Stochastic properties of sums of dependent random variables (risks)

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References

The talk is based on the following references:

- Navarro J., Sarabia J.M. (2022). Copula representations for the sums of dependent risks: models and comparisons. Probability in the Engineering and Informational Sciences 36, 320–340. Doi: 10.1017/S0269964820000649.
- Navarro J., Pellerey F., Mulero J. (2022b). On sums of dependent random lifetimes under the time-transformed exponential model. Test 31, 879–900. Doi: 10.1007/s11749-022-00805-2
- Navarro J., Zapata J.M. (2025). Stochastic dominance of sums of risks under dependence conditions. Submitted. ArXiv: 2503.05348v1.

Outline

Preliminary results

Notation

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- $\bar m_X(t) = E(t - X|X < t)$ and $\bar m_Y(t) = E(t - Y|Y < t)$ mean inactivity time (MIT) functions.

Main stochastic orders

- Stochastic order (or first-order stochastic dominance): $X \leq_{ST} Y \Leftrightarrow \bar{F}_X \leq \bar{F}_Y$ or, equivalently, $E(\phi(X)) \leq E(\phi(X))$ for all increasing functions ϕ.

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- Increasing convex order $X \leq_{ICX} Y \Leftrightarrow E(\phi(X)) \leq E(\phi(X))$ for all increasing and convex functions ϕ.

Main stochastic orders

- Relationships:

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & X \leq_{ICX} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{MIT} Y & \Rightarrow & X \leq_{ICV} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

Distortions

- X has a distorted distribution from a CDF F if $F_X = q(F)$ where $q : [0,1] \to [0,1]$ is a **distortion function**, that is, it is a continuous and increasing function such that $q(0) = 0$ and $q(1) = 1$.

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- Then the respective survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F} = 1 - F$ satisfy $\bar{F}_X = \bar{q}(\bar{F})$, where $\bar{q}(u) = 1 - q(1-u)$ is another distortion function called **dual distortion**.

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- If $F_X = q_1(F)$ and $F_Y = q_2(F)$, then following ordering properties hold:

$$X \leq_{ST} Y \text{ for all } F \Leftrightarrow \bar{q}_1 \leq \bar{q}_2,$$

$$X \leq_{HR} Y \text{ for all } F \Leftrightarrow \bar{q}_2/\bar{q}_1 \text{ decreases in } (0,1)$$

$$X \leq_{RHR} Y \text{ for all } F \Leftrightarrow q_2/q_1 \text{ increases in } (0,1),$$

$$X \leq_{LR} Y \text{ for all } F \Leftrightarrow \bar{q}_2'/\bar{q}_1' \text{ decreases in } (0,1).$$

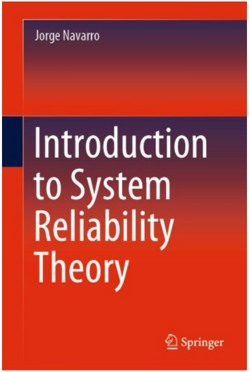

Figure: Publicity of my book on System Reliability Theory.

C-convolutions

- (X, Y) random vector with absolutely continuous CDF

$$\Pr(X \leq x, Y \leq y) = C(F(x), F(y)),$$

where C is a copula and $F(x) = \Pr(X \leq x)$ and $G(y) = \Pr(Y \leq y)$.

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- $S = X + Y$ with CDF $H(z) = \Pr(S \leq z)$.
- If $\lim_{v\to 0^+} \partial_1 C(u, v)) = 0$ for all $v \in (0, 1)$, then

$$H(z) = \int_{-\infty}^{\infty} f(x)\partial_1 C(F(x), G(z-x))dx = \int_0^1 \partial_1 C(F(x), G(z-F^{-1}(u)))du$$

where $f(x) = F'(x)$ is the PDF function and F^{-1} is the quantile function of X, see e.g. Cherubini, Gobbi and Mulinacci (2016) or Navarro and Sarabia (2022).

Special cases

- Of course, if (X, Y) are independent, that is

$$C(u,v) = \Pi(u,v) = uv$$

for all $u, v \in [0,1]$, then $\partial_1 C(u,v) = v$ and

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- Analogously, the notation $H=F*_C G$ is used for the C-convolution stated above for a general copula C.
- It is a mixture of the CDF

$$H_x(z):=\partial_1 C(F(x),G(z-x))=\Pr(Y+x\leq z|X=x)$$

with mixing PDF $f(x)$.

