

Weak Dependence Notions

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References

The conference is based on the following references:

- ▶ Navarro J., Pellerey F., Sordo M.A. Weak dependence notions and their mutual relationships. *Mathematics* 2021, 9(1), 81. <https://doi.org/10.3390/math9010081>.
- ▶ Navarro J, Sordo MA (2018). Stochastic comparisons and bounds for conditional distributions by using copula properties. *Dependence Modeling* 6, 156–177.
- ▶ Navarro J., Durante F., Fernández-Sánchez J. (2021) Connecting copula properties with reliability properties of coherent systems. *Applied Stochastic Models in Business and Industry* 37, 496–512. <https://doi.org/10.1002/asmb.2579>.

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Preliminary results

Notation

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- ▶ $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$ hazard rate functions (HR).

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- ▶ $m_X(t) = E(X - t|X > t)$ and $m_Y(t) = E(Y - t|Y > t)$ mean residual life functions (MRL).

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- ▶ $m_X(t) = E(X - t|X > t)$ and $m_Y(t) = E(Y - t|Y > t)$ mean residual life functions (MRL).
- ▶ $\bar{m}_X(t) = E(t - X|X < t)$ and $\bar{m}_Y(t) = E(t - Y|Y < t)$ mean inactivity time functions (MIT).

Main stochastic orders

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- ▶ Mean residual life order: $X \leq_{MRL} Y \Leftrightarrow m_X \leq m_Y$.
- ▶ Mean inactivity time order: $X \leq_{MIT} Y \Leftrightarrow \bar{m}_X \geq \bar{m}_Y$.

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- ▶ Mean residual life order: $X \leq_{MRL} Y \Leftrightarrow m_X \leq m_Y$.
- ▶ Mean inactivity time order: $X \leq_{MIT} Y \Leftrightarrow \bar{m}_X \geq \bar{m}_Y$.
- ▶ Likelihood ratio order: $X \leq_{LR} Y \Leftrightarrow f_Y/f_X$ increases.

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- ▶ Mean residual life order: $X \leq_{MRL} Y \Leftrightarrow m_X \leq m_Y$.
- ▶ Mean inactivity time order: $X \leq_{MIT} Y \Leftrightarrow \bar{m}_X \geq \bar{m}_Y$.
- ▶ Likelihood ratio order: $X \leq_{LR} Y \Leftrightarrow f_Y/f_X$ increases.
- ▶ Increasing concave order $X \leq_{ICV} Y \Leftrightarrow E(\phi(X)) \leq E(\phi(X))$
for all increasing and concave functions ϕ .

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- ▶ Likelihood ratio order: $X \leq_{LR} Y \Leftrightarrow f_Y/f_X$ increases.
- ▶ Increasing concave order $X \leq_{ICV} Y \Leftrightarrow E(\phi(X)) \leq E(\phi(X))$ for all increasing and concave functions ϕ .
- ▶ Increasing convex order $X \leq_{ICX} Y \Leftrightarrow E(\phi(X)) \leq E(\phi(X))$ for all increasing and convex functions ϕ .

Main stochastic orders

► Relationships:

$$\begin{array}{ccccc}
 X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & X \leq_{ICX} Y \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 X \leq_{MIT} Y & \Rightarrow & X \leq_{ICV} Y & \Rightarrow & E(X) \leq E(Y)
 \end{array}$$

Properties based on residual lifetimes

Proposition

If $X_t = (X - t | X > t)$ and $Y_t = (Y - t | Y > t)$, the following conditions are equivalent:

- i) $X \leq_{HR} Y$;
- ii) $X_t \leq_{HR} Y_t$ for all t ;
- iii) $X_t \leq_{ST} Y_t$ for all t ;
- iv) $h_X \geq h_Y$ (abs. cont. case).

Properties based on residual lifetimes

Proposition

The following conditions are equivalent:

- i) $X \leq_{LR} Y$;
- ii) $X_t \leq_{LR} Y_t$ for all t ;
- iii) $X_t \leq_{RHR} Y_t$ for all t ;
- iv) $(X|s < X < t) \leq_{ST} (Y|s < Y < t)$ for all $0 < s < t$.

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Proposition

The following conditions are equivalent:

- i) $X \leq_{MRL} Y$;
- ii) $X_t \leq_{MRL} Y_t$ for all t ;
- iii) $X_t \leq_{ICX} Y_t$ for all t .

