Are the order statistics ordered? A copula approach

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References

The conference is based on the following references:

- Navarro J., Durante F., Fernández-Sánchez J. (2021) Connecting copula properties with reliability properties of coherent systems. Applied Stochastic Models in Business and Industry 37, 496–512.
- Navarro J., Torrado N., del Águila Y. (2018). Comparisons between largest order statistics from multiple-outlier models with dependence. Methodology and Computing in Applied Probability 20, 411–433.
- Navarro J., Rychlik T. and Shaked M. (2007). Are the order statistics ordered? A Survey of Recent Results. Communications in Statistics Theory and Methods 36 (7), 1273–1290.

Outline

Preliminary results

Notation

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- $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$ hazard rate functions (HR).
- $m_X(t) = E(X - t|X > t)$ and $m_Y(t) = E(Y - t|Y > t)$ mean residual life functions (MRL).

Main stochastic orders

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- Llikelihood ratio order: $X \leq_{LR} Y \Leftrightarrow f_Y/f_X$ increases.
- Relationships:

$$
\begin{array}{ccccc}
X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\
 & & \Downarrow & & \Downarrow \\
 & & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y)
\end{array}
$$

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- $X_{n-k+1:n}$ lifetime of a k-out-of-n system.

Basic properties

- Survival function (IID case):

$$\bar{F}_{i:n}(t) = \Pr(X_{i:n} > t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t).$$

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- Hazard rate function (IID case):

$$h_{i:n}(t) = i \binom{n}{i} \frac{f(t) F^{i-1}(t) \bar{F}^{n-i}(t)}{\sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t)}.$$

Basic references on order statistics and systems

- Arnold B.C., Balakrishnan N., Nagaraja, H.N. *A First Course in Order Statistics.* SIAM, 2008.

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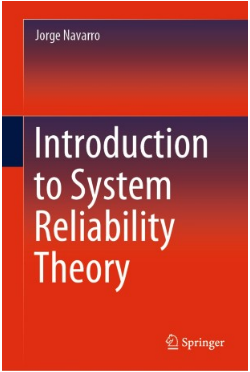
- My new book:



Distortion representations

- As we have seen, in the IID case we have:

$$\bar{F}_{i:n}(t) = \bar{q}(\bar{F}(t))$$

where $\bar{q} : [0,1] \to [0,1]$ is a distortion function, i.e., $\bar{q}$ is continuous, is increasing and satisfies $\bar{q}(0) = 0$ and $\bar{q}(1) = 1$.

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- In the general case, from any $(X_1, \ldots, X_n)$ it can be written as

$$\bar{F}_{i:n}(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \text{ for all } t \in \mathbb{R}, \qquad (1.1)$$

  where $\bar{F}_i(t) = \Pr(X_i > t)$ and $\bar{Q} : [0,1]^n \to [0,1]$ is a distortion function, i.e., $\bar{Q}$ is continuous, increasing and satisfies $\bar{Q}(0, \ldots, 0) = 0$ and $\bar{Q}(1, \ldots, 1) = 1$.

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- $\bar{Q}$ only depends on i, n and the survival copula $\hat{C}$ obtained from Sklar's theorem to get:

$$\Pr(X_1 > x_1, \dots, X_n > x_n) = \widehat{C}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n))$$

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- If the components are independent (IND), then $\bar{Q}$ is a multinomial.
- If the components are IID, then $\bar{q}$ is a polynomial.

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where $\delta_{\hat{C}}(u) = \widehat{C}(u, \dots, u)$ is the diagonal section of $\widehat{C}$.
- IID case:

$$\bar{F}_{1:n}(t) = \bar{F}^n(t) = \bar{q}(\bar{F}(t))$$

with $\bar{q}(u) = u^n$ for $u \in [0, 1]$.

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- $T_1 \leq_{LR} T_2$ for all F iff $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$.
- $T_1 \leq_{MRL} T_2$ for all F such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub shaped in $(0, 1)$.

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- Navarro et al. ASMBI, 2013 and Navarro and Gomis ASMBI, 2016.

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- $T_1 \leq_{RHR} T_2$ for all $\bar{F}_1, \dots, \bar{F}_r$ iff Q_2/Q_1 is increasing in $(0,1)^n$.

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- $T_1 \leq_{HR} T_2$ for all $\bar{F}_1, \dots, \bar{F}_n$ such that

$$F_1 \geq_{HR} \cdots \geq_{HR} F_n \tag{1.3}$$

iff the function

$$H(v_1, \dots, v_n) = \frac{\bar{Q}_2(v_1, v_1 v_2, \dots, v_1 \dots v_n)}{\bar{Q}_1(v_1, v_1 v_2, \dots, v_1 \dots v_n)} \tag{1.4}$$

is decreasing in $(0,1)^n$.

