Predicting system failures

Jorge Navarro[1]
Universidad de Murcia, Murcia, Spain.

MMuR23cia

12th International Conference on Mathematical Methods in Reliability
May 30- June 2, 2023, Murcia (Spain)

[1]Partially supported by Ministerio de Ciencia e Innovación of Spain under grant PID2019-108079GB-C22/AEI/10.13039/501100011033.

References

The conference is based on the following references:

- Navarro J, Calì C, Longobardi M, Durante F. (2022). Distortion representations of multivariate distributions. *Statistical Methods & Applications* 31, 925–954. DOI: 10.1007/s10260-021-00613-2.
- Navarro J., Arriaza A. Suárez-Llorens A. (2023). Predicting system failure times. Submitted.

Outline

Preliminary results
- Coherent systems
- Distortion representations

Predicting system failures
- Case I: $T_1 < T$
- Case II: $T_1 \leq T$
- Case III: $T_1 < T_2 < T$

Examples
- Case I
- Case II
- Case III

Preliminary results

Coherent systems

- A **system** with n components is a Boolean function

$$\phi : \{0,1\}^n \to \{0,1\}$$

where $\phi(x_1, \dots, x_n)$ represents the state of the system when we know the states $x_1, \dots, x_n$ for the components.

Coherent systems

- A **system** with n components is a Boolean function

$$\phi : \{0,1\}^n \to \{0,1\}$$

  where $\phi(x_1,\ldots,x_n)$ represents the state of the system when we know the states $x_1,\ldots,x_n$ for the components.
- A system is **semi-coherent** if ϕ is increasing, $\phi(0,\ldots,0)=0$ and $\phi(1,\ldots,1)=1$.

Coherent systems

- A **system** with n components is a Boolean function

$$\phi : \{0,1\}^n \to \{0,1\}$$

where $\phi(x_1,\dots,x_n)$ represents the state of the system when we know the states $x_1,\dots,x_n$ for the components.

- A system is **semi-coherent** if ϕ is increasing, $\phi(0,\dots,0)=0$ and $\phi(1,\dots,1)=1$.
- A system is **coherent** if ϕ is increasing and it is strictly increasing in at least a point in each variable (i.e. it does not contain irrelevant components).

Basic properties

- A set $P \subseteq \{1,\dots,n\}$ is a **path set** of a system ϕ if

$$\phi(x_1,\dots,x_n)=1$$

when $x_i=1$ for all $i\in P$.

Basic properties

- A set $P \subseteq \{1, \ldots, n\}$ is a **path set** of a system ϕ if

$$\phi(x_1, \ldots, x_n) = 1$$

  when $x_i = 1$ for all $i \in P$.
- A path set is a **minimal pat set** if does not contain other path sets.

Basic properties

- A set $P \subseteq \{1,\dots,n\}$ is a **path set** of a system ϕ if

$$\phi(x_1,\dots,x_n)=1$$

when $x_i=1$ for all $i\in P$.

- A path set is a **minimal pat set** if does not contain other path sets.
- If $P_1,\dots,P_r$ are the minimal path sets of a system ϕ, then

$$\phi(x_1,\dots,x_n)=\max_{i=1,\dots,n}\min_{j\in P_i} x_j.$$

Basic properties

- If T is the lifetime of a system, then $\bar{F}_T(t) = \Pr(T > t)$ is the system reliability function for $t \geq 0$.

Basic properties

- If T is the lifetime of a system, then $\bar{F}_T(t) = \Pr(T > t)$ is the system reliability function for $t \geq 0$.
- If $X_1, \ldots, X_n$ are the lifetimes of the components of a system, then $\bar{F}_i(t) = \Pr(X_i > t)$ is the reliability function of the ith component and

$$\Pr(X_1 > t_1, \ldots, X_n > t_n) = \hat{C}(\bar{F}_1(t_1), \ldots, \bar{F}_n(t_n))$$

is the joint reliability function of the components.

Basic properties

- If T is the lifetime of a system, then $\bar{F}_T(t) = \Pr(T > t)$ is the system reliability function for $t \geq 0$.
- If $X_1, \ldots, X_n$ are the lifetimes of the components of a system, then $\bar{F}_i(t) = \Pr(X_i > t)$ is the reliability function of the ith component and

$$\Pr(X_1 > t_1, \ldots, X_n > t_n) = \hat{C}(\bar{F}_1(t_1), \ldots, \bar{F}_n(t_n))$$

  is the joint reliability function of the components.
- If $P_1, \ldots, P_r$ are the minimal path sets of a system ϕ, then

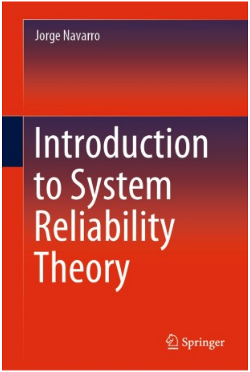
$$T = \phi(X_1, \ldots, X_n) = \max_{i=1,\ldots,n} \min_{j \in P_i} X_j.$$

Basic references on coherent systems

- Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston.

Basic references on coherent systems

- Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston.
- My new book:



Distortion representations

- The system reliability function can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)) \quad \text{for all } t \in \mathbb{R}, \tag{1.1}$$

where $\bar{Q} : [0,1]^n \to [0,1]$ is a distortion function, i.e., $\bar{Q}$ is continuous, is increasing and satisfies $\bar{Q}(0,\dots,0) = 0$ and $\bar{Q}(1,\dots,1) = 1$. $\bar{Q}$ only depends on $P_1, \dots, P_r$ and $\hat{C}$.

Distortion representations

- The system reliability function can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)) \text{ for all } t \in \mathbb{R}, \tag{1.1}$$

where $\bar{Q} : [0,1]^n \to [0,1]$ is a distortion function, i.e., $\bar{Q}$ is continuous, is increasing and satisfies $\bar{Q}(0,\dots,0) = 0$ and $\bar{Q}(1,\dots,1) = 1$. $\bar{Q}$ only depends on $P_1, \dots, P_r$ and $\hat{C}$.

- If the components are identically distributed (ID), then

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)), \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

where $\bar{q}(u) = \bar{Q}(u,\dots,u)$ is a distortion function and $\bar{F} = \bar{F}_1 = \dots = \bar{F}_n$.

Distortion representations

Definition (Navarro, Calì, Longobardi and Durante (2022))

A multivariate distribution function **F** is said to be a *multivariate distorted distribution* (MDD) of the univariate distribution functions $G_1, \dots, G_n$ if there exists a **multivariate distortion function** D such that

$$\mathbf{F}(x_1, \dots, x_n) = D(G_1(x_1), \dots, G_n(x_n)), \; \forall x_1, \dots, x_n \in \mathbb{R}. \quad (1.3)$$

D is a continuous multivariate distribution function with support contained in $[0,1]^n$.

Distortion representations

- If T_1 and T_2 are the lifetimes of two coherent systems with common ID component lifetimes $X_1, \ldots, X_n$ then

$$\Pr(T_1 > t_1, T_2 > t_2) = D(\bar{F}(t_1), \bar{F}(t_2)) \text{ for all } t_1, t_2 \in \mathbb{R}, \tag{1.4}$$

where D is a bivariate distortion function which depends on ϕ_1, ϕ_2 and $\hat{C}$.

Distortion representations

- If T_1 and T_2 are the lifetimes of two coherent systems with common ID component lifetimes $X_1, \dots, X_n$ then

$$\Pr(T_1 > t_1, T_2 > t_2) = D(\bar{F}(t_1), \bar{F}(t_2)) \text{ for all } t_1, t_2 \in \mathbb{R}, \quad (1.4)$$

  where D is a bivariate distortion function which depends on ϕ_1, ϕ_2 and $\hat{C}$.
- The purpose of the paper is to use (1.4) to predict T_2 when we know T_1 and we assume $T_1 \le T_2$.

Distortion representations

- If T_1 and T_2 are the lifetimes of two coherent systems with common ID component lifetimes $X_1, \dots, X_n$ then

$$\Pr(T_1 > t_1, T_2 > t_2) = D(\bar{F}(t_1), \bar{F}(t_2)) \text{ for all } t_1, t_2 \in \mathbb{R}, \quad (1.4)$$

  where D is a bivariate distortion function which depends on ϕ_1, ϕ_2 and $\hat{C}$.
- The purpose of the paper is to use (1.4) to predict T_2 when we know T_1 and we assume $T_1 \leq T_2$.
- To this end we will use quantile regression techniques that also provide prediction intervals for T_2.

Conditional distributions

- All the conditional distributions of a multivariate distorted distribution (MDD) have also MDD representations.

Conditional distributions

- All the conditional distributions of a multivariate distorted distribution (MDD) have also MDD representations.
- In particular $(T_2|T_1 = t_1)$ has a distortion representation, i.e.,

$$\bar{F}_{2|1}(t_2|t_1) = \Pr(T_2 > t_2|T_1 = t_1) = D_{2|1}(\bar{F}(t_2)|\bar{F}(t_1)) \quad (1.5)$$

where

$$D_{2|1}(v|u) = \frac{\partial_1 D(u,v) - \partial_1 D(u,0^+)}{\partial_1 D(u,1)}$$

is a distortion function for all $0 < u < 1$ such that $\partial_1 D(u,1) > 0$.