Properties based on inactivity times

Proposition

If ${}_tX = (t - X | X < t)$ and ${}_tY = (t - Y | Y < t)$, the following conditions are equivalent:

- i) $X \leq_{RHR} Y$;
- ii) ${}_tX \geq_{HR} {}_tY$ for all $t > 0$;
- iii) ${}_tX \geq_{ST} {}_tY$ for all $t > 0$.

Properties based on inactivity times

Proposition

The following conditions are equivalent:

- i) $X \leq_{LR} Y$;
- ii) ${}_tX \geq_{LR} {}_tY$ for all $t > 0$;
- iii) ${}_tX \geq_{RHR} {}_tY$ for all $t > 0$;
- iv) $(X|s < X < t) \leq_{ST} (Y|s < Y < t)$ for all $0 < s < t$.

Properties based on inactivity times

Proposition

The following conditions are equivalent:

- i) $X \leq_{MIT} Y$;
- ii) ${}_tX \geq_{MRL} {}_tY$ for all $t > 0$;
- iii) ${}_tX \geq_{ICX} {}_tY$ for all $t > 0$.

Dependence notions

- ▶ *PQD* (Positively Quadrant Dependent) iff $X \leq_{ST} (X|Y > t)$ for all $t \geq 0$.

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- ▶ *RTI*($X|Y$) (Right Tail Increasing) iff $(X|Y > t)$ ST-increasing.

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- ▶ *LTI*($X|Y$) (Left Tail Increasing) iff $(X|Y \leq t)$ is ST-decreasing.
- ▶ *SI*($X|Y$) (Stochastically Increasing) iff $(X|Y = t)$ is ST-increasing.

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- ▶ *SI*($X|Y$) (Stochastically Increasing) iff $(X|Y = t)$ is ST-increasing.
- ▶ *SD*($X|Y$) (Stochastically Decreasing) iff $(X|Y = t)$ is ST-decreasing.

Dependence notions

- ▶ *RCSI* (Right Corner Set Increasing) iff $\Pr(X > x, Y > y | X > s, Y > t)$ is increasing in s and t for all x, y or, equivalently, if $(X | Y > t)$ is HR-increasing in t .

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- ▶ *RCSD* (Right Corner Set Decreasing) iff $\Pr(X > x, Y > y | X > s, Y > t)$ is decreasing in s and t for all x, y or, equivalently, if $(X|Y > t)$ is HR-decreasing in t .

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- ▶ *RCSD* (Right Corner Set Decreasing) iff $\Pr(X > x, Y > y | X > s, Y > t)$ is decreasing in s and t for all x, y or, equivalently, if $(X|Y > t)$ is HR-decreasing in t .
- ▶ *LCSD* (Left Corner Set Decreasing) iff $\Pr(X \leq x, Y \leq y | X \leq s, Y \leq t)$ is decreasing in s and t for all x, y or, equivalently if $(X|Y \leq t)$ is RHR-increasing in t .

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- ▶ *RCSI* (Right Corner Set Increasing) iff $\Pr(X > x, Y > y | X > s, Y > t)$ is increasing in s and t for all x, y or, equivalently, if $(X|Y > t)$ is HR-increasing in t .
- ▶ *RCSD* (Right Corner Set Decreasing) iff $\Pr(X > x, Y > y | X > s, Y > t)$ is decreasing in s and t for all x, y or, equivalently, if $(X|Y > t)$ is HR-decreasing in t .
- ▶ *LCSD* (Left Corner Set Decreasing) iff $\Pr(X \leq x, Y \leq y | X \leq s, Y \leq t)$ is decreasing in s and t for all x, y or, equivalently if $(X|Y \leq t)$ is RHR-increasing in t .
- ▶ *LCSI* (Left Corner Set Increasing) iff $\Pr(X \leq x, Y \leq y | X \leq s, Y \leq t)$ is increasing in s and t for all x, y or, equivalently if $(X|Y \leq t)$ is RHR-decreasing in t .

Dependence notions

- ▶ $SIRL(X|Y)$ (Stochastically Increasing in Residual Life) iff $(X|Y = t)$ is HR-increasing in t .

