution

$$F_{\mathbf{p}}(t) = p_1 F_1(t) + \cdots + p_n F_n(t), t \in \mathbb{R},$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, $p_i \geq 0$ and $p_1 + \cdots + p_n = 1$.

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- Its PDF is $f_{\mathbf{p}}(t) = p_1 f_1(t) + \cdots + p_n f_n(t), t \in \mathbb{R}$.
- Its HR is

$$h_{\mathbf{p}}(t) = w_1(t) h_1(t) + \cdots + w_n(t) h_n(t), \ w_i(t) = \frac{p_i \bar{F}_i(t)}{\bar{F}_{\mathbf{p}}(t)} \geq 0.$$

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- System lifetime $T = \phi(X_1,\dots,X_n)$, where $X_1,\dots,X_n$ are the component lifetimes.

Systems

- $(X_1, \ldots, X_n)$ with joint distribution

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- Marginal distributions $F_i(x_i) = \Pr(X_i \leq x_i)$, $i = 1, \ldots, n$.
- **Sklar's theorem:** There exist a copula C such that

$$\mathbf{F}(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),\ x_1, \ldots, x_n \in \mathbb{R}.$$

Moreover, if $F_1, \ldots, F_n$ are continuous, then C is unique.

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- A copula C is a multivariate distribution function with uniform marginals over the interval $(0, 1)$ (see Nelsen (2006)).

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- A copula C is a multivariate distribution function with uniform marginals over the interval $(0,1)$ (see Nelsen (2006)).
- Note that we just need C in $[0,1]^n$.

Survival copula representation

- $(X_1, \ldots, X_n)$ with joint reliability (survival) function

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Moreover, if $\bar{F}_1, \dots, \bar{F}_n$ are continuous, then $\widehat{C}$ is unique.

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  Moreover, if $\bar{F}_1, \dots, \bar{F}_n$ are continuous, then $\widehat{C}$ is unique.
- $\widehat{C}$ is a copula (distribution function), not a survival function.

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- Then
$$F_{n:n}(t) = C(F_1(t), \dots, F_n(t)),\ t \in \mathbb{R}.$$

Parallel systems

- Lifetime of a parallel system $X_{n:n} = \max(X_1, \ldots, X_n)$.
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- The reverse is not true.

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Systems

Theorem (Distortion representation, general case)

If T is the lifetime of a semi-coherent system and its component lifetimes $(X_1, \ldots, X_n)$ have the survival copula $\widehat{C}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \tag{1.5}$$

for all t, where $\bar{Q}$ is a distortion function which depends on the structure ϕ of the sytem and on $\widehat{C}$.

(see, e.g., Navarro, del Águila, Sordo and Suárez-Llorens, 2016)

Distortion representation, ID case

Theorem (Distortion representation, ID case)

If T is the lifetime of a semi-coherent system and the component lifetimes $(X_1, \ldots, X_n)$ have the survival copula $\widehat{C}$ and a common reliability $\bar{F}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

for all t, where $\bar{q}$ is a distortion function which only depends on ϕ and on $\widehat{C}$.

Proof. Take $\bar{q}(u) = \bar{Q}(u, \ldots, u)$.

Distortion representation, IID case

Theorem (Distortion representation, IID case)

If T is the lifetime of a semi-coherent system with IID component lifetimes $X_1, \ldots, X_n$ having a common reliability $\bar{F}$, then the reliability function of T can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t))$$

for all t, where $\bar{q}(u) = \sum_{i=1}^{n} a_i u^i$ is a distortion function and $a = (a_1, \ldots, a_n)$ is the minimal signature which only depends on ϕ. Moreover, $q(u) = \sum_{i=1}^{n} b_i u^i$ where $b = (b_1, \ldots, b_n)$ is the maximal signature.