Quantile regression

- The (mean) **regression curve** to predict T_2 from T_1 is

$$\tilde{m}(t_1) = E(T_2 | T_1 = t_1).$$

Quantile regression

- The (mean) **regression curve** to predict T_2 from T_1 is

$$\tilde{m}(t_1) = E(T_2|T_1 = t_1).$$

- Another option to predict T_2 from T_1 is the **conditional median regression curve**

$$m(t_1) := \bar{F}^{-1}_{2|1}(0.5|t_1)$$

(see Koenker (2005) or Nelsen (2006), p. 217).

Quantile regression

- The (mean) **regression curve** to predict T_2 from T_1 is

$$\tilde{m}(t_1) = E(T_2|T_1 = t_1).$$

- Another option to predict T_2 from T_1 is the **conditional median regression curve**

$$m(t_1) := \bar{F}_{2|1}^{-1}(0.5|t_1)$$

(see Koenker (2005) or Nelsen (2006), p. 217).

Quantile regression

- The (mean) **regression curve** to predict T_2 from T_1 is

$$\tilde{m}(t_1) = E(T_2 | T_1 = t_1).$$

- Another option to predict T_2 from T_1 is the **conditional median regression curve**

$$m(t_1) := \bar{F}_{2|1}^{-1}(0.5|t_1)$$

(see Koenker (2005) or Nelsen (2006), p. 217).

- The quantile function $F_{2|1}^{-1}$ can be computed from (1.5).

Quantile regression

- The (mean) **regression curve** to predict T_2 from T_1 is

$$\tilde{m}(t_1) = E(T_2 | T_1 = t_1).$$

- Another option to predict T_2 from T_1 is the **conditional median regression curve**

$$m(t_1) := \bar{F}_{2|1}^{-1}(0.5|t_1)$$

(see Koenker (2005) or Nelsen (2006), p. 217).

- The quantile function $F_{2|1}^{-1}$ can be computed from (1.5).
- Moreover, it can be used to obtain α-prediction bands for T_2

$$\left[\bar{F}_{2|1}^{-1}(\beta_2|t_1), \bar{F}_{2|1}^{-1}(\beta_1|t_1)\right]$$

taking $0 \le \beta_1 < \beta_2 \le 1$ such that $\beta_2 - \beta_1 = \alpha \in (0,1)$.

Predicting system failures

Typical cases

- Let us consider three typical cases:

Typical cases

- Let us consider three typical cases:
- **Case I:** $T_1 < T$. For example, $T_1 = \min(X_1, \ldots, X_n)$ and T is a system that does not fail with the first component failure.

Typical cases

- Let us consider three typical cases:
- **Case I:** $T_1 < T$. For example, $T_1 = \min(X_1, \dots, X_n)$ and T is a system that does not fail with the first component failure.
- **Case II:** $T_1 \leq T$. Here we have two options:

Typical cases

- Let us consider three typical cases:
- **Case I:** $T_1 < T$. For example, $T_1 = \min(X_1, \ldots, X_n)$ and T is a system that does not fail with the first component failure.
- **Case II:** $T_1 \leq T$. Here we have two options:
- **Case II.a:** In the a priori option, i.e., before the failure of T_1, we can provide a protocol to predict T without the knowledge of the event $T_1 = T$. Here we use $(T|T_1 = t \leq T)$.

Typical cases

- Let us consider three typical cases:
- **Case I:** $T_1 < T$. For example, $T_1 = \min(X_1, \dots, X_n)$ and T is a system that does not fail with the first component failure.
- **Case II:** $T_1 \leq T$. Here we have two options:
- **Case II.a:** In the a priori option, i.e., before the failure of T_1, we can provide a protocol to predict T without the knowledge of the event $T_1 = T$. Here we use $(T|T_1 = t \leq T)$.
- **Case II,b:** In the a posteriori option, i.e., when we are at time $t = T_1$, we could assume $T > t$ since if $T = t$ we do not need to predict it. Here we use $(T_2|T_1 = t < T_2)$.

Typical cases

- Let us consider three typical cases:
- **Case I:** $T_1 < T$. For example, $T_1 = \min(X_1, \ldots, X_n)$ and T is a system that does not fail with the first component failure.
- **Case II:** $T_1 \le T$. Here we have two options:
- **Case II.a:** In the a priori option, i.e., before the failure of T_1, we can provide a protocol to predict T without the knowledge of the event $T_1 = T$. Here we use $(T|T_1 = t \le T)$.
- **Case II,b:** In the a posteriori option, i.e., when we are at time $t = T_1$, we could assume $T > t$ since if $T = t$ we do not need to predict it. Here we use $(T_2|T_1 = t < T_2)$.
- **Case III:** We can predict the system lifetime T from two preceding system lifetimes $T_1 < T_2 < T$.

Case I: $T_1 < T$

Theorem

If T_1 and T are the lifetimes of two coherent systems satisfying $T_1 < T$ based on the same ID component lifetimes and (T_1, T) has a joint absolutely continuous distribution, then there exists a bivariate distortion function $\widehat{D} : [0,1]^2 \to [0,1]$ such that

$$\bar{G}(x,y) := \Pr(T_1 > x, T > y) = \widehat{D}(\bar{F}(x), \bar{F}(y)) \qquad (2.1)$$

for all x, y. Moreover, the reliability function of $(T|T_1 = t)$ is

$$\bar{G}_{T|T_1}(y|t) := \Pr(T > y|T_1 = t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y)) - \partial_1 \widehat{D}(\bar{F}(t), 0^+)}{\partial_1 \widehat{D}(\bar{F}(t), 1)} \qquad (2.2)$$

for $y \ge t$, where $\partial_1 \widehat{D}(u, 0^+) := \lim_{v \to 0^+} \partial_1 \widehat{D}(u, v)$.

Case I: $T_1 < T$

- Note that (2.1) is not a copula representation since $\bar{F}$ is neither the reliability function of T_1 nor that of T.

Case I: $T_1 < T$

- Note that (2.1) is not a copula representation since $\bar{F}$ is neither the reliability function of T_1 nor that of T.
- The proof of Theorem 2.1 shows how to get $\widehat{D}$.

Case I: $T_1 < T$

- Note that (2.1) is not a copula representation since $\bar{F}$ is neither the reliability function of T_1 nor that of T.
- The proof of Theorem 2.1 shows how to get $\widehat{D}$.
- In many cases $\partial_1 \widehat{D}(u, 0^+) = 0$ holds and then

$$\bar{G}_{T|T_1}(y|t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\partial_1 \widehat{D}(\bar{F}(t), 1)}. \tag{2.3}$$

Case I: $T_1 < T$

- Note that (2.1) is not a copula representation since $\bar{F}$ is neither the reliability function of T_1 nor that of T.
- The proof of Theorem 2.1 shows how to get $\widehat{D}$.
- In many cases $\partial_1 \widehat{D}(u, 0^+) = 0$ holds and then

$$\bar{G}_{T|T_1}(y|t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\partial_1 \widehat{D}(\bar{F}(t), 1)}. \tag{2.3}$$

- This expression can be used to both compute

$$\tilde{m}(t) = E(T|T_1 = t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = \int_0^\infty \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\partial_1 \widehat{D}(\bar{F}(t), 1)} dy,$$

and to get the quantiles of $(T|T_1 = t)$.

Case I: $T_1 < T$

- Note that (2.1) is not a copula representation since $\bar{F}$ is neither the reliability function of T_1 nor that of T.
- The proof of Theorem 2.1 shows how to get $\widehat{D}$.
- In many cases $\partial_1 \widehat{D}(u, 0^+) = 0$ holds and then

$$\bar{G}_{T|T_1}(y|t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\partial_1 \widehat{D}(\bar{F}(t), 1)}. \tag{2.3}$$

- This expression can be used to both compute

$$\tilde{m}(t) = E(T|T_1 = t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = \int_0^\infty \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\partial_1 \widehat{D}(\bar{F}(t), 1)} dy,$$

  and to get the quantiles of $(T|T_1 = t)$.
- For the latter, we will need the inverse function of $\bar{G}_{T|T_1}(y|t)$, denoted as $\bar{G}^{-1}_{T|T_1}(w|t)$, obtained by solving $\bar{G}_{T|T_1}(y|t) = w$ for $0 < w < 1$.

Case I: $T_1 < T$

- Then, the median regression curve to predict T from T_1 is obtained with $w = 0.5$ as

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) \quad \text{for } t \geq 0$$

Case I: $T_1 < T$

- Then, the median regression curve to predict T from T_1 is obtained with $w = 0.5$ as

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) \quad \text{for } t \geq 0$$

- The centered prediction band for T at level 90% is obtained with $w = 0.05$ and $w = 0.95$ as

$$I_{90}(t) = \left[\bar{G}^{-1}_{T|T_1}(0.95|t), \bar{G}^{-1}_{T|T_1}(0.05|t)\right].$$

Case I: $T_1 < T$

- Then, the median regression curve to predict T from T_1 is obtained with $w = 0.5$ as

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) \quad \text{for } t \geq 0$$

- The centered prediction band for T at level 90% is obtained with $w = 0.05$ and $w = 0.95$ as

$$I_{90}(t) = \left[\bar{G}^{-1}_{T|T_1}(0.95|t), \bar{G}^{-1}_{T|T_1}(0.05|t)\right].$$

- Of course, $\Pr(T \in I_{90}(t)|T_1 = t) = 0.90$ for all $t \geq 0$.