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- ▶ $SIRL(X|Y)$ (Stochastically Increasing in Residual Life) iff $(X|Y = t)$ is HR-increasing in t .
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- ▶ $SIRL(X|Y)$ (Stochastically Increasing in Residual Life) iff $(X|Y = t)$ is HR-increasing in t .
- ▶ $SDRL(X|Y)$ (Stochastically Decreasing in Residual Life) iff $(X|Y = t)$ is HR-decreasing in t .
- ▶ $PRLD$ (Positive Likelihood Ratio Dependent) iff its joint density function is TP_2 or, equivalently, if $(X|Y = t)$ LR-increasing in t .

Dependence notions

- ▶ $SIRL(X|Y)$ (Stochastically Increasing in Residual Life) iff $(X|Y = t)$ is HR-increasing in t .
- ▶ $SDRL(X|Y)$ (Stochastically Decreasing in Residual Life) iff $(X|Y = t)$ is HR-decreasing in t .
- ▶ $PRLD$ (Positive Likelihood Ratio Dependent) iff its joint density function is TP_2 or, equivalently, if $(X|Y = t)$ LR-increasing in t .
- ▶ $NRLD$ (Negative Likelihood Ratio Dependent) iff its joint density function is RR_2 or, equivalently, if $(X|Y = t)$ LR-decreasing in t .

Copula representations

- ▶ (X, Y) non-negative random vector.

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- ▶ The joint distribution function can be written as

$$F(x, y) = \Pr(X \leq x, Y \leq y) = C(F_X(x), F_Y(y))$$

for all $x, y \in \mathbb{R}$, where $C : [0, 1]^2 \rightarrow [0, 1]$ is a copula function.

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for all $x, y \in \mathbb{R}$, where $C : [0, 1]^2 \rightarrow [0, 1]$ is a copula function.

- ▶ The joint survival (or reliability) function can be written as

$$\bar{F}(x, y) = \Pr(X < x, Y < y) = \hat{C}(\bar{F}_X(x), \bar{F}_Y(y))$$

for all $x, y \in \mathbb{R}$, where $\hat{C} : [0, 1]^2 \rightarrow [0, 1]$ is a copula function.

Properties

Proposition

The following conditions are equivalent:

- i) $X \leq_{ST} (X|Y > s)$ for all s and all \bar{F}_X, \bar{F}_Y ;
- ii) $Y \leq_{ST} (Y|X > t)$ for all t and all \bar{F}_X, \bar{F}_Y ;
- iii) $\hat{C}(u, v) \geq uv$ for all $u, v \in (0, 1)$;
- iv) $\text{Cov}(\phi_1(X), \phi_2(Y)) \geq 0$ for all increasing functions ϕ_1 and ϕ_2 ;
- iv) (X, Y) is PQD.

Properties

Proposition

If $X_{s,t} = (X - t | X > t, Y > s)$ and $Y_{s,t} = (Y - s | X > t, Y > s)$, for continuous \bar{F}_X, \bar{F}_Y the following conditions are equivalent:

- i) $X \leq_{HR} (X | Y > s)$ for all s and all \bar{F}_X, \bar{F}_Y ;
- ii) $X_t \leq_{HR} X_{t,s}$ for all t, s and all \bar{F}_X, \bar{F}_Y ;
- iii) $X_t \leq_{ST} X_{t,s}$ for all t, s and all \bar{F}_X, \bar{F}_Y ;
- iv) $\hat{C}(u, v)/u$ is decreasing in u for all $v \in (0, 1)$;
- v) $RTI(Y|X)$.

Properties

Proposition

The following conditions are equivalent:

- i) $X \leq_{LR} (X|Y > s)$ for all s and all \bar{F}_X, \bar{F}_Y ;
- ii) $X_t \leq_{LR} X_{t,s}$ for all t, s and all \bar{F}_X, \bar{F}_Y ;
- iii) $X_t \leq_{RHR} X_{t,s}$ for all t, s and all \bar{F}_X, \bar{F}_Y ;
- iv) $\hat{C}(u, v)$ is concave in u (or $\partial_1 \hat{C}(u, v)$ is decreasing in u) for all $v \in (0, 1)$;
- v) $SI(Y|X)$.

