Preservation of aging classes in coherent systems

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References

The conference is based on the following references:

- M. Alimohammadi, J. Navarro (2023). Resolving an old problem on the preservation of the IFR property under the formation of k-out-of-n systems with discrete distributions. To appear in Journal of Applied Probability.
- J. Navarro (2018). Preservation of DMRL and IMRL aging classes under the formation of order statistics and coherent systems. Statistics and Probability Letters 137, 264–268.
- J. Navarro (2022). Introduction to System Reliability Theory. Springer. Chapter 4.

Basic references on reliability and coherent systems

- Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.

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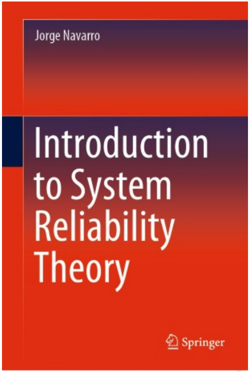
- Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.
- My new book:



Outline

Preliminary results

Notation

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- $m_X(t) = E(X - t|X > t)$ and $m_Y(t) = E(Y - t|Y > t)$ mean residual life functions (MRL).

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- Mean residual life order: $X \leq_{MRL} Y \Leftrightarrow m_X \leq m_Y$.
- Llikelihood ratio order: $X \leq_{LR} Y \Leftrightarrow f_Y/f_X$ increases.
- Relationships:

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ & & \Downarrow & & \Downarrow \\ & & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

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- X is New Better (Worse) than Used in Expectations, NBUE (NWUE), if $E(X) = m(0) \geq m_X(t)$ $(\leq)$ for all $t \geq 0$.

Relationships

- Relationships between natural (or positive) aging classes:

$$\begin{array}{ccccc} ILR & \Rightarrow & IFR & \Rightarrow & NBU \\ & & \Downarrow & & \Downarrow \\ & & DMRL & \Rightarrow & NBUE \end{array}$$

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- A system ϕ is **coherent** if ϕ is increasing and all the components are relevant that is ϕ is not constant in any variable.

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- If $x_{1:n} \leq \dots \leq x_{n:n}$ are the ordered values obtained from $x_1 \dots, x_n$

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- The series system $x_{1:n}$ is a n-out-of-n system.
- The parallel system $x_{n:n}$ is a 1-out-of-n system.

Distortion representations

- If $X_1, \ldots, X_n$ are the lifetimes of the components, then the lifetime of the system T can be written as

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- From any random vector $(X_1,\dots,X_n)$, the reliability function of T can be written as

$$\bar{F}_T(t)=\bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t)) \text{ for all } t\in\mathbb{R}, \tag{1.1}$$

where $\bar{F}_i(t)=\Pr(X_i>t)$ and $\bar{Q}:[0,1]^n\to[0,1]$ is a distortion (or aggregation) function, i.e., $\bar{Q}$ is continuous, increasing and satisfies $\bar{Q}(0,\dots,0)=0$ and $\bar{Q}(1,\dots,1)=1$, see Navarro (2022), p. 54.

Distortion representations, particular cases.

- If the components are identically distributed (ID), that is, $\bar{F}_i = \bar{F}$ for all i, then

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)), \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

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- If the components are independent (IND), then $\bar{Q}$ is a multinomial.
- If the components are IID, then $\bar{q}$ is a polynomial.

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- IID case:

$$\bar{F}_{1:n}(t) = \bar{F}^n(t) = \bar{q}(\bar{F}(t))$$

with $\bar{q}(u) = u^n$ for $u \in [0, 1]$.

Preservation of aging classes

Preservation of aging classes in systems

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- However this is not always the case.
- These preservation properties might depend on the structure of the system and on the dependence between the components (the copula).

Systems with ID components

- Recall that if $\bar{F}_i = \bar{F}$ for all i, then $\bar{F}_T = \bar{q}(\bar{F})$ where $\bar{q} : [0,1] \to [0,1]$ is a distortion function.

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- The NBU class is preserved iff $\bar{q}$ is submultiplicative, that is,

$$\bar{q}(uv) \leq \bar{q}(u)\bar{q}(v) \text{ for all } u, v \in [0,1].$$

Systems with ID components

- Recall that if $\bar{F}_i = \bar{F}$ for all i, then $\bar{F}_T = \bar{q}(\bar{F})$ where $\bar{q} : [0,1] \to [0,1]$ is a distortion function.
- Let $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ be the elasticity function associated to $\bar{q}$. Then we have the following properties:
- The IFR class is preserved iff α is decreasing.
- The DFR class is preserved iff α is increasing.
- The IFR and DFR classes are preserved iff $\bar{q}(u) = u^c$ for a $c > 0$.
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- If the IFR class is preserved, then so is the NBU class.

Examples

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- The same happen for the 2-out-of-3 system $X_{2:3}$.

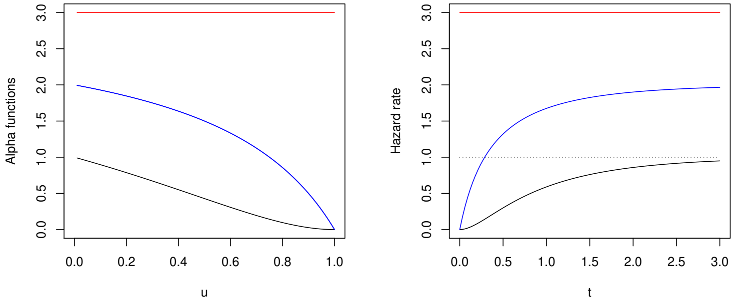

Figure: Alpha functions (left) and hazard rate functions (right) of k-out-of-3 systems for $k = 1, 2, 3$ (black, blue, red) with IID components having a common standard exponential distribution. The dotted line represents the hazard rate of the components.

