Predicting record values by using bivariate distortions

Jorge Navarro[1]
Universidad de Murcia, Murcia, Spain.

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References

The conference is based on the following references:

- Navarro J, Calì C, Longobardi M, Durante F. Distortion representations of multivariate distributions. To appear in *Statistical Methods & Applications.* Published online first Jan. 2022. DOI: 10.1007/s10260-021-00613-2.
- Navarro J. Prediction of record values by using quantile regression curves and distortion functions. To appear in *Metrika*. Published online first Nov. 2021. DOI: 10.1007/s00184-021-00847-w.

Outline

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- Univariate distorted distributions
- Multivariate distorted distributions
- Main properties

Record values
- Representations
- Predictions
- Examples

Distorted distributions

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Definition

The **distorted distribution** (DD) associated to a distribution function (DF) F and to an increasing continuous *distortion function* $q : [0, 1] \to [0, 1]$ such that $q(0) = 0$ and $q(1) = 1$, is given by

$$F_q(t) = q(F(t)), \text{ for all } t \in \mathbb{R}. \qquad (1.1)$$

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where $\bar{q}(u) := 1 - q(1 - u)$ is called the *dual distortion function*.

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  where $\bar{q}(u) := 1 - q(1 - u)$ is called the *dual distortion function*.
- (1.1) and (1.2) are equivalent.

Examples of distorted distributions.

- Proportional Hazard Rate (PHR) Cox model $\bar{F}_\theta(t) = \bar{F}^\theta(t)$, where $\bar{q}(u) = u^\theta$ and $\theta > 0$.

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- Proportional Reversed Hazard Rate (PRHR) model $F_\theta(t) = F^\theta(t)$, where $q(u) = u^\theta$ and $\theta > 0$.
- Order statistics $X_{1:n}, \ldots X_{n:n}$. Then

$$\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t) = \bar{q}_{i:n}(\bar{F}(t)),$$

where $\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$.

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where $\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$.
- Coherent system lifetimes T:

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)), \tag{1.3}$$

where $\bar{Q} : [0,1]^n \to [0,1]$ is a generalized distortion function, see e.g. Navarro (2022).

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where $F_1, \ldots, F_n$ are the marginals.
- A similar representation holds for the joint survival function

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n).$$

Definition

Definition (Navarro, Calì, Longobardi and Durante (2022))

A multivariate distribution function $\mathbf{F}$ is said to be a *multivariate distorted distribution* (MDD) of the univariate distribution functions $G_1, \ldots, G_n$ if there exists a *distortion* function D such that

$$\mathbf{F}(x_1, \ldots, x_n) = D(G_1(x_1), \ldots, G_n(x_n)), \; \forall x_1, \ldots, x_n \in \mathbb{R}. \quad (1.4)$$

We write $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$, when $\mathbf{F}$ is a MDD of $G_1, \ldots, G_n$.

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A continuous function $D : [0,1]^n \to [0,1]$ is called *(n–dimensional) distortion function* (shortly written as $D \in \mathcal{D}_n$) if:

(i) $D(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n) = 0$ for all $u_1, \dots, u_n \in [0,1]$.

(ii) $D(1, \dots, 1) = 1$.

(iii) D is n–increasing, i.e. for all $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ with $x_i \le y_i$, it holds $\triangle_{\mathbf{x}}^{\mathbf{y}} D \ge 0$, where

$$\triangle_{(x_1,\dots,x_n)}^{(y_1,\dots,y_n)} D := \sum_{z_i \in \{x_i, y_i\}} (-1)^{\mathbf{1}(z_1,\dots,z_n)} D(z_1, \dots, z_n),$$

with $\mathbf{1}(z_1, \dots, z_n) = \sum_{i=1}^n 1(z_i = x_i)$ and $1(A) = 1$ (respectively, 0) if A is true (respectively, false).

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is a multivariate distribution function for all DF $G_1, \ldots, G_n$.

- If $\mathbf{F} \equiv MDD(G_1, \ldots, G_n)$, then

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = \widehat{D}(\bar{G}_1(x_1), \ldots, \bar{G}_n(x_n)), \qquad (1.5)$$

where $\bar{G}_i = 1 - G_i$ and $\widehat{D} \in \mathcal{D}_n$.

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$$F_i(x_i) = D(1, \dots, 1, G_i(x_i), 1, \dots, 1) = D_i(G_i(x_i)), \quad (1.6)$$

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where $D_i(u) := D(1, \ldots, 1, u, 1, \ldots, 1)$ and the value u is placed at the ith position.
- Clearly, we have $G_i = F_i$ for a fixed $i \in \{1, \ldots, n\}$ when $D_i(u) = u$ for all $u \in [0, 1]$.

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Proposition

Let (X_1, X_2) *with* $\mathbf{F} \equiv MDD(G_1, G_2)$ *for* $D \in \mathcal{D}_2$*, then*

$$F_{2|1}(x_2|x_1) = D_{2|1}(G_2(x_2)|G_1(x_1)) \tag{1.7}$$

whenever $\lim_{v \to 0^+} \partial_1 D(G_1(x_1), v) = 0$*, where*

$$D_{2|1}(v|G_1(x_1)) = \frac{\partial_1 D(G_1(x_1), v)}{\partial_1 D(G_1(x_1), 1)}$$

for $0 < v < 1$ *and* x_1 *such that* $\partial_1 D(G_1(x_1), 1) > 0$.

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- Another option to predict X_2 from X_1 is the *conditional median regression curve* $m_{2|1}(x_1) := F_{2|1}^{-1}(0.5|x_1)$ (see Koenker (2005) or Nelsen (2006), p. 217).