Properties

Proposition

The following conditions are equivalent:

- i) $(X|Y > s_1) \leq_{HR} (X|Y > s_2)$ for all $s_1 \leq s_2$ and all \bar{F}_X, \bar{F}_Y ;
- ii) $(Y|X > t_1) \leq_{HR} (Y|X > t_2)$ for all $t_1 \leq t_2$ and all \bar{F}_X, \bar{F}_Y ;
- iii) $X_{t,s_1} \leq_{HR} X_{t,s_2}$ for all t , all $s_1 \leq s_2$ and all \bar{F}_X, \bar{F}_Y ;
- iv) $X_{t,s_1} \leq_{ST} X_{t,s_2}$ for all t , all $s_1 \leq s_2$ and all \bar{F}_X, \bar{F}_Y ;
- v) $Y_{t_1,s} \leq_{HR} Y_{t_2,s}$ for all s , all $t_1 \leq t_2$ and all \bar{F}_X, \bar{F}_Y ;
- vi) $Y_{t_1,s} \leq_{ST} Y_{t_2,s}$ for all s , all $t_1 \leq t_2$ and all \bar{F}_X, \bar{F}_Y ;
- vii) $\hat{C}(u, v)$ is TP_2 ;
- viii) (X, Y) is $RCSI$.

Properties

Proposition

The following conditions are equivalent:

- i) $(X|Y > s_1) \leq_{LR} (X|Y > s_2)$ for all $s_1 < s_2$ and all \bar{F}_X, \bar{F}_Y ;
- ii) $(X_t|Y > s_1) \leq_{LR} (X_t|Y > s_2)$ for all $t, s_1 < s_2$ and all \bar{F}_X, \bar{F}_Y ;
- iii) $(X_t|Y > s_1) \leq_{RHR} (X_t|Y > s_2)$ for all $t, s_1 < s_2$ and all \bar{F}_X, \bar{F}_Y ;
- iv) $\partial_1 \hat{C}(u, v)$ is TP_2 ;
- v) $SIRL(Y|X)$.

Properties

Proposition

The following conditions are equivalent:

- i) $(Y|X = t_1) \leq_{LR} (Y|X = t_2)$ for all $t_1 < t_2$ and all \bar{F}_X, \bar{F}_Y ;
- ii) $(X|Y = s_1) \leq_{LR} (X|Y = s_2)$ for all $s_1 < s_2$ and all \bar{F}_X, \bar{F}_Y ;
- iii) $(X_t|Y = s_1) \leq_{LR} (X_t|Y = s_2)$ for all $t, s_1 < s_2$ and all \bar{F}_X, \bar{F}_Y ;
- iv) $(X_t|Y = s_1) \leq_{RHR} (X_t|Y = s_2)$ for all $t, s_1 < s_2$ and all \bar{F}_X, \bar{F}_Y ;
- v) $(Y_s|X = t_1) \leq_{LR} (Y_s|X = t_2)$ for all $s, t_1 < t_2$ and all \bar{F}_X, \bar{F}_Y ;
- vi) $(Y_s|X = t_1) \leq_{RHR} (Y_s|X = t_2)$ for all $s, t_1 < t_2$ and all \bar{F}_X, \bar{F}_Y ;
- vii) $\partial_{1,2}^2 C(u, v)$ is TP_2 in (u, v) ;
- viii) $PLRD$.

New dependence notions

New dependence notions

- ▶ We say that (X, Y) is Stochastically Increasing (Decreasing) in the order ORD , denoted $SI_{ORD}(X|Y)$ ($SD_{ORD}(X|Y)$) if

$$(X|Y = t_1) \leq_{ORD} (X|Y = t_2) \quad (\geq_{ORD})$$

holds for all $t_1 < t_2$ for a given stochastic order ORD .

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$$(X|Y = t_1) \leq_{ORD} (X|Y = t_2) \quad (\geq_{ORD})$$

holds for all $t_1 < t_2$ for a given stochastic order ORD .

- ▶ With this definition $SI(X|Y) = SI_{ST}(X|Y)$,
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- ▶ Note that $SI_{LR}(Y|X)$ is equivalent to $SI_{LR}(X|Y)$ and so we can just write SI_{LR} .

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- ▶ The proofs of these equivalences can be found in Navarro and Sordo (2018) and Longobardi and Pellerey (2019).
- ▶ Some of these notions are actually equivalent, like, e.g., $RTI_{LR}^0(Y|X)$ and $SI_{ST}(X|Y)$, or $RTI_{HR}^0(Y|X)$ and $RTI_{ST}(X|Y)$ (see Foschi and Spizzichino (2013) for details).

