Preservation of aging classes under the formation of coherent systems

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References

The talk is based on the following references:

- J. Navarro, T. Rychlik, M. Szymkowiak (2025). Key distributions in the preservation of aging classes under the construction of systems. Journal of Computational and Applied Mathematics Volume 469, 1 December 2025, 116650.
- M. Alimohammadi, J. Navarro (2024). Resolving an old problem on the preservation of the IFR property under the formation of k-out-of-n systems with discrete distributions. Journal of Applied Probability 61, 644?653.
- A. Arriaza, J. Navarro, P. Ortega-Jiménez (2025). Risk times in mission-oriented systems. Submitted.

Outline

Preliminary results

Notation

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- $\bar{h}_X = f_X/F_X$ and $\bar{h}_Y = f_Y/F_Y$ reversed hazard (or failure) rate (RHR) functions.

Main stochastic orders

- Stochastic order (or first-order stochastic dominance): $X \leq_{ST} Y \Leftrightarrow \bar{F}_X \leq \bar{F}_Y$ or, equivalently, $E(\phi(X)) \leq E(\phi(X))$ for all increasing functions ϕ.

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- Superadditive order, $X \leq_{SU} Y$, if $\phi(t) = F_Y^{-1}(F_X(t))$ is superadditive, that is, $\phi(t+s) \geq \phi(t)\phi(s)$ in the support of X.

Main stochastic orders

- Relationships:

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & X \leq_{ICX} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{MIT} Y & \Rightarrow & X \leq_{ICV} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

and

$$X \leq_{C} Y \Rightarrow X \leq_{*} Y \Rightarrow X \leq_{SU} Y.$$

Distortions

- X has a distorted distribution from a CDF F if $F_X = q(F)$ where $q : [0,1] \to [0,1]$ is a **distortion function**, that is, it is a continuous and increasing function such that $q(0) = 0$ and $q(1) = 1$.

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- Then the respective survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F} = 1 - F$ satisfy $\bar{F}_X = \bar{q}(\bar{F})$, where $\bar{q}(u) = 1 - q(1-u)$ is another distortion function called **dual distortion**.

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- If $F_X = q_1(F)$ and $F_Y = q_2(F)$, then following ordering properties hold:

$$X \le_{ST} Y \text{ for all } F \Leftrightarrow \bar{q}_1 \le \bar{q}_2,$$

$$X \le_{HR} Y \text{ for all } F \Leftrightarrow \bar{q}_2/\bar{q}_1 \text{ decreases in } (0,1)$$

$$X \le_{RHR} Y \text{ for all } F \Leftrightarrow q_2/q_1 \text{ increases in } (0,1),$$

$$X \le_{LR} Y \text{ for all } F \Leftrightarrow \bar{q}_2'/\bar{q}_1' \text{ decreases in } (0,1).$$

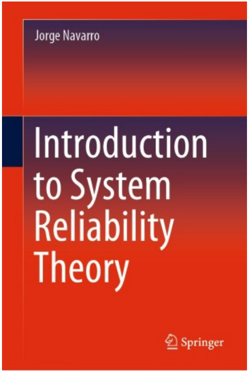

Figure: Publicity of my book on System Reliability Theory.

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- If $X_1,\dots,X_n$ are identically distributed ID with a common reliability function $\bar{F}$, then the reliability function of the system can be written as $\bar{F}_T=\bar{q}(\bar{F})$ where $\bar{q}$ depends on ϕ and on the copula of $(X_1,\dots,X_n)$.

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- The preceding ordering results can be used to compare systems with ID components, see Navarro (2022).
- In this way we obtain distributions-free comparisons results (they hold for all $\bar{F}$).
- The case of non-ID components is also studied in Navarro (2022).