Case I: $T_1 < T$

- Then, the median regression curve to predict T from T_1 is obtained with $w = 0.5$ as

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) \quad \text{for } t \geq 0$$

- The centered prediction band for T at level 90% is obtained with $w = 0.05$ and $w = 0.95$ as

$$I_{90}(t) = \left[\bar{G}^{-1}_{T|T_1}(0.95|t), \bar{G}^{-1}_{T|T_1}(0.05|t)\right].$$

- Of course, $\Pr(T \in I_{90}(t)|T_1 = t) = 0.90$ for all $t \geq 0$.
- Other prediction bands can be obtained similarly.

Case I: $T_1 < T$

- Then, the median regression curve to predict T from T_1 is obtained with $w = 0.5$ as

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) \quad \text{for } t \geq 0$$

- The centered prediction band for T at level 90% is obtained with $w = 0.05$ and $w = 0.95$ as

$$I_{90}(t) = \left[\bar{G}^{-1}_{T|T_1}(0.95|t), \bar{G}^{-1}_{T|T_1}(0.05|t)\right].$$

- Of course, $\Pr(T \in I_{90}(t)|T_1 = t) = 0.90$ for all $t \geq 0$.
- Other prediction bands can be obtained similarly.
- The median regression curve is an excellent alternative to the conditional expectation, and the prediction bands allow us to give more accurate predictions.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.
- In practice, two options can be considered.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.
- In practice, two options can be considered.
- In the first case (II.a), we are at time zero, and we want to know a priori what will happen when the failure of T_1 occurs at a time t.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.
- In practice, two options can be considered.
- In the first case (II.a), we are at time zero, and we want to know a priori what will happen when the failure of T_1 occurs at a time t.
- This case includes when both lifetimes coincide, that is, $T_1 = T = t$.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.
- In practice, two options can be considered.
- In the first case (II.a), we are at time zero, and we want to know a priori what will happen when the failure of T_1 occurs at a time t.
- This case includes when both lifetimes coincide, that is, $T_1 = T = t$.
- In the second case (II.b), we are at a time $t > 0$ and we know that $T_1 = t$ and that $T > t$.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.
- In practice, two options can be considered.
- In the first case (II.a), we are at time zero, and we want to know a priori what will happen when the failure of T_1 occurs at a time t.
- This case includes when both lifetimes coincide, that is, $T_1 = T = t$.
- In the second case (II.b), we are at a time $t > 0$ and we know that $T_1 = t$ and that $T > t$.
- Note that if $T = t$, we do not need to predict T.

Case II: $T_1 \leq T$

- This is the most complex case because (T_1, T) has a singular part over the line $T = T_1$.
- In practice, two options can be considered.
- In the first case (II.a), we are at time zero, and we want to know a priori what will happen when the failure of T_1 occurs at a time t.
- This case includes when both lifetimes coincide, that is, $T_1 = T = t$.
- In the second case (II.b), we are at a time $t > 0$ and we know that $T_1 = t$ and that $T > t$.
- Note that if $T = t$, we do not need to predict T.
- Let us see how these cases can be managed.

Case II.b: $(T|T_1 = t < T)$.

- First, we note that the joint reliability function of (T_1, T) can be written as in (2.1) for this case as well.

Case II.b: $(T|T_1 = t < T)$.

- First, we note that the joint reliability function of (T_1, T) can be written as in (2.1) for this case as well.
- Now we might have a singular part in $T = T_1$.

Case II.b: $(T|T_1 = t < T)$.

- First, we note that the joint reliability function of (T_1, T) can be written as in (2.1) for this case as well.
- Now we might have a singular part in $T = T_1$.
- However, if the components have an absolutely continuous joint distribution, then the joint distribution of (T_1, T) in the set $T > T_1$ is absolutely continuous as well.

Case II.b: $(T|T_1 = t < T)$.

- First, we note that the joint reliability function of (T_1, T) can be written as in (2.1) for this case as well.
- Now we might have a singular part in $T = T_1$.
- However, if the components have an absolutely continuous joint distribution, then the joint distribution of (T_1, T) in the set $T > T_1$ is absolutely continuous as well.
- Then (2.2) holds for $y > t \geq 0$ and can be completed by adding that $\bar{G}_{T|T_1}(y|t) = 1$ for $0 \leq y \leq t$.

Case II.b: $(T|T_1 = t < T)$.

- First, we note that the joint reliability function of (T_1, T) can be written as in (2.1) for this case as well.
- Now we might have a singular part in $T = T_1$.
- However, if the components have an absolutely continuous joint distribution, then the joint distribution of (T_1, T) in the set $T > T_1$ is absolutely continuous as well.
- Then (2.2) holds for $y > t \geq 0$ and can be completed by adding that $\bar{G}_{T|T_1}(y|t) = 1$ for $0 \leq y \leq t$.
- However, note that, in this case

$$\alpha(t) := \Pr(T > t|T_1 = t) = \lim_{y \to t^+} \bar{G}_{T|T_1}(y|t)$$

can be less than 1 and if $\partial_1 \widehat{D}(\bar{F}(t), 0^+) = 0$ then

$$\Pr(T > y|T_1 = t < T) = \frac{\Pr(T > y|T_1 = t)}{\Pr(T > t|T_1 = t)} = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\alpha(t)\partial_1 \widehat{D}(\bar{F}(t), 1)}$$

Case II.a: $(T|T_1 = t \leq T)$.

- This case is actually straightforward, and we can directly use the reliability function given in (2.2).

Case II.a: $(T|T_1 = t \leq T)$.

- This case is actually straightforward, and we can directly use the reliability function given in (2.2).
- Now this function might have a jump at t, that is, it might have a mass $\Pr(T = t|T_1 = t) = 1 - \alpha(t)$ at time t.

Case II.a: $(T|T_1 = t \leq T)$.

- This case is actually straightforward, and we can directly use the reliability function given in (2.2).
- Now this function might have a jump at t, that is, it might have a mass $\Pr(T = t|T_1 = t) = 1 - \alpha(t)$ at time t.
- In this case, it is better to use bottom prediction bands instead of centered ones.

Case II.a: $(T|T_1 = t \leq T)$.

- This case is actually straightforward, and we can directly use the reliability function given in (2.2).
- Now this function might have a jump at t, that is, it might have a mass $\Pr(T = t|T_1 = t) = 1 - \alpha(t)$ at time t.
- In this case, it is better to use bottom prediction bands instead of centered ones.
- For example the bottom prediction band for T at level 90% is obtained with $w = 0.10$ as

$$I_{90}^{bottom}(t) = \left[t, \bar{G}_{T|T_1}^{-1}(0.10|t)\right].$$

Case II.a: $(T|T_1 = t \leq T)$.

- This case is actually straightforward, and we can directly use the reliability function given in (2.2).
- Now this function might have a jump at t, that is, it might have a mass $\Pr(T = t|T_1 = t) = 1 - \alpha(t)$ at time t.
- In this case, it is better to use bottom prediction bands instead of centered ones.
- For example the bottom prediction band for T at level 90% is obtained with $w = 0.10$ as

$$I_{90}^{bottom}(t) = \left[t, \bar{G}_{T|T_1}^{-1}(0.10|t)\right].$$

- It might also happen that the median regression curve satisfies $m(t) = t$ for some values of t.

Case III: $T_1 < T_2 < T$

- Here the purpose is to use all the information available.

Case III: $T_1 < T_2 < T$

- Here the purpose is to use all the information available.
- We consider a simple case where we know a first failure at a time $T_1 = t_1$.

Case III: $T_1 < T_2 < T$

- Here the purpose is to use all the information available.
- We consider a simple case where we know a first failure at a time $T_1 = t_1$.
- Then, we know a second failure $T_2 = t_2$ for $t_2 \geq t_1$, and we assume $T > T_2$ (with probability one).

Case III: $T_1 < T_2 < T$

- Here the purpose is to use all the information available.
- We consider a simple case where we know a first failure at a time $T_1 = t_1$.
- Then, we know a second failure $T_2 = t_2$ for $t_2 \geq t_1$, and we assume $T > T_2$ (with probability one).
- The other options can be solved similarly.

Case III: $T_1 < T_2 < T$

- Here the purpose is to use all the information available.
- We consider a simple case where we know a first failure at a time $T_1 = t_1$.
- Then, we know a second failure $T_2 = t_2$ for $t_2 \geq t_1$, and we assume $T > T_2$ (with probability one).
- The other options can be solved similarly.
- As in the preceding cases, if the components are ID$\sim \bar{F}$, then the joint reliability of (T_1, T_2, T) can be written as

$$\bar{G}(t_1, t_2, t) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))$$

for all t_1, t_2, t, where we assume that this joint reliability is absolutely continuous. Then its PDF is

$$g(t_1, t_2, t) = f(t_1)f(t_2)f(t)\partial_{1,2,3}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))$$

for all $0 \leq t_1 \leq t_2 \leq t$.