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- A similar result holds for the RHR order.

Ordering properties

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- **Case III:** $X_1, \ldots, X_n$ are IND with SF $\bar{F}_1, \ldots, \bar{F}_n$.

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- **Case III:** $X_1, \ldots, X_n$ are IND with SF $\bar{F}_1, \ldots, \bar{F}_n$.
- **Case IV:** $X_1, \ldots, X_n$ are arbitrary (GENeral case) with SF $\bar{F}_1, \ldots, \bar{F}_n$ and SC $\widehat{C}$.

Case I: IID, ST order

- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered?

Case I: IID, ST order

- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered?
- Yes, since

$$\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t) \leq \sum_{j=0}^{i} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t) = \bar{F}_{i+1:n}(t)$$

for $i = 1, \ldots, n-1$ and all n, F.

Case I: IID, HR order

- Are $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ ordered?

Case I: IID, HR order

- Are $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ ordered?
- Yes, since

$$\frac{\bar{F}_{i+1:n}(t)}{\bar{F}_{i:n}(t)} = \frac{\sum_{j=0}^{i} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t)}{\sum_{j=0}^{i-1} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t)} = 1 + \frac{\binom{n}{i} F^i(t)\bar{F}^{n-i}(t)}{\sum_{j=0}^{i-1} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t)}$$

is increasing in t for $i = 1, \dots, n-1$ and all n, F because

$$\sum_{j=0}^{i-1} \binom{n}{j} F^{j-i}(t)\bar{F}^{i-j}(t) = \sum_{j=0}^{i-1} \binom{n}{j} \bar{H}^{i-j}(t)$$

where $H(t) = \bar{F}(t)/F(t) = -1 + 1/F(t)$ is decreasing.

Case I: IID, LR order

- Are $X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}$ ordered?

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- Are $X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}$ ordered?
- Yes, since

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is increasing in t for $i = 1, \ldots, n-1$ and all n, F (because H is decreasing).

Case II: ID, ST order

- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered?

Case II: ID, ST order

- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered?
- Yes, since

$$\bar{F}_{i:n}(t) = \bar{q}_{i:n}(\bar{F}(t)) \leq \bar{F}_{i+1:n}(t) = \bar{q}_{i+1:n}(\bar{F}(t))$$

and $\bar{q}_{i:n} \leq \bar{q}_{i+1:n}$ for all copula $\widehat{C}$.

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- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered?
- Yes, since

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and $\bar{q}_{i:n} \leq \bar{q}_{i+1:n}$ for all copula $\widehat{C}$.

- For example, for $n = 2$

$$\bar{F}_{1:2}(t) = \bar{q}_{1:2}(\bar{F}(t)) \leq \bar{F}_{2:2}(t) = \bar{q}_{2:2}(\bar{F}(t))$$

where $\bar{q}_{1:2}(u) = \widehat{C}(u, u)$, and

$$\bar{F}_{2:2}(t) = \Pr(\max(X_1, X_2) > t) = \Pr(X_1 > t) + \Pr(X_2 > t) - \Pr(X_{1:2} > t)$$

that is, $\bar{q}_{2:2}(u) = 2u - \widehat{C}(u, u)$ with

$$\widehat{C}(u, u) \leq 2u - \widehat{C}(u, u)$$

since $\widehat{C}(u, u) \leq \widehat{C}(1, u) = u$ for all copula $\widehat{C}$.

Case II: ID, HR order

- Are $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ ordered?

Case II: ID, HR order

- Are $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ ordered?
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Case II: ID, HR order

- Are $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ ordered?
- Are $X_{1:2} \leq_{HR} X_{2:2}$ ordered?
- It holds if and only if

$$\frac{\bar{q}_{2:2}(u)}{\bar{q}_{1:2}(u)} = \frac{2u - \widehat{C}(u,u)}{\widehat{C}(u,u)} = -1 + \frac{2u}{\widehat{C}(u,u)}$$

is decreasing, that is, if and only if

$$r(u) = \frac{\widehat{C}(u,u)}{u} = \frac{\delta_{\widehat{C}}(u)}{u}$$

is increasing in $(0, 1)$.

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- Are $X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n}$ ordered?
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  is increasing in $(0,1)$.
- Is this property true for any copula?

Case II: ID, HR order

- Yes for the product copula

$$r(u) = \frac{\widehat{C}(u,u)}{u} = \frac{u^2}{u} = u.$$

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- If we consider the following Clayton copula

$$\widehat{C}(u,v) = \frac{uv}{u+v-uv}$$

then

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- Hence $X_{1:2} \leq_{HR} X_{2:2}$ are ordered for any F and this copula.