Aging classes

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- X has an New Better (Worse) than Used, NBU (NWU) distribution if $X \geq_{ST} X_t$ for all $t \geq 0$, that is,

$$\bar{F}(x)\bar{F}(t) - \bar{F}(x+t) \geq 0 \ (\leq) \quad \text{for all } x, t \geq 0.$$

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- X has an Increasing (Decreasing) Density, ID (DD), if $\bar{F}$ is concave (convex) in its support.
- If X has an absolutely continuous distribution, it is ID (DD) if the PDF f is increasing (decreasing) for all $t \in (\ell, u)$ and $f(t) = 0$ elsewhere.

Aging classes

- The relationships between these classes are as follows:

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- The uniform model is a member of the ensuing classes: IFR, NBU, IFRA, DD and ID.
- Note that these key distributions belong, in some cases, to both positive and negative aging classes. In that sense, they are the extreme elements of the above classes.

Preservation results

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- A class C is preserved by a system T is $\bar{F}_T \in C$ for all $\bar{F} \in C$, where $\bar{F}$ is the common reliability function of the components.

Preservation results IFR/DFR

Theorem

Let T be the lifetime of a system with homogeneous components and distortion function $\bar{q}$. Then the ensuing requirements are equivalent:

(i) *T preserves the IFR (DFR) class.*

(ii) *T preserves the IFR (DFR) class for the standard exponential distribution.*

(iii) *The following condition holds*

$$\bar{q}(zu)\bar{q}(v) \leq \bar{q}(zv)\bar{q}(u) \ (\geq) \ 0 \leq u \leq v \leq 1, \ 0 \leq z \leq 1.$$

(iv) *The function $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ is decreasing (increasing).*

(v) *$T_Z \leq_C Z$ $(\geq_{CX})$, where $Z \sim Exp(\mu = 1)$.*

Preservation results IFRA/DFRA

Theorem

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(iii) *The following condition holds*

$$\bar{q}^c(u) \leq \bar{q}(u^c)\ (\geq)\ 0 \leq u \leq 1,\ 0 \leq c \leq 1.$$

(iv) *The function $\beta(u) = \log \bar{q}(u)/u$ is decreasing (increasing).*

(v) *$T_Z \leq_* Z$ $(\geq_*)$, where $Z \sim Exp(\mu = 1)$.*

Preservation results NBU/NWU

Theorem

Let T be the lifetime of a system with homogeneous components and distortion function $\bar{q}$. Then the ensuing requirements are equivalent:

(i) *T preserves the NBU (NWU) class.*

(ii) *T preserves the NBU (NWU) class for the standard exponential distribution.*

(iii) *The function*

$$\gamma(u,v) = \bar{q}(uv) - \bar{q}(u)\bar{q}(v) \leq 0 \ (\geq)\ 0 \leq u \leq 1,\ 0 \leq v \leq 1.$$

(iv) *$T_Z \leq_{SU} Z$ ($\geq_{SU}$), where $Z \sim Exp(\mu = 1)$.*

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- These facts follow from the relations *IFR* $\subseteq$ *IFRA* $\subseteq$ *NBU*.
- Also, if γ is not nonpositive, then α and β are not decreasing.
- Similar properties holds for the reversed classes DFR, DFRA, and NWU.

Preservation results DD/ID

Theorem

Let T be the lifetime of a system with homogeneous components and distortion function $\bar{q}$. Then the ensuing requirements are equivalent:

(i) *T preserves the DD (ID) class.*

(ii) *T preserves the DD (ID) class for the standard uniform distribution.*

(iii) *The function $\bar{q}$ is concave (convex) in $(0,1)$.*

(iv) *$T_U \leq_{cx} U$ $(\geq_{cx})$.*

Examples

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- The IFRA and NBU property is preserved in coherent systems with IID components, Esary, Marshall and Proschan (1970).

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- Then IFRA and NBU are preserved but the DFR property is not preserved in k-out-of-n systems with IID components except in the case of series systems where α is constant, Navarro (2022).
- The IFRA and NBU property is preserved in coherent systems with IID components, Esary, Marshall and Proschan (1970).
- The IFR property is preserved in all coherent systems with IID components.