Examples

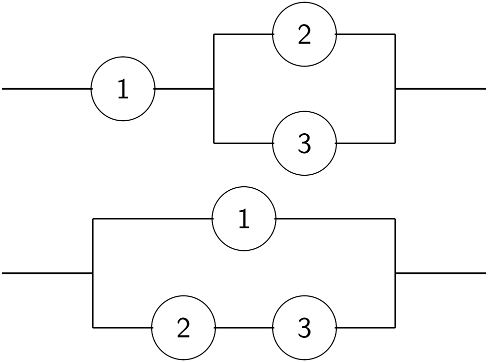

Figure: Two coherent systems of order 3.

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- The DFR class is not preserved in these systems.

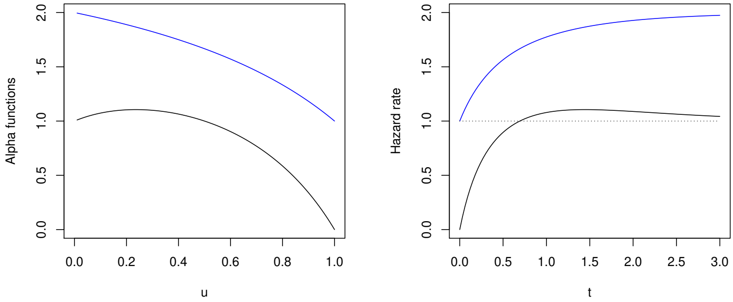

Figure: Alpha functions (left) and hazard rate functions (right) for the systems T_1 (blue) and T_2 (black) with IID components having a common standard exponential distribution. The dotted line represents the hazard rate of the components.

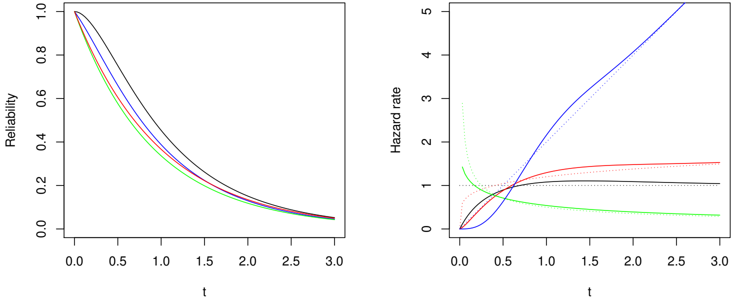

Figure: Reliability functions (left) for the system T_2 with IID components having a common standard exponential distribution when new (black line) and with ages $t = 0.2, 1.444, 5$ (blue, green, red). Hazard rate functions (right) for T_2 when the components have Weibull distributions with shape parameter $\beta = 0.5, 1, 1.2, 2$ (green, black, red, blue). The dotted lines represent the hazard rate of the components.

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- Consider now the series and parallel systems with dependent components (X_1, X_2) having standard exponential distributions and the Clayton-Oakes survival copula

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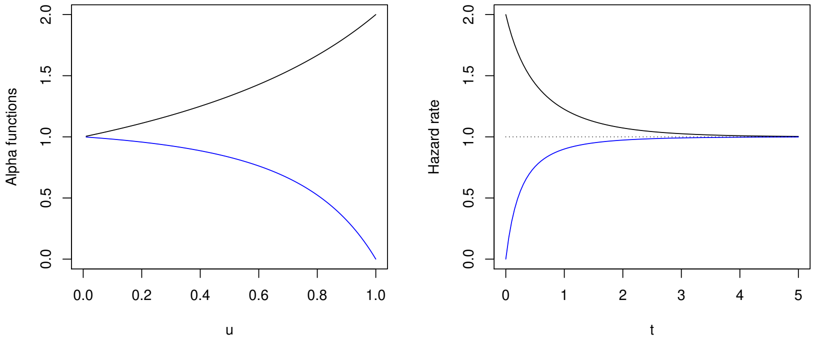
- Therefore, the IFR class is preserved in $X_{2:2}$ but it is not preserved in $X_{1:2}$. We have the opposite for the DFR class.



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Preservation of DMRL/IMRL classes

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- This result was extended in Navarro (2018) with the following:

Theorem

(i) *The DMRL class is preserved if*

$$\sup_{u\in(0,v]} \frac{\bar{q}(u)}{u} \leq \frac{\bar{q}^2(v)}{v^2\bar{q}'(v)} \text{ for all } v \in (0,1). \tag{2.1}$$

(ii) *The IMRL class is preserved if*

$$\inf_{u\in(0,v]} \frac{\bar{q}(u)}{u} \geq \frac{\bar{q}^2(v)}{v^2\bar{q}'(v)} \text{ for all } v \in (0,1). \tag{2.2}$$

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- Therefore (2.1) holds and the DMRL class is preserved.

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- That is, in the IID case, $\bar{q}$ is always submultiplicative.

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Let X be a discrete random variable with ordered support $\{x_i\}_{i\in I}$. Let $X_1,\dots,X_n$ be IID random variables with the same distribution of X. If X is IFR, then $X_{k:n}$ is IFR for $k=1,\dots,n$.

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- Some extensions to the general case are obtained as well.

Further results

- For preservation properties based on the signature of the systems (IID or EXC cases) see Samaniego (1985); Rychlik and Szymkowiak (2021) and the references therein.

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- The NBU is preserved if $\bar{Q}$ is submiultiplicative, that is,

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- The NBU property is preserved if the components are independent, see Esary, Marshall and Proschan (1970) or Navarro (2022), p. 131.

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- Questions?