Table 1: Minimal and maximal signatures

Table: Minimal **a** and maximal **b** signatures of all the coherent systems with 1-4 IID components.

i	T_i	$\mathbf{a}$	$\mathbf{b}$
1	$X_{1:1} = X_1$	(1)	(1)
2	$X_{1:2} = \min(X_1, X_2)$ (2-series)	$(0, 1)$	$(2, -1)$
3	$X_{2:2} = \max(X_1, X_2)$ (2-parallel)	$(2, -1)$	$(0, 1)$
4	$X_{1:3} = \min(X_1, X_2, X_3)$ (3-series)	$(0, 0, 1)$	$(3, -3, 1)$
5	$\min(X_1, \max(X_2, X_3))$	$(0, 2, -1)$	$(1, 1, -1)$
6	$X_{2:3}$ (2-out-of-3)	$(0, 3, -2)$	$(0, 3, -2)$
7	$\max(X_1, \min(X_2, X_3))$	$(1, 1, -1)$	$(0, 2, -1)$
8	$X_{3:3} = \max(X_1, X_2, X_3)$ (3-parallel)	$(3, -3, 1)$	$(0, 0, 1)$
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$ (series)	$(0, 0, 0, 1)$	$(4, -6, 4, -1)$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(0, 0, 2, -1)$	$(2, 0, -2, 1)$
11	$\min(X_{2:3}, X_4)$	$(0, 0, 3, -2)$	$(1, 3, -5, 2)$

Table 1: Minimal and maximal signatures

i	T_i	**a**	**b**
12	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$(0, 1, 1, -1)$	$(1, 2, -3, 1)$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$(0, 3, -3, 1)$	$(1, 0, 1, -1)$
14	$X_{2:4}$ (3-out-of-4)	$(0, 0, 4, -3)$	$(0, 6, -8, 3)$
15	$\max(\min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_2, X_3, X_4))$	$(0, 1, 2, -2)$	$(0, 5, -6, 2)$
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	$(0, 2, 0, -1)$	$(0, 4, -4, 1)$
17	$\max(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_3, X_4))$	$(0, 2, 0, -1)$	$(0, 4, -4, 1)$
18	$\max(\min(X_1, X_2), \min(X_2, X_3), \min(X_3, X_4))$	$(0, 3, -2, 0)$	$(0, 3, -2, 0)$
19	$\max(\min(X_1, \max(X_2, X_3, X_4)), \min(X_2, X_3, X_4))$	$(0, 3, -2, 0)$	$(0, 3, -2, 0)$
20	$\min(\max(X_1, X_2), \max(X_1, X_3), \max(X_2, X_3, X_4))$	$(0, 4, -4, 1)$	$(0, 2, 0, -1)$
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	$(0, 4, -4, 1)$	$(0, 2, 0, -1)$

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i	T_i	**a**	**b**
22	$\min(\max(X_1,X_2),\max(X_1,X_3,X_4),\max(X_2,X_3,X_4))$	$(0,5,-6,2)$	$(0,1,2,-2)$
23	$X_{3:4}$ (2-out-of-4)	$(0,6,-8,3)$	$(0,0,4,-3)$
24	$\max(X_1,\min(X_2,X_3,X_4))$	$(1,0,1,-1)$	$(0,3,-3,1)$
25	$\max(X_1,\min(X_2,X_3),\min(X_3,X_4))$	$(1,2,-3,1)$	$(0,1,1,-1)$
26	$\max(X_{2:3},X_4)$	$(1,3,-5,2)$	$(0,0,3,-2)$
27	$\min(\max(X_1,X_2,X_3),\max(X_2,X_3,X_4))$	$(2,0,-2,1)$	$(0,0,2,-1)$
28	$X_{4:4}=\max(X_1,X_2,X_3,X_4)$ (4-parallel)	$(4,-6,4,-1)$	$(0,0,0,1)$

Applications

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- Stochastic comparisons for systems with non-ID components can be seen in Navarro, del Águila, Sordo and Suárez-Llorens (2016); Navarro and del Águila (2017).
- Preservation of aging classes were studied in Navarro, del Águila, Sordo and Suárez-Llorens (2014, 2016).

Multivariate distorted distributions

Notation

- $(X_1, \ldots, X_n)$ random vector over $(\Omega, \mathcal{S}, \Pr)$.