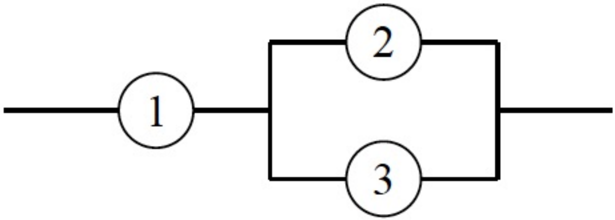

Figure: A coherent system with structure function $T_1 = \min(X_1, \max(X_2, X_3))$.

Examples-IID case

- If the components of $T_1 = \min(X_1, \max(X_2, X_3)$ are IID, the dual distortion function is

$$\bar{q}_1(u) = 2u^2 - u^3.$$

Examples-IID case

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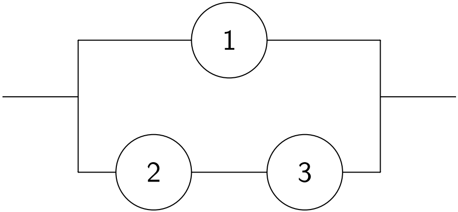

Figure: A coherent system with structure function $T_2 = \max(X_1, \min(X_2, X_3))$.

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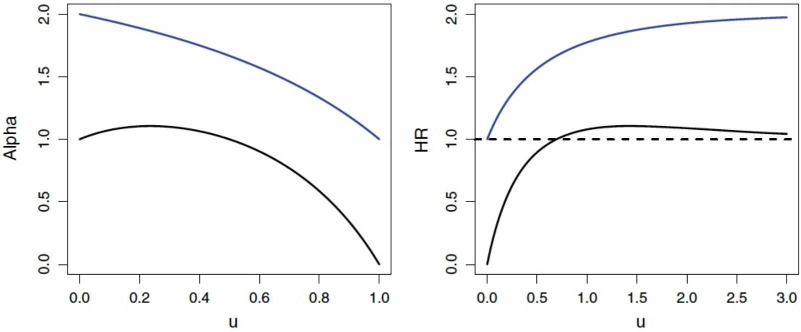

Figure: Alpha (left) and hazard rate (right) functions for the systems T_1 (blue) and T_2 (black) with three IID components with standard exponential distributions.

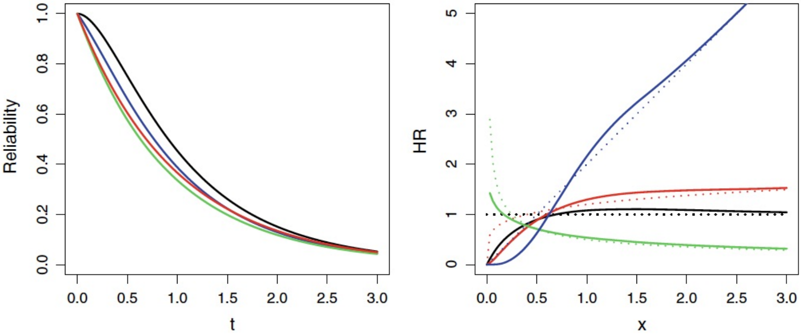

Figure: Reliability functions (left) for the system T_2 with IID components having a standard exponential distribution when new (black line) and with ages $t = 0.2, 1.444, 5$ (blue, green, red). Hazard rate functions (right) for T_2 when the components have Weibull distributions with shape parameter $0.5, 1, 1.2, 2$ (green, black, red, blue). The dotted lines represent the hazard rate functions of the components.