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$$\Pr(X > x, Y > y) = \widehat{C}(\bar{F}(x), \bar{G}(y))$$
where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, then
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$$\bar{H}(z) := \Pr(S > z) = \bar{F}(z) + \int_0^z f(x)\partial_1 \widehat{C}(\bar{F}(x), \bar{G}(z-x))dx.$$
- For standard exponential distributions we get
$$\bar{H}(z) = \bar{F}(z) + \int_{\bar{F}(z)}^1 \partial_1 \widehat{C}(v, \bar{F}(z)/v)dv = \bar{q}(\bar{F}(z))$$
which is a distortion representation.

Special cases

- If (X, Y) are non-negative and satisfy the *minimal repair model*, that is,

$$\Pr(Y > y | X = x) = \Pr(X - x > y | X = x) = \frac{\bar{F}(x + y)}{\bar{F}(x)}$$

for all $x, y \geq 0$, then

$$\bar{H}(z) = \bar{F}(z) + \int_0^z f(x) \frac{\bar{F}(z)}{\bar{F}(x)} dx = \bar{F}(z) - \bar{F}(z) \log \bar{F}(z) = \bar{q}(\bar{F}(z))$$

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- This is also a distortion model.
- It is also the model for the second record value and it is represented as $\bar{H} = \bar{F} \# \bar{F}$.

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- If (X, Y) are non-negative and satisfy the *minimal repair model* with a common exponential distribution, then

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for all $x, y \geq 0$, then

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- Hence X and Y are independent and $\bar{H} = \bar{F} \# \bar{F} = \bar{F} * \bar{F}$.
- It is a gamma (Erlang) distribution with PDF

$$h(z) = \lambda z e^{-\lambda z}$$

for $z \geq 0$ (a well known result).

Positive dependence notions

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- $PQDE(Y|X)$ (Positively Quadrant Dependent in Expectations) $E(Y) \leq E(Y|X \leq x)$ for all x.

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- $PQDE(Y|X)$ (Positively Quadrant Dependent in Expectations) $E(Y) \leq E(Y|X \leq x)$ for all x.
- See Nelsen (2006) and Navarro, Pellerey and Sordo (2021) for more dependence notions and properties.

Dependence notions

$$
\begin{array}{ccccc}
SI_{ST}(Y|X) & \Rightarrow & PQD(X,Y) & \Rightarrow & PQDE(Y|X) \\
 & & \Downarrow & & \Downarrow \\
 & & \tau_C, \rho_C, \gamma_C \geq 0 & & \rho = Cor(X,Y) \geq 0
\end{array}
$$

Table: Relationships between positive dependence notions where τ_C is the Kendall's tau, ρ_C is the Spearman's correlation and γ_C is the Gini measure of association $\gamma_C = 4\int_0^1 [C(u,u) + C(u,1-u) - u]\, du$.

Stochastic properties

Ordering properties

Properties in:

- Navarro J., Sarabia J.M. (2022). Copula representations for the sums of dependent risks: models and comparisons. Probability in the Engineering and Informational Sciences 36, 320–340. Doi: 10.1017/S0269964820000649.

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for all increasing functions ϕ_1 and ϕ_2.

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for all increasing functions ϕ_1 and ϕ_2.

- They cannot be compared if they have the same marginals and different copulas see Vernic (2016).

Ordering properties

Proposition

If (X, Y) and (X^*, Y^*) have marginals F, G, F^*, G^*, copulas C and C^* and the following properties hold:

i) $X \leq_{ST} X^*$;

ii) $\partial_1 C(u, G(x)) \geq \partial_1 C^*(u, G^*(x))$ *for all* $u \in (0, 1)$ *and all* $x \in \mathbb{R}$;

then $X + Y \leq_{ST} X^* + Y^*$.

Ordering properties

- If $C^* = \Pi$, then $\partial_1 C^*(u, G^*(x)) = G^*(x)$. Hence, if $X \leq_{ST} X^*$ and

$$G_1(x) = \inf_{u \in (0,1)} \partial_1 C(u, G(x)) \geq G^*(x),$$

then $X + Y \leq_{ST} X^* + Y^*$.

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- If $C^* = \Pi$, then $\partial_1 C^*(u, G^*(x)) = G^*(x)$. Hence, if $X \leq_{ST} X^*$ and

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- If G_1, G_2 are CDF, then $F * G_2 \leq_{ST} F *_C G \leq_{ST} F * G_1$.