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where $F_1, \dots, F_n$ are the marginals.
- A similar representation holds for the joint survival function

$$\overline{\mathbf{F}}(x_1, \dots, x_n) = \Pr(X_1 > x_1, \dots, X_n > x_n).$$

Definition

Definition (Navarro, Calì, Longobardi and Durante (2021))

A multivariate distribution function $\mathbf{F}$ is said to be a *multivariate distorted distribution* (MDD) of the univariate distribution functions $G_1, \ldots, G_n$ if there exists a *distortion* function D such that

$$\mathbf{F}(x_1, \ldots, x_n) = D(G_1(x_1), \ldots, G_n(x_n)), \ \forall x_1, \ldots, x_n \in \mathbb{R}. \quad (2.1)$$

We write $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$, when $\mathbf{F}$ is a MDD of $G_1, \ldots, G_n$.

Definition

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A continuous function $D : [0,1]^n \to [0,1]$ is called *(n–dimensional) distortion function* (shortly written as $D \in \mathcal{D}_n$) if:

(i) $D(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$ for all $u_1, \ldots, u_n \in [0,1]$.

(ii) $D(1, \ldots, 1) = 1$.

(iii) D is n–increasing, i.e. for all $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ with $x_i \leq y_i$, it holds $\triangle_{\mathbf{x}}^{\mathbf{y}} D \geq 0$, where

$$\triangle_{(x_1,\ldots,x_n)}^{(y_1,\ldots,y_n)} D := \sum_{z_i \in \{x_i, y_i\}} (-1)^{\mathbf{1}(z_1,\ldots,z_n)} D(z_1, \ldots, z_n),$$

with $\mathbf{1}(z_1, \ldots, z_n) = \sum_{i=1}^{n} 1(z_i = x_i)$ and $1(A) = 1$ (respectively, 0) if A is true (respectively, false).

Main properties

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- According to Sklar's theorem, any multivariate distribution function can be expressed in terms of its univariate marginal distributions via a copula representation.
- If the marginals are continuous then this representation (copula) is unique.
- In the following result, we state a similar Sklar-type theorem for MDD under mild conditions.

Sklar-type theorem

Proposition

Let $(X_1, \ldots, X_n)$ be a random vector with joint continuous distribution function $\mathbf{F}$. Let $G_1, \ldots, G_n$ be arbitrary continuous distribution functions and let us assume that G_i is strictly increasing in the support of X_i for $i = 1, \ldots, n$. Then there exists a unique distortion $D \in \mathcal{D}_n$ such that

$$\mathbf{F}(x_1, \ldots, x_n) = D(G_1(x_1), \ldots, G_n(x_n))$$

holds for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Construction of new multivariate models

The converse of the preceding proposition can be stated as follows.

Proposition

If $D \in \mathcal{D}_n$, then

$$D(G_1(x_1), \dots, G_n(x_n))$$

is a multivariate distribution function for all univariate distribution functions $G_1, \dots, G_n$.

Relationship with the copula

Proposition

Let $(X_1,\dots,X_n)$ be a random vector with joint continuous distribution function $\mathbf{F}$. Let $G_1,\dots,G_n$ be arbitrary continuous distribution functions. Suppose that $\mathbf{F} \equiv MDD(G_1,\dots,G_n)$ with distortion D. Then,

$$D(u_1,\dots,u_n) = C(F_1(G_1^{-1}(u_1)),\dots,F_n(G_n^{-1}(u_n)))$$

for all $(u_1,\dots,u_n) \in [0,1]^n$, where G_i^{-1} is the quasi-inverse of G_i and F_i is the ith marginal of $\mathbf{F}$ for $i = 1,\dots,n$.

Joint survival function.

Proposition

Let $(X_1, \ldots, X_n)$ be a random vector with distribution function $\mathbf{F}$*. If* (2.1) *holds for $G_1, \ldots, G_n$ and $D \in \mathcal{D}_n$, then the joint survival function of $(X_1, \ldots, X_n)$ can be written as*

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \hat{D}(\bar{G}_1(x_1), \ldots, \bar{G}_n(x_n)) \tag{2.2}$$

for all $x_1, \ldots, x_n$, where $\bar{G}_i = 1 - G_i$ is the survival function associated to G_i for $i = 1, \ldots, n$ and $\hat{D} \in \mathcal{D}_n$.

Marginal distributions

- A relevant property of the MDD representation $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$ is that all the multivariate marginal distributions of $\mathbf{F}$ are also MDD from $G_1, \ldots, G_n$.

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- Let $F_{1,\dots,m}$ be the distribution function of $(X_1, \dots, X_m)$.