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- Now we assume that the components are DID with the following M-O survival copula

$$\widehat{C}(u,v,w) = u^{1-a}v^{1-b}w^{1-c}\min(u^a, v^b, w^c)$$

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- For $1 > a > b = 0.8 > c = 0.1$ we get the following alpha function:

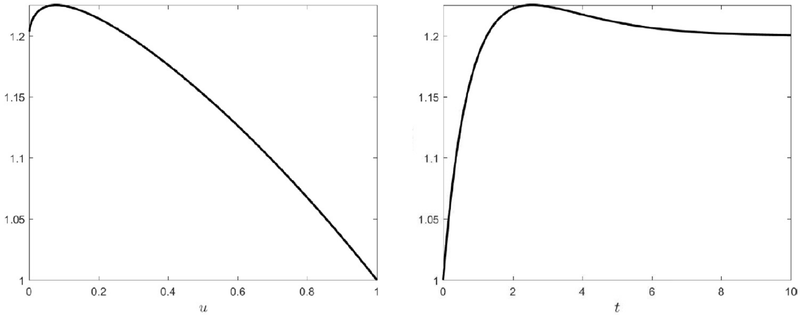

Figure: Alpha (left) and hazard rate (right) functions for T_1 under a M-O copula with $1 > a > b = 0.8 > c = 0.1$ and standard exponential components.

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Comments

- The limit behaviour of the reliability and hazard rate function can be determined from the dual distortion function, see Navarro and Sarabia (2024).

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- Preservation results for heterogeneous components based on generalized distortions can be seen in Navarro (2022).

Preservation of the IFR property in discrete distributions

Properties included in:

- M. Alimohammadi, J. Navarro (2024). Resolving an old problem on the preservation of the IFR property under the formation of k-out-of-n systems with discrete distributions. Journal of Applied Probability 61, 644–653.

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- The preservation of the IFR property under the formation of k-out-of-n systems has been an open question since it was proved for the continuous case in Esary and Proschan (1963).

Preservation of the IFR property in discrete distributions

Theorem

Let $X_1, \ldots, X_n$ be independent random variables with the same discrete distribution with ordered support. If X_1 is IFR, then $X_{k:n}$ is IFR for $k = 1, \ldots, n$.

Preservation of the IFR property in discrete distributions

- The proof is based on the distortion representation $\bar{F}_{k:n}(t) = \bar{q}(\bar{F}(t))$ with

$$\bar{q}(u) = k\binom{n}{k}\int_0^u x^{n-k}(1-x)^{k-1}dx$$

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where α is the function used in the first theorem.
- So, from the classic result, we know that α is decreasing for k-out-of-n systems.
- We can use this technique to extend the first theorem to systems with discrete ID components as follows:

Preservation of the IFR property in discrete distributions

Theorem

Let $X_1, \ldots, X_n$ be independent random variables with the same discrete distribution with ordered support and let T be the lifetime of a coherent system based on these components with dual distortion $\bar{q}$. Then the following conditions are equivalent:

(i) *T preserves the IFR class.*

(ii) *The function $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ is decreasing in $(0, 1)$.*

Comments

- The IFR class can be defined int he general case with the property $X_s \geq_{ST} X_t$ for all $s \leq t$, that is,

$$\bar{F}(x+s)\bar{F}(t) - \bar{F}(x+t)\bar{F}(s) \geq 0 \quad \text{for all } s \leq t,\ x \geq 0.$$

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- The definition cannot be extended to the DFR class in discrete times.
- Note that the preceding theorem can be applied to systems with dependent components.
- The IFR property is not preserved in k-out-of-n systems when the components are dependent.

Results included in:

- A. Arriaza, J. Navarro, P. Ortega-Jiménez (2025). Risk times in mission-oriented systems. Submitted.

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- The results are base on distortion representations and on the PELCoV concept introduced in Ortega-Jimeénez et al. (2022) in the context of Risk Theory.

References

Esary, J.D., Marshall, A.W. and Proschan, F. (1970). Some reliability applications of the hazard transform. SIAM Journal on Applied Mathematics 18, 849–860.

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Navarro J., Sarabia J.M. (2024). A note on the limiting behaviour of hazard rate functions of generalized mixtures. Journal of Computational and Applied Mathematics 435, 114653.

Ortega-Jiménez P., Pellerey F., Sordo M.A. Suárez-Llorens, A. (2022). Probability equivalent level for Covar and VAR. Insurance: Mathematics and Economics 115, 22–35.

Final slide

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- Questions?