Case III: $T_1 < T_2 < T$

- The joint reliability function of (T_1, T_2) can be written as

$$\bar{G}_{1,2}(t_1, t_2) = \bar{G}(t_1, t_2, 0) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)$$

for all t_1, t_2, t and its PDF is

$$g_{1,2}(t_1, t_2) = f(t_1)f(t_2)\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1) \text{ for all } 0 \leq t_1 \leq t_2.$$

Case III: $T_1 < T_2 < T$

- The joint reliability function of (T_1, T_2) can be written as
$$\bar{G}_{1,2}(t_1, t_2) = \bar{G}(t_1, t_2, 0) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)$$
for all t_1, t_2, t and its PDF is
$$g_{1,2}(t_1, t_2) = f(t_1)f(t_2)\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1) \text{ for all } 0 \le t_1 \le t_2.$$
- Hence, the PDF of $(T|T_1 = t_1, T_2 = t_2)$ is
$$g_{3|1,2}(t|t_1, t_2) = \frac{g(t_1, t_2, t)}{g_{1,2}(t_1, t_2)} = \frac{\partial_{1,2,3}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))}{\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)} f(t)$$
for $0 \le t_1 \le t_2 \le t$ such that $f(t_1)f(t_2) \ne 0$.

Case III: $T_1 < T_2 < T$

- The joint reliability function of (T_1, T_2) can be written as

$$\bar{G}_{1,2}(t_1, t_2) = \bar{G}(t_1, t_2, 0) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)$$

for all t_1, t_2, t and its PDF is

$$g_{1,2}(t_1, t_2) = f(t_1)f(t_2)\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1) \text{ for all } 0 \leq t_1 \leq t_2.$$

- Hence, the PDF of $(T|T_1 = t_1, T_2 = t_2)$ is

$$g_{3|1,2}(t|t_1, t_2) = \frac{g(t_1, t_2, t)}{g_{1,2}(t_1, t_2)} = \frac{\partial_{1,2,3}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))}{\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)} f(t)$$

for $0 \leq t_1 \leq t_2 \leq t$ such that $f(t_1)f(t_2) \neq 0$.

- Therefore, the conditional reliability function is

$$\bar{G}_{3|1,2}(t|t_1, t_2) = \frac{\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t)) - \partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 0^+)}{\partial_{1,2}\widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)} \quad (2.5)$$

Examples

Example 1, case I

- We consider the system lifetime $T = \max(X_1, \min(X_2, X_3))$.

Example 1, case I

- We consider the system lifetime $T = \max(X_1, \min(X_2, X_3))$.
- Its minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.

Example 1, case I

- We consider the system lifetime $T = \max(X_1, \min(X_2, X_3))$.
- Its minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.
- Thence

$$\bar{F}_T(t) = \Pr(X_1 > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t). \quad (3.1)$$

- If the components are IID$\sim \bar{F}$, then

$$\bar{F}_T(t) = \bar{F}(t) + \bar{F}^2(t) - \bar{F}^3(t) = \bar{q}(\bar{F}(t))$$

for $t \geq 0$, where $\bar{q}(u) = u + u^2 - u^3$ for $u \in [0, 1]$.

Example 1, case I

- We consider the system lifetime $T = \max(X_1, \min(X_2, X_3))$.
- Its minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.
- Thence

$$\bar{F}_T(t) = \Pr(X_1 > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t). \quad (3.1)$$

- If the components are IID$\sim \bar{F}$, then

$$\bar{F}_T(t) = \bar{F}(t) + \bar{F}^2(t) - \bar{F}^3(t) = \bar{q}(\bar{F}(t))$$

for $t \geq 0$, where $\bar{q}(u) = u + u^2 - u^3$ for $u \in [0, 1]$.

- Then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dy.$$

Example 1, case I

- We consider the system lifetime $T = \max(X_1, \min(X_2, X_3))$.
- Its minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.
- Thence

$$\bar{F}_T(t) = \Pr(X_1 > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t). \quad (3.1)$$

- If the components are IID$\sim \bar{F}$, then

$$\bar{F}_T(t) = \bar{F}(t) + \bar{F}^2(t) - \bar{F}^3(t) = \bar{q}(\bar{F}(t))$$

for $t \geq 0$, where $\bar{q}(u) = u + u^2 - u^3$ for $u \in [0, 1]$.
- Then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dy.$$

- If $\bar{F}(t) = e^{-t/\mu}$ for $t \geq 0$, then $E(T) = 7\mu/6 = 1.166667\mu$.

Example 1, case I

- We consider the system lifetime $T = \max(X_1, \min(X_2, X_3))$.
- Its minimal path sets are $P_1 = \{1\}$ and $P_2 = \{2, 3\}$.
- Thence

$$\bar{F}_T(t) = \Pr(X_1 > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t). \quad (3.1)$$

- If the components are IID$\sim \bar{F}$, then

$$\bar{F}_T(t) = \bar{F}(t) + \bar{F}^2(t) - \bar{F}^3(t) = \bar{q}(\bar{F}(t))$$

  for $t \geq 0$, where $\bar{q}(u) = u + u^2 - u^3$ for $u \in [0, 1]$.
- Then

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dy.$$

- If $\bar{F}(t) = e^{-t/\mu}$ for $t \geq 0$, then $E(T) = 7\mu/6 = 1.166667\mu$.
- This is the prediction (expected value) at time $t = 0$.

Example 1, case I

- Now let us predict T at the first component failure $T_1 = X_{1:3} = t$ for $t \geq 0$.

Example 1, case I

- Now let us predict T at the first component failure $T_1 = X_{1:3} = t$ for $t \geq 0$.
- From Theorem 2.1, the joint reliability function of (T_1, T) is

$$\bar{G}(x,y) = \Pr(T_1 > x, T > y) = \Pr(T_1 > x) = \bar{F}^3(x)$$

for $0 \leq y < x$ and

$$\bar{G}(x,y) = \bar{F}^2(x)\bar{F}(y) + \bar{F}(x)\bar{F}^2(y) - \bar{F}^3(y)$$

for $0 \leq x \leq y$.

Example 1, case I

- Now let us predict T at the first component failure $T_1 = X_{1:3} = t$ for $t \geq 0$.
- From Theorem 2.1, the joint reliability function of (T_1, T) is

$$\bar{G}(x,y) = \Pr(T_1 > x, T > y) = \Pr(T_1 > x) = \bar{F}^3(x)$$

for $0 \leq y < x$ and

$$\bar{G}(x,y) = \bar{F}^2(x)\bar{F}(y) + \bar{F}(x)\bar{F}^2(y) - \bar{F}^3(y)$$

for $0 \leq x \leq y$.

- Hence $\bar{G}(x,y) = \widehat{D}(\bar{F}(x), \bar{F}(y))$ for all x, y, where

$$\widehat{D}(u,v) = \begin{cases} u^3 & \text{for} \quad 0 \leq u < v \leq 1; \\ u^2v + uv^2 - v^3 & \text{for} \quad 0 \leq v \leq u \leq 1. \end{cases}$$

Example 1, case I

- Hence

$$\partial_1 \widehat{D}(u,v) = \begin{cases} 3u^2 & \text{for} \quad 0 \leq u < v \leq 1; \\ 2uv + v^2 & \text{for} \quad 0 \leq v \leq u \leq 1. \end{cases}$$

Example 1, case I

- Hence

$$\partial_1 \widehat{D}(u,v) = \begin{cases} 3u^2 & \text{for} \quad 0 \le u < v \le 1; \\ 2uv + v^2 & \text{for} \quad 0 \le v \le u \le 1. \end{cases}$$

- Note that $\lim_{v \to 0^+} \partial_1 \widehat{D}(u,v) = 0$.

Example 1, case I

- Hence

$$\partial_1 \widehat{D}(u,v) = \begin{cases} 3u^2 & \text{for} \quad 0 \le u < v \le 1; \\ 2uv + v^2 & \text{for} \quad 0 \le v \le u \le 1. \end{cases}$$

- Note that $\lim_{v \to 0^+} \partial_1 \widehat{D}(u,v) = 0$.
- Then, from (2.3), we get

$$\bar{G}_{T|T_1}(y|t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\partial_1 \widehat{D}(\bar{F}(t), 1)} = \frac{2\bar{F}(y)\bar{F}(t) + \bar{F}^2(y)}{3\bar{F}^2(t)}$$

for $0 \le t \le y$ (1 for $y \le t$).