- Proposition

If $\mathbf{F} \equiv MDD(G_1, \dots, G_n)$ *and* $1 \le m \le n$, *then*

$$F_{1,\dots,m}(x_1, \dots, x_m) = D_{1,\dots,m}(G_1(x_1), \dots, G_m(x_m)) \qquad (2.3)$$

for all $(x_1, \dots, x_m) \in \mathbb{R}^m$, *where*

$$D_{1,\dots,m}(u_1, \dots, u_m) := D(u_1, \dots, u_m, 1, \dots, 1)$$

for all $(u_1, \dots, u_m) \in [0,1]^m$ *and* $D_{1,\dots,m} \in \mathcal{D}_m$.

Univariate marginal distributions.

- In particular, the ith marginal distribution function of X_i can be written as

$$F_i(x_i) = D(1, \dots, 1, G_i(x_i), 1, \dots, 1) = D_i(G_i(x_i)) \qquad (2.4)$$

for all $x_i \in \mathbb{R}$, where

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and the value u is placed at the ith position.

- Clearly, we have $G_i = F_i$ for a fixed $i \in \{1, \dots, n\}$ when $D_i(u) = u$ for all $u \in [0, 1]$.

Probability density function

Let us assume that $\mathbf{F}$ is absolutely continuous with joint probability density function (PDF) $\mathbf{f}$, where

$$\mathbf{f}(x_1,\dots,x_n)=\partial_{1,\dots,n}\mathbf{F}(x_1,\dots,x_n)\ (a.e.).$$

Proposition

If $\mathbf{F}\equiv MDD(G_1,\dots,G_n)$ for absolutely continuous distribution functions $G_1,\dots,G_n$ with PDFs $g_1,\dots,g_n$, respectively, and a distortion function D that admits continuous mixed derivatives of order n, then

$$\mathbf{f}(x_1,\dots,x_n)=g_1(x_1)\dots g_n(x_n)\ \partial_{1,\dots,n}D(G_1(x_1),\dots,G_n(x_n)). \tag{2.5}$$

Conditional distributions

- All the conditional distributions of $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$ have MDD representations.

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- Proposition

Let (X_1, X_2) with $\mathbf{F} \equiv MDD(G_1, G_2)$ for a distortion function $D \in \mathcal{D}_2$ that admits continuous mixed derivatives of order 2, then

$$F_{2|1}(x_2|x_1) = D_{2|1}(G_2(x_2)|G_1(x_1)) \tag{2.6}$$

whenever $\lim_{v \to 0^+} \partial_1 D(G_1(x_1), v) = 0$, *where*

$$D_{2|1}(v|G_1(x_1)) = \frac{\partial_1 D(G_1(x_1), v)}{\partial_1 D(G_1(x_1), 1)}$$

for $0 < v < 1$ *and* x_1 *such that* $\partial_1 D(G_1(x_1), 1) > 0$.

Theoretical Quantile Regression

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- Another option is the *conditional median regression curve*

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(see Koenker (2005) or Nelsen (2006), p. 217).

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- This quantile function $F_{2|1}^{-1}$ can be computed from (2.6) as

$$F_{2|1}^{-1}(q|x_1) = G_2^{-1}(D_{2|1}^{-1}(q|G_1(x_1))), \; 0 < q < 1.$$

Confidence bands

- Moreover, we can obtain α-confidence bands in a similar way with
$$\left[F_{2|1}^{-1}(\beta_1|x_1), F_{2|1}^{-1}(\beta_2|x_1)\right]$$
taking $0 \leq \beta_1 < \beta_2 \leq 1$ such that $\beta_2 - \beta_1 = \alpha$.

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taking $0 \leq \beta_1 < \beta_2 \leq 1$ such that $\beta_2 - \beta_1 = \alpha$.
- For example, the centered 50% and 90% quantile-confidence bands for $(X_2|X_1 = x_1)$ are determined, respectively, by
$$\left[F_{2|1}^{-1}(0.25|x_1), F_{2|1}^{-1}(0.75|x_1)\right]$$
and
$$\left[F_{2|1}^{-1}(0.05|x_1), F_{2|1}^{-1}(0.95|x_1)\right].$$

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- $(X_i - t | X_i > t)$ denotes the univariate residual lifetimes at time $t > 0$ with

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for every $x \geq 0$, whenever $\bar{F}_i(t) > 0$.