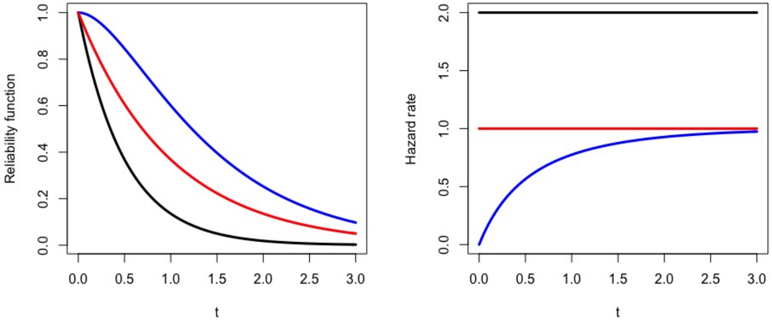
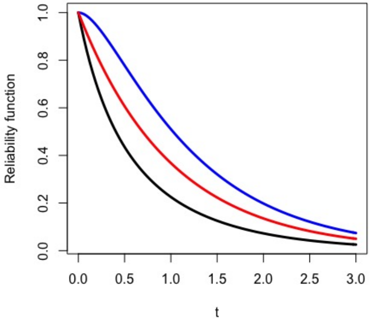
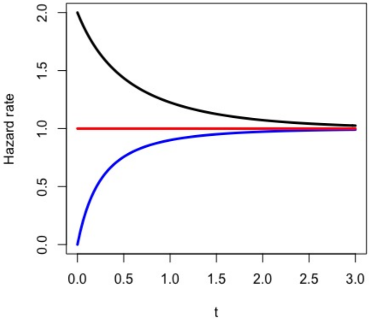


Figure: Reliability and hazard rate functions of $X_{1:2}$ (black line), X_i (red line) and $X_{2:2}$ (blue line) when X_1 and X_2 are IID and have a common exponential distribution with mean one.





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Case II: ID, HR order

- If we choose the following copula extracted from Example 4.1 in Navarro, Torrado and del Águila (2018)

$$\widehat{C}(u,v) = \min(u, v, 0.5\delta(u) + 0.5\delta(v))$$

with

$$\delta(u) = \begin{cases} u & \text{for} \quad 0 \le u \le 1/3 \\ 1/3 & \text{for} \quad 1/3 \le u \le 2/3 \\ 2u-1 & \text{for} \quad 2/3 \le u \le 1 \end{cases}$$

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- Then $\delta_{\hat{C}}(u)=\delta(u)$ and $r(u)=\delta(u)/u$ is not increasing.

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- So the correct answer is NO (it depends on the copula).

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- Then $\delta_{\hat{C}}(u)=\delta(u)$ and $r(u)=\delta(u)/u$ is not increasing.
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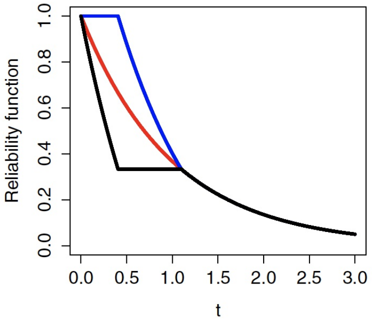
- This surprising fact was proved first in Navarro and Shaked, Journal of Applied Probability 43 (2006), 391-408.



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Proposition (Navarro, Torrado and del Águila (2018))

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(ii) $X_1 \leq_{HR} X_{2:2}$ *for all* F;
(iii) $X_{1:2} \leq_{HR} X_{2:2}$ *for all* F;
(iv) $\widehat{C}(u,u)/u$ *is increasing in* $(0,1)$;
(v) $(1 - C(u,u))/(1-u)$ *is increasing in* $(0,1)$.

Case II: ID, LR order

- Are $X_{1:2} \leq_{LR} X_{2:2}$ ordered?

Case II: ID, LR order

- Are $X_{1:2} \leq_{LR} X_{2:2}$ ordered?
- It holds iff

$$\frac{\bar{q}'_{2:2}(u)}{\bar{q}'_{1:2}(u)} = \frac{2 - \delta'_{\widehat{C}}(u)}{\delta'_{\widehat{C}}(u)}$$

is decreasing in $(0, 1)$.

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- So it holds iff $\delta'_{\widehat{C}}(u)$ is increasing in $(0, 1)$, i.e., iff $\delta_{\widehat{C}}(u) = \widehat{C}(u, u)$ is convex in $(0, 1)$.

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- Then the correct answers is NO (it depends on the copula).

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(iii) $X_{1:2} \leq_{LR} X_{2:2}$ *for all* F*;*
(iv) $\widehat{C}(u,u)$ *is convex in* $(0,1)$*;*
(v) $C(u,u)$ *is convex in* $(0,1)$*.*

Cases III & IV: ST order, GEN case

- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered in the GENeral case?

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- Are $X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}$ ordered in the GENeral case?
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$$X_{1:n} \leq \cdots \leq X_{n:n}$$

implies

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see Shaked and Shanthikumar (2007), p. 5.