Example 1, case I

- By solving the quadratic equation

$$\bar{F}^2(y) + 2\bar{F}(t)\bar{F}(y) - 3w\bar{F}^2(t) = 0,$$

for $0 < w < 1$ we get the quantile function

$$\bar{G}^{-1}_{T|T_1}(w|t) = \bar{F}^{-1}\left(-\bar{F}(t) + \bar{F}(t)\sqrt{1+3w}\right).$$

Example 1, case I

- By solving the quadratic equation

$$\bar{F}^2(y) + 2\bar{F}(t)\bar{F}(y) - 3w\bar{F}^2(t) = 0,$$

for $0 < w < 1$ we get the quantile function

$$\bar{G}^{-1}_{T|T_1}(w|t) = \bar{F}^{-1}\left(-\bar{F}(t) + \bar{F}(t)\sqrt{1+3w}\right).$$

- Therefore, the median regression curve to predict T is

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) = \bar{F}^{-1}\left(\bar{F}(t)\left(\sqrt{2.5} - 1\right)\right)$$

Example 1, case I

- By solving the quadratic equation

$$\bar{F}^2(y) + 2\bar{F}(t)\bar{F}(y) - 3w\bar{F}^2(t) = 0,$$

for $0 < w < 1$ we get the quantile function

$$\bar{G}^{-1}_{T|T_1}(w|t) = \bar{F}^{-1}\left(-\bar{F}(t) + \bar{F}(t)\sqrt{1+3w}\right).$$

- Therefore, the median regression curve to predict T is

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) = \bar{F}^{-1}\left(\bar{F}(t)\left(\sqrt{2.5} - 1\right)\right)$$

- The centered 90% prediction band for T are

$$I_{90}(t) = \left[\bar{F}^{-1}\left(\bar{F}(t)\left(\sqrt{3.85} - 1\right)\right), \bar{F}^{-1}\left(\bar{F}(t)\left(\sqrt{1.15} - 1\right)\right)\right].$$

Example 1, case I

- If the components have an exponential distribution, then

$$m(t) = t - \mu \log\left(\sqrt{2.5} - 1\right) = t + 0.5427656\mu,$$

and the mean regression curve is

$$\tilde{m}(t) = E(T \,|\, T_1 = t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + 0.8333333\mu.$$

Example 1, case I

- If the components have an exponential distribution, then

$$m(t) = t - \mu \log\left(\sqrt{2.5} - 1\right) = t + 0.5427656\mu,$$

and the mean regression curve is

$$\tilde{m}(t) = E(T|T_1 = t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + 0.8333333\mu.$$

- The quantile regression curves are also straight lines.

Example 1, case I

- If the components have an exponential distribution, then

$$m(t) = t - \mu \log\left(\sqrt{2.5} - 1\right) = t + 0.5427656\mu,$$

and the mean regression curve is

$$\tilde{m}(t) = E(T|T_1 = t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + 0.8333333\mu.$$

- The quantile regression curves are also straight lines.
- As expected from the independence assumption and the lack of memory property of the exponential distribution, the predictions for the residual lifetime $(T - t|T_1 = t)$ do not depend on t.

Example 1, case I

- In the following figure (left) we provide the plots of the median (red) and mean (green) regression curves and the prediction bands for a standard exponential distribution jointly with a scatterplot of a simulated sample from (T_1, T) of size 100.

Example 1, case I

- In the following figure (left) we provide the plots of the median (red) and mean (green) regression curves and the prediction bands for a standard exponential distribution jointly with a scatterplot of a simulated sample from (T_1, T) of size 100.
- In the right plot we estimate these curves (lines) by using linear quantile regression (LQR) (for m and the prediction band limits) and linear regression (for $\tilde{m}$). The basic theory for LQR can be seen in Koenker (2005).

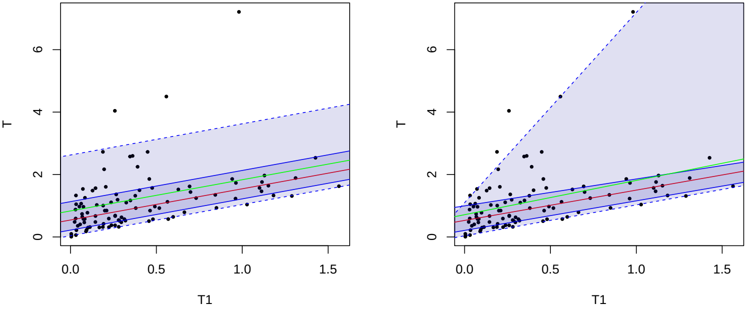


Figure: Scatterplots of a sample from (T_1, T) for the systems in Example 1 jointly with the theoretical (left) and estimated (right) median (red) and mean (green) regression curves and prediction bands with confidence levels 50% (dark grey) and 90% (light grey).

Example 1, case I

- Note that the prediction bands explain better the uncertainty in these predictions than the single mean or median regression curves.

Example 1, case I

- Note that the prediction bands explain better the uncertainty in these predictions than the single mean or median regression curves.
- For example, the first data in our sample is $T_1 = 0.4632196$ and $T = 0.8434573$.

Example 1, case I

- Note that the prediction bands explain better the uncertainty in these predictions than the single mean or median regression curves.
- For example, the first data in our sample is $T_1 = 0.4632196$ and $T = 0.8434573$.
- The predictions for T at this failure time for T_1 are $m(T_1) = 1.105407$ and $\tilde{m}(T_1) = 1.296553$, which are quite far from the exact value.

Example 1, case I

- Note that the prediction bands explain better the uncertainty in these predictions than the single mean or median regression curves.
- For example, the first data in our sample is $T_1 = 0.4632196$ and $T = 0.8434573$.
- The predictions for T at this failure time for T_1 are $m(T_1) = 1.105407$ and $\tilde{m}(T_1) = 1.296553$, which are quite far from the exact value.
- However, the centered prediction intervals for this value are $I_{50} = [0.8554071, 1.355407]$ and $I_{90} = [0.6554071, 1.555407]$.

Example 1, case I

- Note that the prediction bands explain better the uncertainty in these predictions than the single mean or median regression curves.
- For example, the first data in our sample is $T_1 = 0.4632196$ and $T = 0.8434573$.
- The predictions for T at this failure time for T_1 are $m(T_1) = 1.105407$ and $\tilde{m}(T_1) = 1.296553$, which are quite far from the exact value.
- However, the centered prediction intervals for this value are $I_{50} = [0.8554071, 1.355407]$ and $I_{90} = [0.6554071, 1.555407]$.
- The first one does not contain the exact value (it is close to the left margin) but the second does.

Example 2, case I

- Let us assume now that the components in this system are dependent with the following Clayton type survival copula

$$\widehat{C}(u_1, u_2, u_3) = \frac{u_1 u_2 u_3}{u_2 + u_3 - u_2 u_3}$$

for $u_1, u_2, u_3 \in [0, 1]$.

Example 2, case I

- Let us assume now that the components in this system are dependent with the following Clayton type survival copula

$$\widehat{C}(u_1, u_2, u_3) = \frac{u_1 u_2 u_3}{u_2 + u_3 - u_2 u_3}$$

for $u_1, u_2, u_3 \in [0, 1]$.

- Then we get the following regression curves.

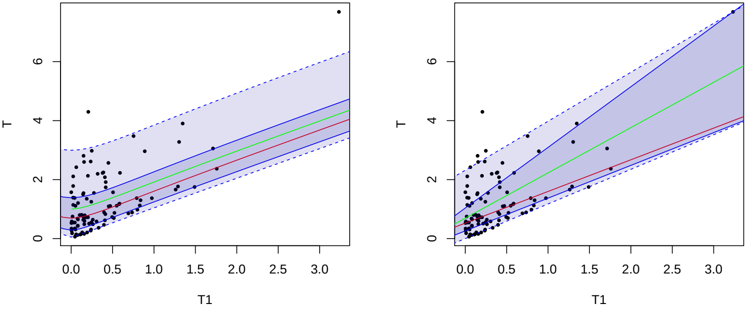


Figure: Scatterplots of a sample from (T_1, T) for the systems in Example 2 jointly with the theoretical (left) and estimated (right) median (red) and mean (green) regression curves and prediction bands with confidence levels 50% (dark grey) and 90% (light grey).

Example 3, case II

- Let us study the system $T = \min(X_1, \max(X_2, X_3))$.

Example 3, case II

- Let us study the system $T = \min(X_1, \max(X_2, X_3))$.
- The minimal path sets are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.

Example 3, case II

- Let us study the system $T = \min(X_1, \max(X_2, X_3))$.
- The minimal path sets are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.
- Hence we get

$$\bar{F}_T(t) = \Pr(X_{\{1,2\}} > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t).$$

Example 3, case II

- Let us study the system $T = \min(X_1, \max(X_2, X_3))$.
- The minimal path sets are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.
- Hence we get

$$\bar{F}_T(t) = \Pr(X_{\{1,2\}} > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t).$$

- If we assume that the component lifetimes are IID$\sim \bar{F}$, then $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$, where $\bar{q}(u) = 2u^2 - u^3$ for $u \in [0, 1]$.

Example 3, case II

- Let us study the system $T = \min(X_1, \max(X_2, X_3))$.
- The minimal path sets are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.
- Hence we get

$$\bar{F}_T(t) = \Pr(X_{\{1,2\}} > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t).$$

- If we assume that the component lifetimes are IID$\sim \bar{F}$, then $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$, where $\bar{q}(u) = 2u^2 - u^3$ for $u \in [0, 1]$.
- Hence

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = 2\int_0^\infty \bar{F}^2(t)dt - \int_0^\infty \bar{F}^3(t)dt.$$

Example 3, case II

- Let us study the system $T = \min(X_1, \max(X_2, X_3))$.
- The minimal path sets are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.
- Hence we get

$$\bar{F}_T(t) = \Pr(X_{\{1,2\}} > t) + \Pr(X_{\{2,3\}} > t) - \Pr(X_{\{1,2,3\}} > t).$$

- If we assume that the component lifetimes are IID$\sim \bar{F}$, then $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$, where $\bar{q}(u) = 2u^2 - u^3$ for $u \in [0, 1]$.
- Hence

$$E(T) = \int_0^\infty \bar{q}(\bar{F}(t))dt = 2\int_0^\infty \bar{F}^2(t)dt - \int_0^\infty \bar{F}^3(t)dt.$$

- If $\bar{F}(t) = e^{-t/\mu}$ for $t \geq 0$, then $E(T) = 2\mu/3 = 0.666667\mu$.