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for every $x \geq 0$, whenever $\bar{F}_i(t) > 0$.
- The mean residual lifetime is $m_i(t) = E(X_i - t | X_i > t)$.

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- $X_1, \ldots, X_n$ represent the lifetimes of n components.
- $(X_i - t|X_i > t)$ denotes the univariate residual lifetimes at time $t > 0$ with

$$\bar{F}_{i,t}(x) := \Pr(X_i - t > x|X_i > t) = \frac{\bar{F}_i(t+x)}{\bar{F}_i(t)}$$

  for every $x \geq 0$, whenever $\bar{F}_i(t) > 0$.
- The mean residual lifetime is $m_i(t) = E(X_i - t|X_i > t)$.
- From $\mathbf{X} = (X_1, \ldots, X_n)$, we can consider

$$\mathbf{X}_t = (X_1 - t, \ldots, X_n - t|X_1 > t, \ldots, X_n > t)$$

  whose survival function for $x_1, \ldots, x_n \geq$ is

$$\bar{F}_t(x_1, \ldots, x_n) := \Pr(X_1 > x_1+t, \ldots, X_n > x_n+t|X_1 > t, \ldots, X_n > t).$$

Example 1: Residual lifetimes

Proposition

If $\bar{F}(t,\dots,t) > 0$ *for some* $t \geq 0$, *then*

$$\bar{F}_t(x_1,\dots,x_n) = \widehat{D}_t(\bar{F}_{1,t}(x_1),\dots,\bar{F}_{n,t}(x_n)) \tag{2.7}$$

for all $x_1,\dots,x_n \geq t$ *and distortion function*

$$\widehat{D}_t(u_1,\dots,u_n) := \frac{\widehat{C}(\bar{F}_1(t)u_1,\dots,\bar{F}_n(t)u_n)}{\widehat{C}(\bar{F}_1(t),\dots,\bar{F}_n(t))},\ \ u_1,\dots,u_n \in [0,1], \tag{2.8}$$

which depends on $\bar{F}_1(t),\dots,\bar{F}_n(t)$.

Example 1: Residual lifetimes

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- Hence (2.7) is not a copula representation and $\widehat{D}_t$ is not always a copula.
- If $X_1, \ldots, X_n$ are exponential, then $\bar{F}_{i,t} = \bar{F}_i \neq \bar{H}_{i,t}$.

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- Let us assume that X and Y have a common absolutely continuous distribution function F. Then

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- Note that both F and C can be estimated from the training sample by using parametric models or empirical or kernel type estimators.
- So, we want to obtain a MDD representation for the random vector (L, U) in terms of F and C.

Example 2: Ordered paired data

- The joint distribution $\mathbf{G}(x,y)=\Pr(L\leq x, U\leq y)$ of (L,U) is

$$\mathbf{G}(x,y)=\begin{cases} C(F(y),F(y)) & \text{for} \quad y\leq x;\\ C(F(x),F(y))+C(F(y),F(x))-C(F(x),F(x)) & \text{for} \quad y>x.\end{cases}$$

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- Therefore, $\mathbf{G}\equiv MDD(F,F)$, i.e.

$$\mathbf{G}(x,y)=D(F(x),F(y)) \tag{2.9}$$

with the following distortion function

$$D(u,v)=\begin{cases} C(v,v) & \text{for} \quad v\le u;\\ C(u,v)+C(v,u)-C(u,u) & \text{for} \quad u<v. \end{cases} \tag{2.10}$$

Example 2: Ordered paired data

- Then the marginal distributions of (L, U) can be written as

$$G_1(x) := \Pr(L \leq x) = D(F(x), 1) = D_1(F(x)),$$

$$G_2(y) := \Pr(U \leq y) = D(1, F(y)) = D_2(F(y)),$$

where

$$D_1(u) = D(u, 1) = 2u - C(u, u)$$

and

$$D_2(v) = D(1, v) = C(v, v)$$

for all $u, v \in [0, 1]$.