Ordering properties-Example 1

- If $C^* = \Pi$ and C is a Clayton copula

$$C(u,v) = \frac{uv}{u+v-uv},\ \ u,v \in (0,1),$$

then $\partial_1 C^*(u,v) = v$ and

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$$G_2(x) = \sup_{u\in(0,1)} \partial_1 C(u, G(x)) = 1 \le G^*(x),$$

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- If $C^* = \Pi$, C is the above Clayton copula and

$$F(t) = G(t) = F^*(t) = G^*(t) = 1 - \exp(-t) \; t \geq 0,$$

then

$$H^*(t) = \int_0^t e^{-x}(1 - e^{x-t})dx = 1 - e^{-t} - te^{-t}, \; t \geq 0$$

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$$H(t) = \int_0^t e^{-x} \frac{(1 - e^{x-t})^2}{(1 - e^{-t})^2} dx = \frac{1 - 2te^{-t} - e^{-2t}}{1 - 2e^{-t} - e^{-2t}}, \; t \geq 0.$$

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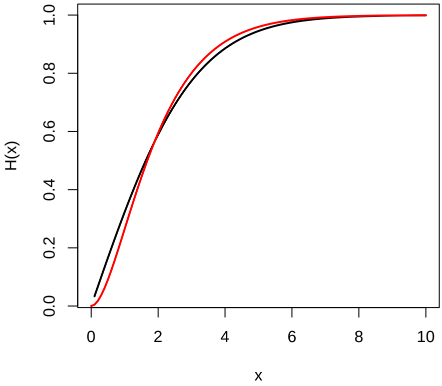

Figure: Distribution functions for the sum of two standard exponential distributions with a Clayton copula (black) and a product copula (red).

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- However, if $C^* = \Pi$, C is the above Clayton copula,

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- In this case, a trivial upper bound is F, that is

$$F \leq_{ST} F *_C G \leq_{ST} F * G^2$$

or $X \leq_{ST} X + Y \leq_{ST} X^* + Y_{2:2}$.

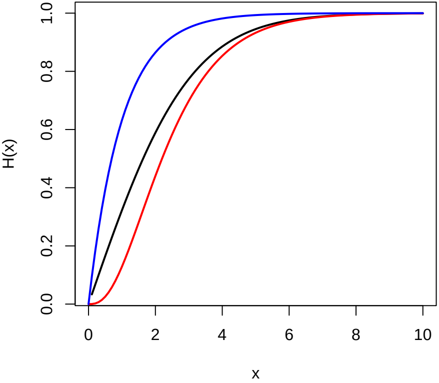

Figure: Distribution functions for the sum of two standard exponential distributions $F = G$ with a Clayton copula (black), a product copula with $F^* = F$ and $G^* = G^2$ (red) and F (blue).

Ordering properties-Example 1

- If $F = G = F^* = G^*(t) = 1 - \exp(-t)$ $t \geq 0$, then

$$H^*(t) = 1 - e^{-t} - te^{-t} = q^*(1 - e^{-t}), \ \ t \geq 0$$

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$$H(t) = \frac{1 - 2te^{-t} - e^{-2t}}{1 - 2e^{-t} - e^{-2t}} = q(1 - e^{-t})$$

with $q^*(u) = u + (1-u)\ln(1-u)$ and

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- Then the dual distortions are $\bar{q}^*(u) = u - u\ln(u)$ and

$$\bar{q}(u) = \frac{2u^2 - 2u - 2u\ln(u)}{1 - 2u + u^2},\ \ u \in [0, 1].$$

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- Analogously if $F = G = F^*$ and $G^* = G^2$, then

$$H^{**}(t) = 1 - e^{-2t} - 2te^{-t} = q^{**}(1 - e^{-t})$$

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- As $\bar{q}^{**}/\bar{q}$ is not monotone, then $X + Y$ and $X^* + Y^{**}$ are not HR (LR) ordered.
- As $\bar{q}^{**}/\bar{q}^*$ is decreasing, then $X^* + Y^* \leq_{HR,MRL} X^* + Y^{**}$.