Example 3, case II

- As in the preceding examples we choose $T_1 = X_{1:3}$, that is, it is the first component failure.

Example 3, case II

- As in the preceding examples we choose $T_1 = X_{1:3}$, that is, it is the first component failure.
- However, now

$$\Pr(T = T_1) = \Pr(T_1 = X_1) = 1/3$$

and (T_1, T) have a singular part at $T = T_1$ with probability $1/3$ (even when the component lifetimes are IID and absolutely continuous).

Example 3, case II

- As in the preceding examples we choose $T_1 = X_{1:3}$, that is, it is the first component failure.
- However, now

$$\Pr(T = T_1) = \Pr(T_1 = X_1) = 1/3$$

and (T_1, T) have a singular part at $T = T_1$ with probability $1/3$ (even when the component lifetimes are IID and absolutely continuous).

- Therefore, we are in case II.

Example 3, case II

- The joint reliability function of (T_1, T) is

$$\bar{G}(x,y) = \Pr(T_1 > x, T > y) = \Pr(T_1 > x) = \bar{F}^3(x)$$

for $0 \leq y \leq x$, and

$$\bar{G}(x,y) = 2\bar{F}(x)\bar{F}^2(y) - \bar{F}^3(y)$$

for $0 \leq x < y$.

Example 3, case II

- The joint reliability function of (T_1, T) is

$$\bar{G}(x,y) = \Pr(T_1 > x, T > y) = \Pr(T_1 > x) = \bar{F}^3(x)$$

for $0 \le y \le x$, and

$$\bar{G}(x,y) = 2\bar{F}(x)\bar{F}^2(y) - \bar{F}^3(y)$$

for $0 \le x < y$.

- Note that $\bar{G}$ is continuous but not absolutely continuous.

Example 3, case II

- The joint reliability function of (T_1, T) is

$$\bar{G}(x,y) = \Pr(T_1 > x, T > y) = \Pr(T_1 > x) = \bar{F}^3(x)$$

for $0 \leq y \leq x$, and

$$\bar{G}(x,y) = 2\bar{F}(x)\bar{F}^2(y) - \bar{F}^3(y)$$

for $0 \leq x < y$.

- Note that $\bar{G}$ is continuous but not absolutely continuous.
- Moreover, $\bar{G}(x,y) = \widehat{D}(\bar{F}(x), \bar{F}(y))$ for all x, y, where

$$\widehat{D}(u,v) = \begin{cases} u^3 & \text{for} \quad 0 \leq u \leq v \leq 1; \\ 2uv^2 - v^3 & \text{for} \quad 0 \leq v < u \leq 1; \end{cases}$$

and

$$\partial_1 \widehat{D}(u,v) = \begin{cases} 3u^2 & \text{for} \quad 0 \leq u < v \leq 1; \\ 2v^2 & \text{for} \quad 0 \leq v < u \leq 1. \end{cases}$$

Example 3, case II.a

- To solve case II.a, we use (2.2) obtaining

$$\bar{G}_{T|T_1}(y|t) = \Pr(T > y|T_1 = t) = \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}$$

for $y > t$ (one for $0 \leq y \leq t$).

Example 3, case II.a

- To solve case II.a, we use (2.2) obtaining

$$\bar{G}_{T|T_1}(y|t) = \Pr(T > y|T_1 = t) = \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}$$

for $y > t$ (one for $0 \leq y \leq t$).

- Note that

$$\alpha(t) = \Pr(T > T_1|T_1 = t) = \lim_{y \to t^+} \bar{G}_{T|T_1}(y|t) = \frac{2}{3},$$

and that $1 - \alpha(t) = \Pr(T = T_1|T_1 = t) = 1/3$.

Example 3, case II.a

- To solve case II.a, we use (2.2) obtaining

$$\bar{G}_{T|T_1}(y|t) = \Pr(T > y|T_1 = t) = \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}$$

for $y > t$ (one for $0 \leq y \leq t$).

- Note that

$$\alpha(t) = \Pr(T > T_1|T_1 = t) = \lim_{y \to t^+} \bar{G}_{T|T_1}(y|t) = \frac{2}{3},$$

and that $1 - \alpha(t) = \Pr(T = T_1|T_1 = t) = 1/3$.

- In this case, they do not depend on t and so they coincide with $\Pr(T > T_1)$ and $\Pr(T = T_1)$, respectively.

Example 3, case II.a

- To solve case II.a, we use (2.2) obtaining

$$\bar{G}_{T|T_1}(y|t) = \Pr(T > y|T_1 = t) = \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}$$

  for $y > t$ (one for $0 \leq y \leq t$).
- Note that

$$\alpha(t) = \Pr(T > T_1|T_1 = t) = \lim_{y \to t^+} \bar{G}_{T|T_1}(y|t) = \frac{2}{3},$$

  and that $1 - \alpha(t) = \Pr(T = T_1|T_1 = t) = 1/3$.
- In this case, they do not depend on t and so they coincide with $\Pr(T > T_1)$ and $\Pr(T = T_1)$, respectively.
- Then the median regression curve is

$$m(t) = \bar{G}^{-1}_{T|T_1}(0.5|t) = \bar{F}^{-1}\left(\sqrt{0.75}\bar{F}(t)\right)$$

  for $t > 0$.

Example 3, case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.75) = t + 0.143841\mu.$$

Example 3, case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.75) = t + 0.143841\mu.$$

- The regression curve is

$$\tilde{m}(t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + \int_0^\infty \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}dy = t + \frac{1}{3}\mu.$$

Example 3, case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.75) = t + 0.143841\mu.$$

- The regression curve is

$$\tilde{m}(t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + \int_0^\infty \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}dy = t + \frac{1}{3}\mu.$$

- The prediction bands can be obtained in a similar way.

Example 3, case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.75) = t + 0.143841\mu.$$

- The regression curve is

$$\tilde{m}(t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + \int_0^\infty \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}dy = t + \frac{1}{3}\mu.$$

- The prediction bands can be obtained in a similar way.
- For example, the 90% bottom prediction band is

$$I_{90}^{bottom}(t) = \left[t, \bar{F}^{-1}\left(\sqrt{0.15}\bar{F}(t)\right)\right] = [t, t + 0.94856\mu].$$

Example 3, case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.75) = t + 0.143841\mu.$$

- The regression curve is

$$\tilde{m}(t) = \int_0^\infty \bar{G}_{T|T_1}(y|t)dy = t + \int_0^\infty \frac{2\bar{F}^2(y)}{3\bar{F}^2(t)}dy = t + \frac{1}{3}\mu.$$

- The prediction bands can be obtained in a similar way.
- For example, the 90% bottom prediction band is

$$I_{90}^{bottom}(t) = \left[t, \bar{F}^{-1}\left(\sqrt{0.15}\bar{F}(t)\right)\right] = [t, t + 0.94856\mu].$$

- The 50% bottom prediction band is $I_{50}^{bottom}(t) = [t, m(t)]$.

Example 3, case II.b

- To solve case II.b we assume that the first component failure happens at a time t ($T_1 = t$) and that at this time we know that the system is still alive ($T > t$).

Example 3, case II.b

- To solve case II.b we assume that the first component failure happens at a time t ($T_1 = t$) and that at this time we know that the system is still alive ($T > t$).
- Then we want to predict T under these assumptions.

Example 3, case II.b

- To solve case II.b we assume that the first component failure happens at a time t ($T_1 = t$) and that at this time we know that the system is still alive ($T > t$).
- Then we want to predict T under these assumptions.
- To this end, we need to solve

$$\Pr(T > y | T_1 = t, T > t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\alpha(t)\partial_1 \widehat{D}(\bar{F}(t), 1)} = \frac{\bar{F}^2(y)}{\bar{F}^2(t)} = w \tag{3.2}$$

for $y > t$ and $0 < w < 1$.

Example 3, case II.b

- To solve case II.b we assume that the first component failure happens at a time t ($T_1 = t$) and that at this time we know that the system is still alive ($T > t$).
- Then we want to predict T under these assumptions.
- To this end, we need to solve

$$\Pr(T > y | T_1 = t, T > t) = \frac{\partial_1 \widehat{D}(\bar{F}(t), \bar{F}(y))}{\alpha(t)\partial_1 \widehat{D}(\bar{F}(t), 1)} = \frac{\bar{F}^2(y)}{\bar{F}^2(t)} = w \tag{3.2}$$

for $y > t$ and $0 < w < 1$.

- Thus the median regression curve is

$$m(t) = \bar{F}^{-1}\left(\sqrt{0.5}\bar{F}(t)\right).$$

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

- The mean regression curve in the exponential case is

$$\tilde{m}(t) = t + \int_0^\infty \frac{\bar{F}^2(y)}{\bar{F}^2(t)} dy = t + 0.5\mu.$$

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

- The mean regression curve in the exponential case is

$$\tilde{m}(t) = t + \int_0^\infty \frac{\bar{F}^2(y)}{\bar{F}^2(t)} dy = t + 0.5\mu.$$

- The bottom prediction bands are obtained similarly.