Example 2: Ordered paired data, IID case

- For example, if X and Y are independent, then

$$D_1(u) = D(u,1) = 2u - u^2 \neq u$$

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- The distortion function is

$$D(u,v) = \begin{cases} v^2 & \text{for} \quad v \leq u; \\ 2uv - u^2 & \text{for} \quad u < v. \end{cases} \tag{2.11}$$

- Note that it is not a copula and that the marginals G_1 and G_2 of $\mathbf{G}$ do not appear in (2.9) (we use F instead).

Example 2: Ordered paired data, conditional distribution

- From (2.6) and (2.9), the distribution function of $(U|L = x)$ is

$$G_{2|1}(y|x) = D_{2|1}(F(y)|F(x)) \tag{2.12}$$

for $y \geq x$, where

$$D_{2|1}(v|F(x)) := \frac{\partial_1 D(F(x), v)}{\partial_1 D(F(x), 1)},$$

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- In the EXC case, we have

$$\partial_1 D(u,v) = 2\partial_1 C(u,v) - 2\partial_1 C(u,u), \; u \leq v \leq 1.$$

Example 3: Coherent systems

Theorem

If T_1 and T_2 are two coherent systems with ID$\sim F$ common components $(X_1, \ldots, X_n)$, then its joint distribution can be written as

$$\mathbf{G}(t_1, t_2) = D(F(t_1), F(t_2)), \quad \forall t_1, t_2.$$

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- In particular, it can be applied to the k-out-of-n systems (order statistics).
- In a more particular case, for $X_{1:2}$ and $X_{2:2}$ we obtain the distortion D of Example 2.
- Other examples: Sequential order statistics, record values, convolutions, ...

Exact QR curves for paired ordered data. IID case.

- Let (X_i, Y_i) be a sample from (X, Y) where X, Y are IID$\sim F$.

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- Note that L_i and U_i are dependent.

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- Let $L_i = \min(X_i, Y_i)$ and $U_i = \max(X_i, Y_i)$.
- Note that L_i and U_i are dependent.
- From (2.12), the distribution function of $(U|L = x)$ is

$$G_{2|1}(y|x) = D_{2|1}(F(y)|F(x)) \tag{2.13}$$

for $y \geq x$, where

$$D_{2|1}(v|F(x)) = \frac{v - F(x)}{\bar{F}(x)}$$

for $F(x) \leq v \leq 1$.

Paired ordered data. Exp IID case.

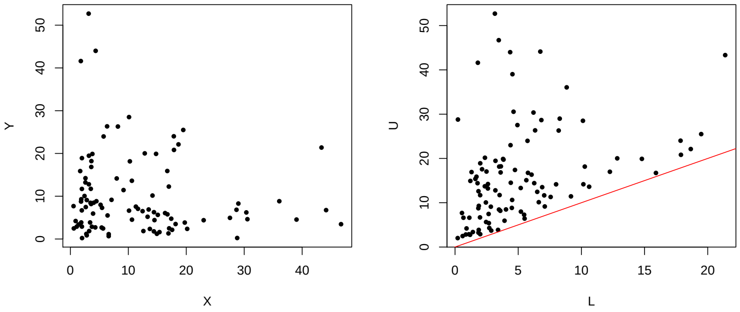

Figure: Independent data from two exponential distributions with mean $\mu = 10$ (left) and the associated paired ordered data (right).

QR for paired ordered data. Exp IID case.

- The quantile function $F_{2|1}^{-1}$ can be computed as

$$F_{2|1}^{-1}(q|x) = F^{-1}(D_{2|1}^{-1}(q|F(x)))$$

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- If $\bar{F}(x) = \exp(-x/\mu)$, then $F^{-1}(y) = -\mu \log(1-y)$ and

$$F_{2|1}^{-1}(q|x) = -\mu \log\left((1-q)e^{-x/\mu}\right) = x - \mu \log(1-q).$$

- Therefore, the exact QR curve is

$$m(x) = x - \mu \log(0.5).$$

QR for paired ordered data. Exp IID case.

- Analogously, the exact QR centered 90% confidence band is

$$[x-\mu\log(0.05),x-\mu\log(0.95)]\,.$$

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- The 50% centered confidence band is obtained in a similar way.

QR for paired ordered data. Exp IID case.