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- $X_{i:m} \leq_{ST} X_{j:n}$ holds iff $i \leq j$ and $m - i \leq n - j$, Arcones, Kvam and Samaniego, JASA, 2002

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- NO (proved before).

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- $X_{1:2} \leq_{HR} X_{2:2}$ holds iff

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that is, iff $\widehat{C}(u,v)/(u+v)$ is increasing.

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- If X_1 and X_2 are IND, then it holds iff

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- So the correct answer is YES (Boland, El-Neweihi and Proschan, JAP, 1994).

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that is, iff $\widehat{C}(u,v)/u$ is increasing in u, Navarro, Durante and Fernández-Sánchez (2021).

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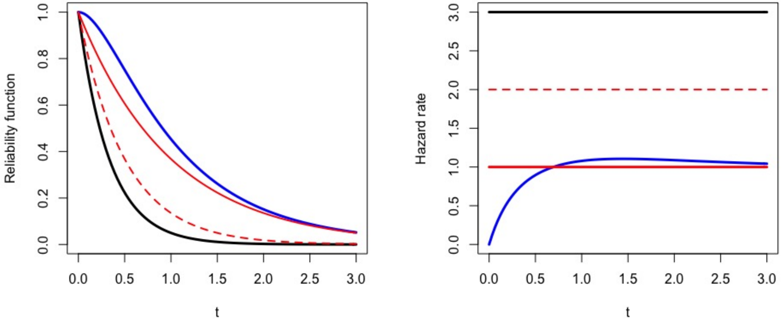
- So the correct answer is NO ($v(-1+1/u)$ is increasing in v).



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- The set $\mathcal{C}_H = \{C \text{ copulas satisfying } (ii)\}$ is a closed set in the class of bivariate copulas $\mathcal{C}$ equipped with the supremum distance d_∞.

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- The set $\mathcal{C}_H$ is nowhere dense in $(\mathcal{C}, d_\infty)$ (i.e. it is a small set from a topological point of view).
- We also have the following negative result:

Proposition (Navarro, Durante and Fernández-Sánchez (2021))

For any possible copula C, $X_1 \leq_{HR} X_{2:2}$ does not hold for all F_1 and F_2.

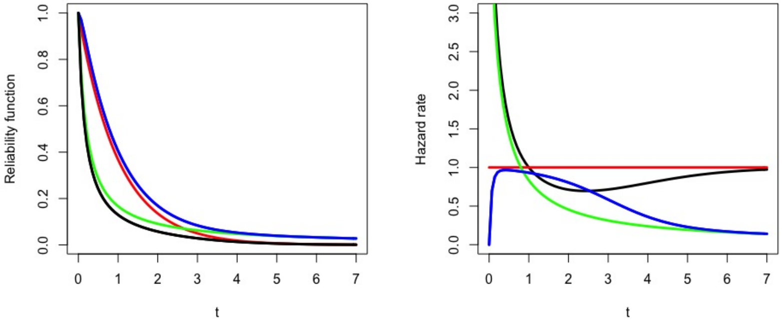

Figure: Reliability and hazard rate functions of $X_{1:2}$ (black line), X_1 (red line), X_2 (green line) and $X_{2:2}$ (blue line) when X_1 and X_2 are dependent with a survival Clayton copula and X_1 has an exponential distribution with mean one and X_2 has a Pareto distribution.

Cases III & IV: HR order, IND case, ordered components

- Are $X_1 \leq_{HR} X_{2:2}$ ordered in the IND case when $X_1 \geq_{HR} X_2$?

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- If X_1 and X_2 are IND, then it holds iff

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- So the correct answer is NO.

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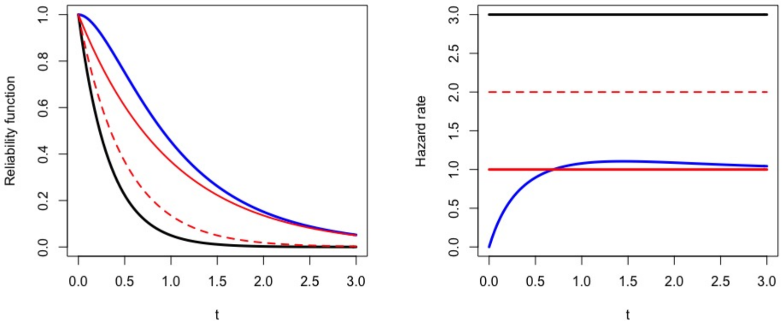

Figure: Reliability and hazard rate functions of $X_{1:2}$ (black line), X_i (red lines) and $X_{2:2}$ (blue line) when X_1 and X_2 are IND and have exponential distributions with hazard rate one and two.

Main references

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- Questions?