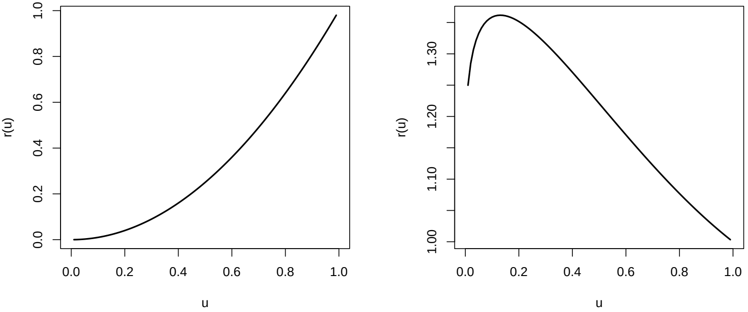

Figure: Ratios of the distortion functions $r = q^{**}/q$ (left) and $r = \bar{q}^{**}/\bar{q}$ (right). The first one leads to the RHR ordering $X + Y \leq_{RHR} X^* + Y^{**}$. The second one shows that the HR ordering does not hold.

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- Ordering properties for $X_1 + \cdots + X_n$ can be seen in the paper.

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- The distortion representations can be used to obtain comparisons results with other distortion of exponential distributions (e.g. order statistics or coherent systems).
- They can also be used to study preservation of aging classes (IFR,DFR, DRFR) under the formation of C-convolutions. In this example, $X + Y$ is IFR (and not DFR).
- Even more, we can determine the limiting behavior of the hazard rate of the sum, see Navarro and Sarabia (2024). In this example $\lim_{t\to\infty} h_{X+Y}(t) = 1 = h_X$.

The Time Transformed Exponential (TTE) model

Properties included in:

- Navarro J., Pellerey F., Mulero J. (2022b). On sums of dependent random lifetimes under the time-transformed exponential model. Test 31, 879–900. Doi: 10.1007/s11749-022-00805-2

The Time Transformed Exponential (TTE) model

- (X_1, X_2) satisfy the TTE model if

$$\Pr(X_1 > x_1, X_2 > x_2) = \bar{G}(R_1(x_1) + R_2(x_2)), \ x_1, x_2 \geq 0,$$

where $\bar{G}$ is a continuous and convex survival function with support $(0, \infty)$ and R_i are continuous and strictly increasing functions with $R_i(0) = 0$ and $R_i(\infty) = \infty$.

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- Then they have a strict bivariate Archimedean survival copula

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with additive generator $\bar{G}^{-1}$.
- It is equivalent to the frailty model where the two lifetimes are conditionally independent given a random parameter that represents the risk due to a common environment.

The Time Transformed Exponential (TTE) model

- It contains the model proposed in 2021 by Genest and Kolev (GK-model) with

$$\Pr(X_1 > x_1, X_2 > x_2) = \bar{G}(c_1 x_1 + c_2 x_2),\ x_1, x_2 \geq 0,\ c_1, c_2 > 0.$$

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- This model is an extension of the well-known Schur-constant model which is obtained when $c_1 = c_2$.
- The TTE model has also a distortion representation

$$\Pr(X_1 > x_1, X_2 > x_2) = \widehat{D}(\bar{H}_1(x_1), \bar{H}_2(x_2)),\ x_1, x_2 \geq 0,$$

where $\bar{H}_i(t) = \exp(-R_i(t))$ are survival functions and

$$\widehat{D}(u_1, u_2) = \bar{G}(-\ln(u_1 u_2))$$

is a bivariate SF with support $(0,1)^2$, see Navarro et al. (2022a,b). In the G-K model $\bar{H}_i(t) = \exp(-c_i t)$ for $t \geq 0$.

Joint PDF

Proposition

If (X_1, X_2) satisfies the TTE model and $S = X_1 + X_2$, then the joint PDF (X_1, S) is

$$\mathrm{g}(x,s) = r_1(x) r_2(s-x) \bar{G}''(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s-x)),\ 0 < x < s$$

where $r_i = R_i' = (-\ln \bar{H}_i)'$.

Joint PDF

Proposition

If (X_1, X_2) satisfies the GK-model and $S = X_1 + X_2$, then the joint PDF (X_1, S) is

$$\mathrm{g}(x,s) = c_1 c_2 \bar{G}''((c_1 - c_2)x + c_2 s),\ 0 < x < s$$

and the joint CDF is

$$\mathrm{G}(x,s) = \Pr(X \le x, S \le s) = G(c_1 x) - c_1 x G'(c_1 s)$$

when $c_1 = c_2$ (SC-model) and

$$\mathrm{G}(x,s) = G(c_1 x) + \frac{c_1}{c_1 - c_2} G(c_2 s) - \frac{c_1}{c_1 - c_2} G((c_1 - c_2)x + c_2 s)$$

when $c_1 \neq c_2$, for $0 < x < s$.