Relationships

$$\begin{array}{ccccc}
 PQD(X, Y) & \Leftarrow & RTI(Y|X) & \Leftarrow & SI(Y|X) \\
 \uparrow & & \uparrow & & \uparrow \\
 RTI(X|Y) & \Leftarrow & RCSI(X, Y) & \Leftarrow & SIRL(Y|X) \\
 \uparrow & & \uparrow & & \uparrow \\
 SI(X|Y) & \Leftarrow & SIRL(X|Y) & \Leftarrow & PLRD(X, Y)
 \end{array}$$

Table: Relationships among positive dependence properties.

Relationships

$$\begin{array}{ccccc}
 SI(Y|X) & \Rightarrow & LTD(Y|X) & \Rightarrow & PQD(X, Y) \\
 \uparrow & & \uparrow & & \uparrow \\
 SI_{RHR}(Y|X) & \Rightarrow & LCSD(X, Y) & \Rightarrow & LTD(X|Y) \\
 \uparrow & & \uparrow & & \uparrow \\
 PLRD(X, Y) & \Rightarrow & SI_{RHR}(X|Y) & \Rightarrow & SI(X|Y)
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Table: Relationships among reversed positive dependence properties.

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- ▶ For that reason to they can be considered as "weak positive dependence notions".
- ▶ The negative dependence properties are defined in a similar way.

Weak dependence notions

Proposition

The following conditions are equivalent:

- i) $X \leq_{ICX} (X|Y > y)$ for all y (i.e. $RTI_{ICX}^0(X|Y)$);
- ii) The survival copula \hat{C} satisfies

$$\int_0^z [\hat{C}(u, v) - uv] d\bar{F}_X^{-1}(u) \leq 0, \quad \forall z \in [0, 1], \quad \forall v \in [0, 1]. \quad (2.1)$$

Weak dependence notions

Proposition

The following conditions are equivalent:

- i) $X \leq_{MRL} (X|Y > y)$ for all y (i.e. $RTI_{MRL}^0(X|Y)$);
- ii) $X_t \leq_{MRL} X_{t,s}$ for all t, s ;
- iii) $X_t \leq_{ICX} X_{t,s}$ for all t, s ;
- iv) The survival copula \hat{C} satisfies

$$\int_0^z [z\hat{C}(u, v) - u\hat{C}(z, v)] d\bar{F}_X^{-1}(u) \leq 0 \quad \forall z, v \in [0, 1] \quad (2.2)$$

Weak dependence notions

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- ▶ Then $PQD \Rightarrow PQDE(X|Y) \Rightarrow r_{X,Y} \geq 0$.

Relationships between weak dependence notions

$$\begin{array}{ccccc}
 r_{X,Y} \geq 0 & & & & \\
 \uparrow & & & & \\
 PQDE(X|Y) & \Leftarrow & RTI_{ICX}^0(X|Y) & \Leftarrow & RTI_{MRL}^0(X|Y) \\
 \uparrow & & \uparrow & & \uparrow \\
 RTI_{ICX}^0(Y|X) & \Leftarrow & PQD(X, Y) & \not\Leftarrow & RTI_{MRL}(X|Y) \\
 \uparrow & & \nexists & & \uparrow \\
 RTI_{MRL}^0(Y|X) & \Leftarrow & RTI_{MRL}(Y|X) & \Leftarrow & RCSI(X, Y)
 \end{array}$$

Table: Relationships among weak positive dependence properties.

Relationships between weak dependence notions

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 r_{X,Y} \geq 0 & & & & \\
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 PQDE(X|Y) & \Leftarrow & RTI_{ICX}^0(X|Y) & \Leftarrow & RTI_{MRL}^0(X|Y) \\
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 RTI_{ICX}^0(X|Y) & \Leftarrow & RTI_{ICX}(X|Y) & \Leftarrow & RTI_{MRL}(X|Y) \\
 \uparrow & & \uparrow & & \uparrow \\
 PQD(X,Y) & \Leftarrow & RTI_{ST}(X|Y) & \Leftarrow & RCSI(X, Y)
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- ▶ However, interesting properties of the survival copula of the vector are introduced when we assume uniform marginals.
- ▶ They can be used to define new weak dependence properties that do not depend on the marginals.
- ▶ In fact, letting P denote the property

$$\int_0^z [\hat{C}(u, v) - uv] du \geq 0, \forall z \in [0, 1], \forall v \in [0, 1],$$

and letting \tilde{P} denote the property

$$X \leq_{ICX} (X|Y > y) \quad \forall y$$

one immediately observes that both are positive dependence properties weaker than PQD .