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

- The mean regression curve in the exponential case is

$$\tilde{m}(t) = t + \int_0^\infty \frac{\bar{F}^2(y)}{\bar{F}^2(t)} dy = t + 0.5\mu.$$

- The bottom prediction bands are obtained similarly.
- In the following figure we provide the predictions for both cases jointly with a simulated sample.

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

- The mean regression curve in the exponential case is

$$\tilde{m}(t) = t + \int_0^\infty \frac{\bar{F}^2(y)}{\bar{F}^2(t)} dy = t + 0.5\mu.$$

- The bottom prediction bands are obtained similarly.
- In the following figure we provide the predictions for both cases jointly with a simulated sample.
- We have 39 data satisfying $T = T_1$.

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

- The mean regression curve in the exponential case is

$$\tilde{m}(t) = t + \int_0^\infty \frac{\bar{F}^2(y)}{\bar{F}^2(t)} dy = t + 0.5\mu.$$

- The bottom prediction bands are obtained similarly.
- In the following figure we provide the predictions for both cases jointly with a simulated sample.
- We have 39 data satisfying $T = T_1$.
- For the 61 remaining points, we get 54 in the bottom 90% prediction band. Only 7 data are not contained in this band.

Example 3, Case II.a

- In the exponential case, we get

$$m(t) = t - 0.5\mu \ln(0.5) = t + 0.3465736\mu.$$

- The mean regression curve in the exponential case is

$$\tilde{m}(t) = t + \int_0^\infty \frac{\bar{F}^2(y)}{\bar{F}^2(t)} dy = t + 0.5\mu.$$

- The bottom prediction bands are obtained similarly.
- In the following figure we provide the predictions for both cases jointly with a simulated sample.
- We have 39 data satisfying $T = T_1$.
- For the 61 remaining points, we get 54 in the bottom 90% prediction band. Only 7 data are not contained in this band.
- The bottom 50% prediction band contains 32 out of 61 data.

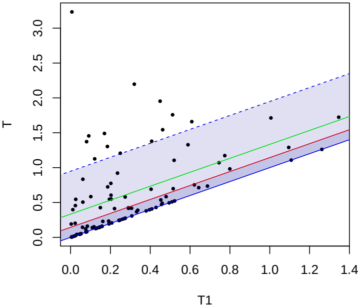


Figure: Scatterplots of a sample from (T_1, T) for the systems in Example 3 jointly with the plots of theoretical median (red) and mean (green) regression curves and the bottom prediction bands with confidence levels 50% (dark grey) and 90% (light grey) for cases II.a (left) and II.b (right).

Example 4, Case II

- Let us consider the same system but with dependent ID components having the following Farlie-Gumbel-Morgenstern (FGM) survival copula

$$\widehat{C}(u_1, u_2, u_3) = u_1 u_2 u_3 + \theta u_1 u_2 u_3 (1 - u_1)(1 - u_2)(1 - u_3) \quad (3.3)$$

for $u_1, u_2, u_3 \in [0, 1]$ and $\theta \in [-1, 1]$.

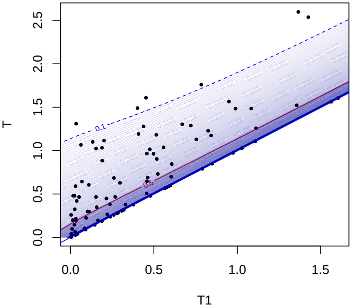
Example 4, Case II

- Let us consider the same system but with dependent ID components having the following Farlie-Gumbel-Morgenstern (FGM) survival copula

$$\widehat{C}(u_1,u_2,u_3) = u_1u_2u_3+\theta u_1u_2u_3(1-u_1)(1-u_2)(1-u_3) \quad (3.3)$$

for $u_1, u_2, u_3 \in [0,1]$ and $\theta \in [-1,1]$.

- Then we get the following curves.



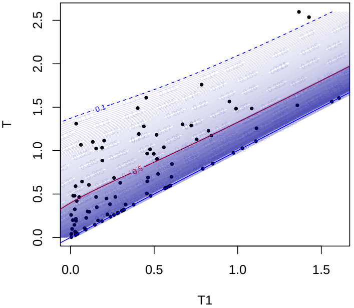


Figure: Scatterplots of a sample from (T_1, T) for the systems in Example 4 jointly with the plots of theoretical median (red) and mean (green) regression curves and the bottom prediction bands with confidence levels 50% (dark grey) and 90% (light grey) for cases II.a (left) and II.b (right).

Example 5, case III

- Let us consider $T = X_{3:3} = \max(X_1, X_2, X_3)$.

Example 5, case III

- Let us consider $T = X_{3:3} = \max(X_1, X_2, X_3)$.
- Let us assume that we know $T_1 = X_{1:3} = t_1$ and $T_2 = X_{2:3} = t_2$ for $0 < t_1 < t_2$.
- We assume that (X_1, X_2, X_3) are exchangeable.

Example 5, case III

- Let us consider $T = X_{3:3} = \max(X_1, X_2, X_3)$.
- Let us assume that we know $T_1 = X_{1:3} = t_1$ and $T_2 = X_{2:3} = t_2$ for $0 < t_1 < t_2$.
- We assume that (X_1, X_2, X_3) are exchangeable.
- Then the joint reliability function $\bar{G}$ of (T_1, T_2, T) is

$$\bar{G}(t_1, t_2, t) = 6\overline{\mathbf{F}}(t_1, t_2, t) - 3\overline{\mathbf{F}}(t_2, t_2, t) - 3\overline{\mathbf{F}}(t_1, t, t) + \overline{\mathbf{F}}(t, t, t)$$

for $0 \le t_1 \le t_2 \le t$, where
$\overline{\mathbf{F}}(x_1, x_2, x_3) = \Pr(X_1 > x_1, X_2 > x_2, X_3 > x_3)$.

Example 5, case III

- Let us consider $T = X_{3:3} = \max(X_1, X_2, X_3)$.
- Let us assume that we know $T_1 = X_{1:3} = t_1$ and $T_2 = X_{2:3} = t_2$ for $0 < t_1 < t_2$.
- We assume that (X_1, X_2, X_3) are exchangeable.
- Then the joint reliability function $\bar{G}$ of (T_1, T_2, T) is

  $$\bar{G}(t_1, t_2, t) = 6\overline{\mathbf{F}}(t_1, t_2, t) - 3\overline{\mathbf{F}}(t_2, t_2, t) - 3\overline{\mathbf{F}}(t_1, t, t) + \overline{\mathbf{F}}(t, t, t)$$

  for $0 \leq t_1 \leq t_2 \leq t$, where
  $\overline{\mathbf{F}}(x_1, x_2, x_3) = \Pr(X_1 > x_1, X_2 > x_2, X_3 > x_3)$.
- Therefore, $\bar{G}(t_1, t_2, t) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))$, where

  $$\widehat{D}(u, v, w) = 6\widehat{C}(u, v, w) - 3\widehat{C}(v, v, w) - 3\widehat{C}(u, w, w) + \widehat{C}(w, w, w)$$

  for $0 \leq w \leq v \leq u \leq 1$.

Example 5, case III

- Let us consider $T = X_{3:3} = \max(X_1, X_2, X_3)$.
- Let us assume that we know $T_1 = X_{1:3} = t_1$ and $T_2 = X_{2:3} = t_2$ for $0 < t_1 < t_2$.
- We assume that (X_1, X_2, X_3) are exchangeable.
- Then the joint reliability function $\bar{G}$ of (T_1, T_2, T) is

  $$\bar{G}(t_1, t_2, t) = 6\overline{\mathbf{F}}(t_1, t_2, t) - 3\overline{\mathbf{F}}(t_2, t_2, t) - 3\overline{\mathbf{F}}(t_1, t, t) + \overline{\mathbf{F}}(t, t, t)$$

  for $0 \le t_1 \le t_2 \le t$, where
  $\overline{\mathbf{F}}(x_1, x_2, x_3) = \Pr(X_1 > x_1, X_2 > x_2, X_3 > x_3)$.
- Therefore, $\bar{G}(t_1, t_2, t) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))$, where

  $$\widehat{D}(u, v, w) = 6\widehat{C}(u, v, w) - 3\widehat{C}(v, v, w) - 3\widehat{C}(u, w, w) + \widehat{C}(w, w, w)$$

  for $0 \le w \le v \le u \le 1$.
- The expressions for $\widehat{D}$ in the other cases can be obtained similarly.

Example 5, case III

- The joint reliability function of (T_1, T_2) is

$$\bar{G}_{1,2}(t_1, t_2) = 3\overline{\mathbf{F}}(t_1, t_2, t_2) - 2\overline{\mathbf{F}}(t_2, t_2, t_2)$$

for $0 \le t_1 \le t_2$, that is, $\bar{G}_{1,2}(t_1, t_2) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)$ with

$$\widehat{D}(u, v, 1) = 3\widehat{C}(u, v, v) - 2\widehat{C}(v, v, v)$$

for $0 \le v \le u \le 1$.