- Analogously, the exact QR centered 90% confidence band is

$$[x-\mu\log(0.05), x-\mu\log(0.95)] .$$

- The 50% centered confidence band is obtained in a similar way.
- Similarly, the exact QR bottom 90% confidence band is

$$[x, x-\mu\log(0.90)] .$$

QR for paired ordered data. Exp IID case.

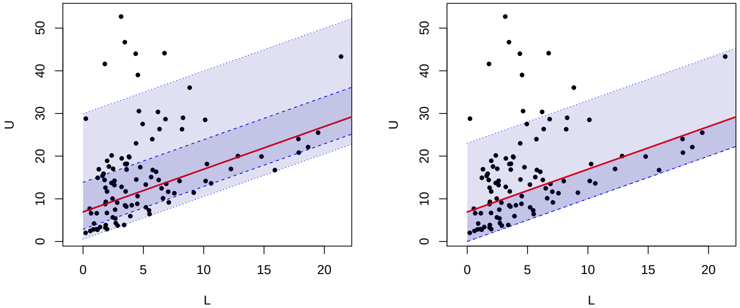

Figure: QR for the paired ordered data (L, U) associated to independent data (X, Y) from two exponential distributions with mean $\mu = 10$ jointly with 50% and 90% centered (left) or bottom (right) confidence bands.

Predictions

- The first ordered pair in our sample is $L_1 = 10.15771$ and $U_1 = 14.17195$.

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- The centered 50% confidence interval for this prediction is $[13.03453, 24.02065]$.

QR for dependent EXC data

- Let us consider now that (X, Y) are DID with an EXC copula C and a common marginal distribution F.

QR for dependent EXC data

- Let us consider now that (X, Y) are DID with an EXC copula C and a common marginal distribution F.
- We consider again the exponential model

$$\bar{F}(t) = \exp(-t/\mu), \ \ t \geq 0$$

and now the Clayton copula

$$C(u, v) = \frac{uv}{u + v - uv}, \ \ u, v \in [0, 1]. \tag{2.14}$$

QR for dependent EXC data

- To get the QR curves we need the distribution $G_{2|1}(y|x)$ of $(U|L=x)$. From (2.12) we need

$$\partial_1 D(u,v) = 2\partial_1 C(u,v) - 2\partial_1 C(u,u) = \frac{2v^2}{(u+v-uv)^2} - \frac{2}{(2-u)^2}$$

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- Hence, we get $G_{2|1}(y|x)=D_{2|1}(F(y)|F(x))$, where

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$$D_{2|1}(v|u) = \frac{\frac{v^2}{(u+v-uv)^2} - \frac{1}{(2-u)^2}}{1 - \frac{1}{(2-u)^2}}, \quad v \geq u.$$

- To compute the inverse, we need to solve for $y \geq x$ the equation $G_{2|1}(y|x) = q$ for a given $q \in (0,1)$.

QR for dependent EXC data

- This leads to

$$G_{2|1}^{-1}(q|x) = y = F^{-1}\left(\frac{F(x)}{F(x) - 1 + \frac{2-F(x)}{\sqrt{1-q+q(2-F(x))^2}}}\right)$$

for $0 < q < 1$.

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- The other confidence bands can be obtained in a similar way.
- For an exponential distribution with $\mu = 10$ we get:

QR for dependent EXC data

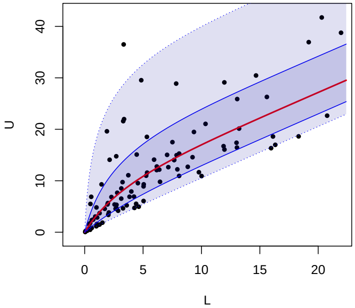

Figure: QR curves for paired ordered data (L, U) associated to dependent data (X, Y) from two exponential distributions with centered (left) and bottom (right) confidence bands.

Parametric QR curves

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- They can be both in F or in C.
- In that case we can use the training sample (X_i, Y_i) from (X, Y) to estimate the unknown parameters.
- Then we can use the MDD representation with the estimated parameters to get the estimated QR curves.