Other dependence properties

- ▶ Property P verifies

$$PQD \Rightarrow P \Rightarrow \rho_{X,Y} \geq 0,$$

where

$$\rho_{X,Y} = 12 \int_0^1 \int_0^1 C(u,v) du dv - 3$$

is the Spearman's rho coefficient for X and Y (for the formula of $\rho_{X,Y}$, see (5.1.15c) in Nelsen (2006)), and the first implication follows from (2.1).

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- ▶ Property \tilde{P} verifies

$$PQD \Rightarrow \tilde{P} \Rightarrow r_{X,Y} \geq 0. \quad (2.4)$$

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- ▶ One can define weak dependence properties as it has been done for P , by letting the margins to be uniformly distributed on $(0, 1)$ in the definitions above.

Reversed weak positive dependence properties

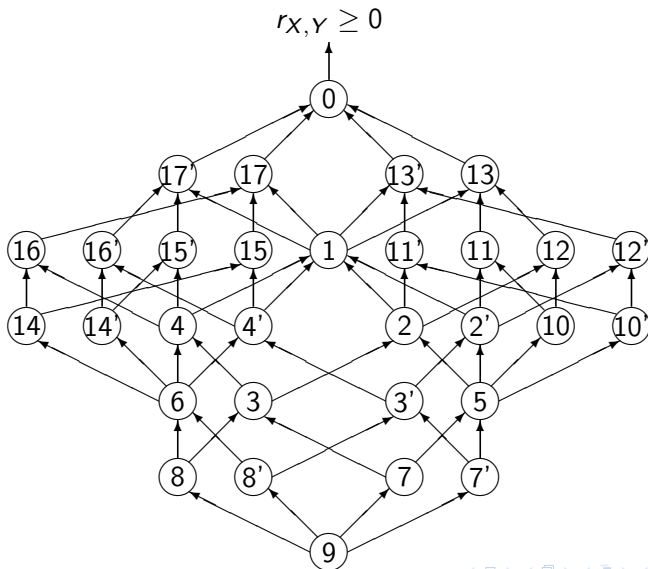
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- ▶ In the paper we also include counterexamples showing that these classes are different.
- ▶ Finally, we conclude with a diagram that connect all these properties.



N	Name			
0	$PQDE$	$E(X) \leq E(X Y > s)$	$E(X) \geq E(X Y \leq s)$	
1	PQD	$RTI_{ST}^0(Y X)$	$RTI_{ST}^0(X Y)$	$LTD_{ST}^\infty(Y X)$
2	$RTI(Y X)$	$RTI_{ST}(Y X)$	$RTI_{HR}^0(X Y)$	
2'	$RTI(X Y)$	$RTI_{ST}(X Y)$	$RTI_{HR}^0(Y X)$	
3	$SI(Y X)$	$SI_{ST}(Y X)$	$RTI_{LR}^0(X Y)$	$LTD_{LR}^\infty(X Y)$
3'	$SI(X Y)$	$SI_{ST}(X Y)$	$RTI_{LR}^0(Y X)$	$LTD_{LR}^\infty(Y X)$
4	$LTD(Y X)$	$LTD_{ST}(Y X)$	$LTD_{RHR}^\infty(X Y)$	
4'	$LTD(X Y)$	$LTD_{ST}(X Y)$	$LTD_{RHR}^\infty(Y X)$	
5	$RCSI$	$RTI_{HR}(Y X)$	$RTI_{HR}(X Y)$	
6	$LCSD$	$LTD_{RHR}(Y X)$	$LTD_{RHR}(X Y)$	
7	$SIRL(Y X)$	$SI_{HR}(Y X)$	$RTI_{LR}(X Y)$	
7'	$SIRL(X Y)$	$SI_{HR}(X Y)$	$RTI_{LR}(Y X)$	
8		$SI_{RHR}(Y X)$	$LTD_{LR}(X Y)$	
8'		$SI_{RHR}(X Y)$	$LTD_{LR}(Y X)$	
9	$PLRD$	$SI_{LR}(Y X)$	$SI_{LR}(X Y)$	