Example 5, case III

- The joint reliability function of (T_1, T_2) is
$$\bar{G}_{1,2}(t_1, t_2) = 3\overline{\mathbf{F}}(t_1, t_2, t_2) - 2\overline{\mathbf{F}}(t_2, t_2, t_2)$$
for $0 \le t_1 \le t_2$, that is, $\bar{G}_{1,2}(t_1, t_2) = \widehat{D}(\bar{F}(t_1), \bar{F}(t_2), 1)$ with
$$\widehat{D}(u, v, 1) = 3\widehat{C}(u, v, v) - 2\widehat{C}(v, v, v)$$
for $0 \le v \le u \le 1$.
- Therefore, by differentiating these expressions we get
$$\partial_{1,2}\widehat{D}(u, v, w) = 6\partial_{1,2}\widehat{C}(u, v, w),$$
$$\partial_{1,2}\widehat{D}(u, v, 1) = 6\partial_{1,2}\widehat{C}(u, v, v)$$
and the reliability function of $(T|T_1 = t_1, T_2 = t_2)$ is
$$\bar{G}_{3|1,2}(t|t_1, t_2) = \frac{\partial_{1,2}\widehat{C}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t))}{\partial_{1,2}\widehat{C}(\bar{F}(t_1), \bar{F}(t_2), \bar{F}(t_2))}$$

Example 5, case III

- If $\widehat{C}$ is the product copula (independent components), we have $\partial_{1,2}\widehat{C}(u,v,w) = w$ and

$$\bar{G}_{3|1,2}(t|t_1,t_2) = \frac{\partial_{1,2}\widehat{C}(\bar{F}(t_1),\bar{F}(t_2),\bar{F}(t))}{\partial_{1,2}\widehat{C}(\bar{F}(t_1),\bar{F}(t_2),\bar{F}(t_2))} = \frac{\bar{F}(t)}{\bar{F}(t_2)}$$

for $t \geq t_2$ (Markovian property of the OS).

Example 5, case III

- If $\widehat{C}$ is the product copula (independent components), we have $\partial_{1,2}\widehat{C}(u,v,w) = w$ and

$$\bar{G}_{3|1,2}(t|t_1,t_2) = \frac{\partial_{1,2}\widehat{C}(\bar{F}(t_1),\bar{F}(t_2),\bar{F}(t))}{\partial_{1,2}\widehat{C}(\bar{F}(t_1),\bar{F}(t_2),\bar{F}(t_2))} = \frac{\bar{F}(t)}{\bar{F}(t_2)}$$

for $t \geq t_2$ (Markovian property of the OS).
- If assume the FGM copula of Example 4, then

$$\partial_{1,2}\widehat{C}(u_1,u_2,u_3) = u_3 + \theta u_3(1-u_3)(1-2u_1)(1-2u_2)$$

for all $u_1,u_2,u_3 \in [0,1]$, and we get

$$\bar{G}_{3|1,2}(t|t_1,t_2) = \frac{\bar{F}(t)}{\bar{F}(t_2)} \cdot \frac{1+\theta F(t)(1-2\bar{F}(t_1))(1-2\bar{F}(t_2))}{1+\theta F(t_2)(1-2\bar{F}(t_1))(1-2\bar{F}(t_2))}$$

for $t \geq t_2$ (one for $0 \leq t < t_2$).

Example 5, case III

- For $\theta = 0$, it coincides with the expression for the IID case.

Example 5, case III

- For $\theta = 0$, it coincides with the expression for the IID case.
- For $\theta \neq 0$, it depends on t_1.

Example 5, case III

- For $\theta = 0$, it coincides with the expression for the IID case.
- For $\theta \neq 0$, it depends on t_1.
- To get its inverse function, we need to solve

$$\theta a(t_1, t_2)\bar{F}^2(t) - (1 + \theta a(t_1, t_2))\bar{F}(t) + wc(t_1, t_2) = 0,$$

where $a(t_1, t_2) = (1 - 2\bar{F}(t_1))(1 - 2\bar{F}(t_2)) \in [-1, 1]$ and

$c(t_1, t_2) = \bar{F}(t_2) + \theta\bar{F}(t_2)F(t_2)(1 - 2\bar{F}(t_1))(1 - 2\bar{F}(t_2)) \in [0, 1]$.

Example 5, case III

- For $\theta = 0$, it coincides with the expression for the IID case.
- For $\theta \neq 0$, it depends on t_1.
- To get its inverse function, we need to solve

$$\theta a(t_1,t_2)\bar{F}^2(t) - (1+\theta a(t_1,t_2))\bar{F}(t) + wc(t_1,t_2) = 0,$$

where $a(t_1,t_2) = (1-2\bar{F}(t_1))(1-2\bar{F}(t_2)) \in [-1,1]$ and

$c(t_1,t_2) = \bar{F}(t_2)+\theta\bar{F}(t_2)F(t_2)(1-2\bar{F}(t_1))(1-2\bar{F}(t_2)) \in [0,1]$.

- This equation has a unique solution in $[0,1]$ given by

$$\bar{F}(t) = \frac{1+\theta a(t_1,t_2) - \sqrt{(1+\theta a(t_1,t_2))^2 - 4\theta w a(t_1,t_2)c(t_1,t_2)}}{2\theta a(t_1,t_2)}$$

for $\theta a(t_1,t_2) \neq 0$.

Example 5, case III

- For $\theta = 0$, it coincides with the expression for the IID case.
- For $\theta \neq 0$, it depends on t_1.
- To get its inverse function, we need to solve

$$\theta a(t_1,t_2)\bar{F}^2(t) - (1+\theta a(t_1,t_2))\bar{F}(t) + wc(t_1,t_2) = 0,$$

where $a(t_1,t_2) = (1-2\bar{F}(t_1))(1-2\bar{F}(t_2)) \in [-1,1]$ and

$c(t_1,t_2) = \bar{F}(t_2)+\theta\bar{F}(t_2)F(t_2)(1-2\bar{F}(t_1))(1-2\bar{F}(t_2)) \in [0,1]$.

- This equation has a unique solution in $[0,1]$ given by

$$\bar{F}(t) = \frac{1+\theta a(t_1,t_2) - \sqrt{(1+\theta a(t_1,t_2))^2 - 4\theta w a(t_1,t_2)c(t_1,t_2)}}{2\theta a(t_1,t_2)}$$

for $\theta a(t_1,t_2) \neq 0$.

- From this expression we can compute $\bar{G}^{-1}_{3|1,2}(w|t_1,t_2)$ for $0 < w < 1$, $0 \leq t_1 \leq t_2$ and $\theta \in [-1,1]$.

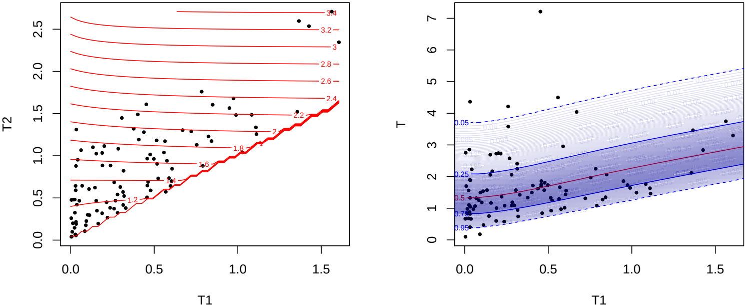


Figure: Scatterplots of a sample from (T_1, T_2) (left) and (T_1, T) (right) for the systems in Example 5 jointly with the theoretical median regression curve (red) and the centered prediction bands (right plot) with levels 50% (dark grey) and 90% (light grey). In the left plot, we only give the level curves (predictions) of the median regression map $m(t_1, t_2)$.

References

Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston.

Buono F, Navarro J (2022). Simulations and predictions of future values in the time homogeneous load-sharing model. To appear in Statistical Papers. Published online first Feb, 2023. DOI: 10.1007/s00362-023-01404-5.

Koenker R (2005). Quantile Regression. Cambridge University Press.

Navarro, J (2021). Prediction of record values by using quantile regression curves and distortion functions. To appear in *Metrika*. Published online first Nov. 2021. DOI: 10.1007/s00184-021-00847-w.

Navarro J (2022). Introduction to System Reliability Theory Springer.

Navarro J., Arriaza A., Suárez-Llorens (2023). Predicting system failure times. Submitted.

Navarro J, Buono F (2022). Predicting future failure times by using quantile regression. To appear in *Metrika*. Published online first Sept 2022. Open access.
DOI: 10.1007/s00184-022-00884-z

Navarro J, Calì C, Longobardi M, Durante F (2022). Distortion representations of multivariate distributions. *Statistical Methods & Applications.* 31, 925–954.
DOI: 10.1007/s10260-021-00613-2.

Nelsen RB (2006). An introduction to copulas. Second edition. Springer, New York.

Final slide

- More references in my web page

  https://webs.um.es/jorgenav/miwiki/doku.php

Final slide

- More references in my web page

  https://webs.um.es/jorgenav/miwiki/doku.php

- That's all. Thank you for your attention!!

Final slide

- More references in my web page

  https://webs.um.es/jorgenav/miwiki/doku.php

- That's all. Thank you for your attention!!
- Questions?