Parametric QR for paired ordered data IID case

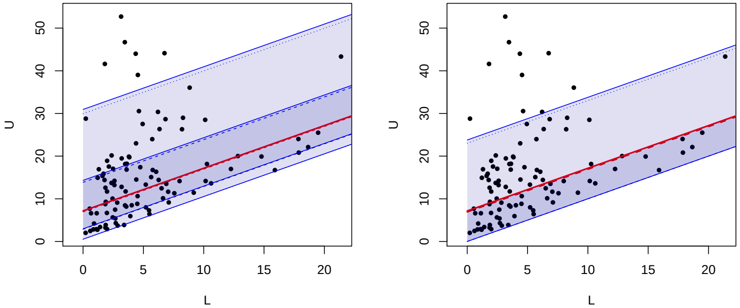
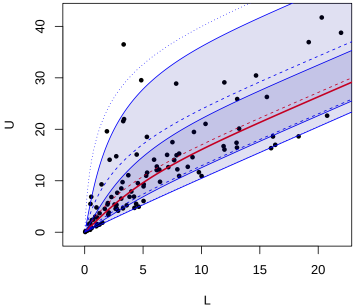


Figure: Parametric QR curves for (L, U) associated to IID data (X, Y) from an exponential distribution jointly with centered (left) and bottom (right) confidence bands. The dashed lines are the exact curves.

Parametric QR for paired ordered data EXC case



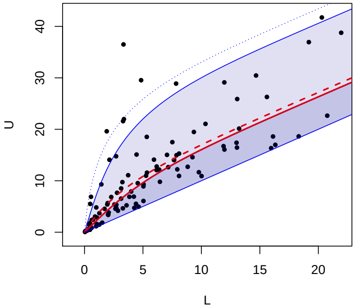


Figure: Parametric QR curves for (L, U) associated to data (X, Y) from an exponential distribution with unknown mean μ and a Clayton copula with unknown parameter θ. The dashed lines are the exact curves.

Non-parametric QR curves.

- If we do not have a parametric model, we can use non-parametric estimators for F and C.

Non-parametric QR curves.

- If we do not have a parametric model, we can use non-parametric estimators for F and C.
- Instead, we can also use the statistical program R with `library('quantreg')` to estimate the exact curves from the training sample (see Koenker, 2005; Koenker and Bassett, 1978) and $L = x$, with $x, x^2, \ldots, x^k$ as predictors.

Non-parametric QR for paired ordered data, IID case

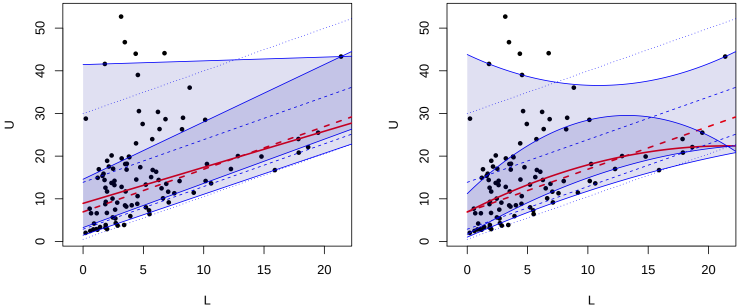
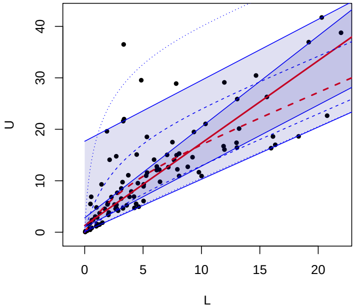


Figure: Non-parametric QR curves for paired ordered data (L, U) associated to IID data (X, Y) from an exponential distribution with $\mu = 10$ and $k = 1$ (left) or $k = 2$ (right).

Non-parametric QR for paired ordered data, EXC case



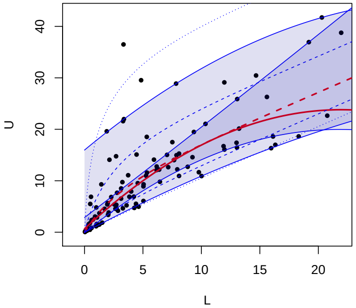


Figure: Non-parametric QR curves for (L, U) associated to data (X, Y) from an exponential distribution and a Clayton copula with $\theta = 1$ and $k = 1$ (left) or $k = 2$ (right). The dashed lines are the exact curves.

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The slides and more references can be seen in my webpage:

https://webs.um.es/jorgenav/miwiki/doku.php

- That's all.

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- Questions?