N	Name	New dependence notions
10	$RTI_{MRL}(Y X)$	$(Y X > s) \leq_{MRL} (Y X > t)$ for all $s < t$
10'	$RTI_{MRL}(X Y)$	$(X Y > s) \leq_{MRL} (X Y > t)$ for all $s < t$
11	$RTI_{MRL}^0(Y X)$	$Y \leq_{MRL} (Y X > t)$ for all $0 < t$
11'	$RTI_{MRL}^0(X Y)$	$X \leq_{MRL} (X Y > t)$ for all $0 < t$
12	$RTI_{ICX}(Y X)$	$(Y X > s) \leq_{ICX} (Y X > t)$ for all $s < t$
12'	$RTI_{ICX}(X Y)$	$(X Y > s) \leq_{ICX} (X Y > t)$ for all $s < t$
13	$RTI_{ICX}^0(Y X)$	$Y \leq_{ICX} (Y X > t)$ for all $0 < t$
13'	$RTI_{ICX}^0(X Y)$	$X \leq_{ICX} (X Y > t)$ for all $0 < t$
14	$LTD_{MIT}(Y X)$	$(Y X \leq s) \leq_{MIT} (Y X \leq t)$ for all $s < t$
14'	$LTD_{MIT}(X Y)$	$(X Y \leq s) \leq_{MIT} (X Y \leq t)$ for all $s < t$
15	$LTD_{MIT}^\infty(Y X)$	$(Y X \leq s) \leq_{MIT} Y$ for all s
15'	$LTD_{MIT}^\infty(X Y)$	$(X Y \leq s) \leq_{MIT} X$ for all s
16	$LTD_{ICV}(Y X)$	$(Y X \leq s) \leq_{ICV} (Y X \leq t)$ for all $s < t$
16'	$LTD_{ICV}(X Y)$	$(X Y \leq s) \leq_{ICV} (X Y \leq t)$ for all $s < t$
17	$LTD_{ICV}^\infty(Y X)$	$(Y X \leq s) \leq_{ICV} Y$ for all s
17'	$LTD_{ICV}^\infty(X Y)$	$(X Y \leq s) \leq_{ICV} X$ for all s

- Simétrica XY
- Simétrica Reversa

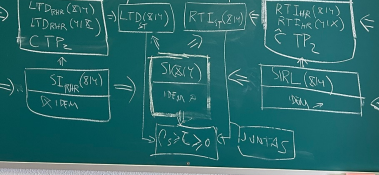
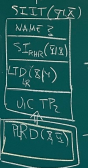
NO BARRER

$$\begin{aligned}
 \text{MIN} &\leq \text{LC} < \text{MID} < \text{TR} \\
 \text{FCV} &\leq \text{ST} < \text{CR} < \text{RR} \\
 \text{MIT} &\leq \text{RR} < \text{LR}
 \end{aligned}$$

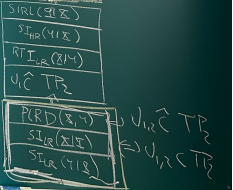
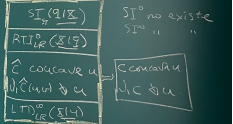
$$\begin{aligned}
 \text{ST} &\leq \text{FCV} < \text{ST} \\
 \text{ST} &\leq \text{FCV} < \text{ST}
 \end{aligned}$$

FALTAN

$$\begin{aligned}
 \hat{F}_M &\Rightarrow \\
 \hat{F}_E &\Downarrow \\
 \hat{F}_S &\leq \hat{F}_M & \hat{F}_S &\leq \hat{F}_E \\
 & & \text{ST} &
 \end{aligned}$$



- $RTI_{ORD}^0(8|4) \Leftrightarrow \exists s \leq 8|4 > 5 \quad \forall s$
- $LTD_{OBY}^{\infty}(8|9) \Leftrightarrow \exists \sum_{OBY} \sum 8|9 < 5 \quad \forall s$
- $RTI_{ORD}(8|4) \Leftrightarrow \sum 8|9 > 5, \leq 8|9 > 5_2$
- $LTD_{ORD}(8|4) \Leftrightarrow 8|9 < 9 \leq \sum 8|9 < 5_2$
- $SI_{ORD}(8|7) \Leftrightarrow 8|7 = 5_1 \Rightarrow 8|4 = 5_2$



SI no existe
SI " "

Concave u
u < u

$\hat{V}_{1,2} \hat{C} TP_2$
 $\Leftrightarrow \hat{V}_{1,2} \hat{C} TP_2$



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- ▶ They can also be related with properties of coherent systems (see Navarro, Durante and Fernández-Sánchez (2021) and Navarro (2022)).

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- ▶ Questions?