Consequences

- In the GK-model the C-convolution is

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Joint survival function

Proposition

If (X_1, X_2) satisfies the TTE model and $S = X_1 + X_2$, then the joint survival function of (X_1, S) is

$$\overline{G}(x,s) = \bar{G}(-\ln \bar{H}_1(s)) + \int_x^s r_1(y) g(-\ln \bar{H}_1(y) - \ln \bar{H}_2(s-y)) dy,$$

for $0 \leq x \leq s$, where $g = -\bar{G}'$ and $r_i = (-\ln \bar{H}_i)'$ is the hazard rate function of $\bar{H}_i$ for $i = 1, 2$.

Consequences

- Then the C-convolution is $F_S(s) = \mathrm{G}(0, s)$ i.e.

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- For example, if $\bar{H}_1(t) = e^{-t}$ for $t \geq 0$, then

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Conditional survival function

Proposition

If (X_1, X_2) satisfies the TTE model and $S = X_1 + X_2$, then the survival function of $(S|X_1 = x)$ is

$$\Pr(S > s|X_1 = x) = \frac{g(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s-x))}{g(-\ln \bar{H}_1(x))},$$

for $0 \le x \le s$. In particular, for the GK-model we get

$$\Pr(S > s|X_1 = x) = \frac{g((c_1 - c_2)x + c_2 s)}{g(c_1 x)}$$

for $0 \le x \le s$.

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- Then by using quantile regression techniques we get

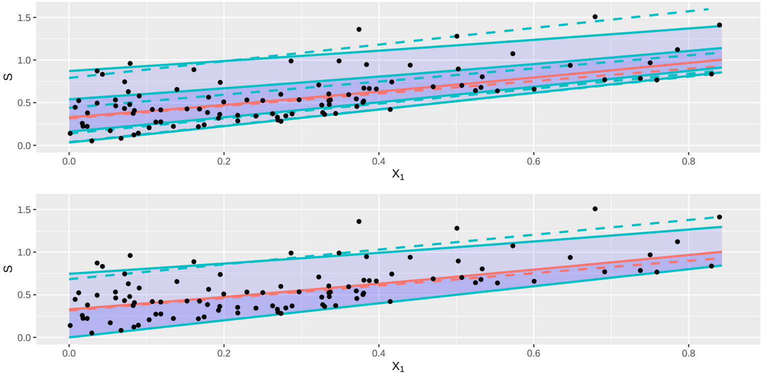

Figure: Scatterplot of a simulated sample from (X_1, S) in Example 2 jointly with the median regression curve (red) and the centered (top) or bottom (bottom) 50% and 90% confidence bands (blue). The dashed lines represent the estimated values when we use a linear quantile regression estimator.

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- Confidence prediction bands are provided as well.
- In the examples, the parameters in the model are estimated from the sample.

Results included in:

- Navarro J., Zapata J.M. (2025). Stochastic dominance of sums of risks under dependence conditions. Submitted. ArXiv: 2503.05348v1.

Comparisons of risks

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- The comparisons are based on recent results obtained in Guan et al. (2024).

Main results

Proposition

If Z is symmetric around zero (i.e. $G(-z) = \bar{G}(z)$ for all z) and the survival copula $\widehat{C}$ of (X, Z) satisfies

$$\widehat{C}(u, v) + \widehat{C}(u, 1 - v) \geq u \text{ for all } u, v \in [0, 1], \qquad (2.1)$$

then $X \leq_{CX} X + Z$.

Proposition

If Z is right skewed (i.e. $G(-z) \leq \bar{G}(z)$ for all $z > 0$) and the survival copula $\widehat{C}$ of (X, Z) satisfies (2.1), then $X \leq_{ICX} X + Z$.

Main results

- The condition (2.1) defines a weak dependence notion, called weakly Positively Quadrant Dependent ($wPQD(Z|X)$).

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- Another dependence condition is proposed as well ($sPQD(Z|X)$).
- The connections with other dependence notions are:

$$\begin{array}{ccccccc} SI_{ST}(Z|X) & \Rightarrow & PQD(X,Z) & \Rightarrow & PQDE(Z|X) & \Rightarrow & \rho \geq 0 \\ \Downarrow & & \Downarrow & & & & \\ sPQD(Z|X) & \Rightarrow & wPQD(Z|X) & \Rightarrow & \rho_C, \gamma_C \geq 0 & & \end{array}$$

where γ_C is the Gini measure of association.

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- Questions?