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- This quantile function $F^{-1}_{2|1}$ can be computed from (1.7) as

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- Moreover, it can be used to obtain α-prediction bands for X_2

$$\left[F^{-1}_{2|1}(\beta_1|x_1), F^{-1}_{2|1}(\beta_2|x_1)\right]$$

taking $0 \le \beta_1 < \beta_2 \le 1$ such that $\beta_2 - \beta_1 = \alpha \in (0,1)$.

Examples of MDD

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- **Coherent systems** with ID components.
- **Order statistics** (k-out-of-n systems).
- Other examples: Sequential order statistics, record values, convolutions, ...

Record values

Univariate representations

- Let us consider the upper record values $R_1, R_2, \ldots$ from a sequence of IID r.v. $X_1, X_2, \ldots$ with a common abs. cont. distribution function F and $\bar{F} = 1 - F$.

Univariate representations

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- It is well known (see, e.g., Nevzorov, 2001, p. 65) that the survival function $\bar{G}_n(t) = \Pr(R_n > t)$ of R_n is given by

$$\bar{G}_n(t) = \bar{F}(t) \sum_{k=0}^{n-1} \frac{(-\log(\bar{F}(t)))^k}{k!} = \bar{q}_n(\bar{F}(t)), \qquad (2.1)$$

where

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- The function $\bar{q}_n$ is a distortion function.

Multivariate representations

Proposition

The joint survival function $\overline{\mathbf{G}}$ *of* $(R_1, \ldots, R_n)$ *can be written as*

$$\overline{\mathbf{G}}(x_1, \ldots, x_n) = \hat{D}(\bar{F}(x_1), \ldots, \bar{F}(x_n)) \tag{2.2}$$

for a continuous distortion function $\hat{D} : [0, 1]^n \to [0, 1]$. *The probability density function* $\hat{d} = \partial_{1,\ldots,n} \hat{D}$ *of* $\hat{D}$ *is given by*

$$\hat{d}(u_1, \ldots, u_n) = \frac{1}{u_1 \ldots u_{n-1}} \tag{2.3}$$

for $1 > u_1 > \cdots > u_n > 0$ *(zero elsewhere).*

Bivariate representation

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$$\overline{\mathbf{G}}_{i,j}(x_i, x_j) = \hat{D}_{i,j}(\bar{F}(x_i), \bar{F}(x_j)), \tag{2.4}$$

where $\hat{D}_{i,j}(u, v) = \hat{D}(1, \ldots, 1, u, 1, \ldots, 1, v, 1, \ldots, 1)$ and u and v are placed at the i-th and j-th variables, respectively.

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- This expression can be used to predict R_j from R_i for $i < j$ and to get prediction bands for this prediction. The result can be stated as follows.

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for $x_j \geq x_i$ *whenever* $f(x_i) > 0$, $0 < \bar{F}(x_i) < 1$ *and* $\lim_{v \to 0^+} \partial_1 \widehat{D}_{i,j}(u, v) = 0$ *for all* $0 < u < 1$.

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- Hence, the *median regression curve* to predict R_j from R_i is

$$m_{j|i}(x_i) = \overline{\mathbf{G}}_{j|i}^{-1}(0.5|x_i), \tag{2.6}$$

where $\overline{\mathbf{G}}_{j|i}^{-1}$ is the inverse function of $\overline{\mathbf{G}}_{j|i}$.

Case $i = 1$ and $j = 2$

- The joint survival function $\overline{\mathbf{G}}_{1,2}$ of (R_1, R_2) can be written as

$$\overline{\mathbf{G}}_{1,2}(x_1, x_2) = \bar{F}(x_2) + \bar{F}(x_2)\log\frac{\bar{F}(x_1)}{\bar{F}(x_1)} = \widehat{D}_{1,2}(\bar{F}(x_1), \bar{F}(x_2)) \quad (2.7)$$

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- Actually $\widehat{D}_{1,2}(1, v) = \bar{q}_2(v)$, that is, it is the dual distortion function of the second upper record given in (2.1).
- $\bar{F}$ is equal to the first marginal survival function since $\widehat{D}_{1,2}(u, 1) = u$, but it is not equal to the second one.

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$$\overline{\mathbf{G}}_{1,2}(x_1, x_2) = \bar{F}(x_2) + \bar{F}(x_2) \log \frac{\bar{F}(x_1)}{\bar{F}(x_1)} = \widehat{D}_{1,2}(\bar{F}(x_1), \bar{F}(x_2)) \quad (2.7)$$

for $x_1 \leq x_2$, where

$$\widehat{D}_{1,2}(u, v) = v + v \log \frac{u}{v}; \ 1 > u \geq v > 0.$$

- $\widehat{D}_{1,2}$ is not a copula since $\widehat{D}_{1,2}(1, v) = v - v \log v \neq v$.
- Actually $\widehat{D}_{1,2}(1, v) = \bar{q}_2(v)$, that is, it is the dual distortion function of the second upper record given in (2.1).
- $\bar{F}$ is equal to the first marginal survival function since $\widehat{D}_{1,2}(u, 1) = u$, but it is not equal to the second one.
- To get the copula representation of (R_1, R_2) we need the inverse of the distribution function of R_2.

Case $i = 1$ and $j = 2$

- Then, the median regression curve to predict R_2 from R_1 is

$$m_{2|1}(x_1) = \overline{\mathbf{G}}_{2|1}^{-1}(0.5|x_1) = \bar{F}^{-1}(0.5\overline{F}(x_1)), \qquad (2.8)$$

where $\bar{F}^{-1}$ is the inverse function of $\bar{F}$.

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- Analogously, the centered 50% and 90% quantile prediction bands for R_2 are given by

$$[\overline{\mathbf{G}}_{2|1}^{-1}(0.75|x_1), \overline{\mathbf{G}}_{2|1}^{-1}(0.25|x_1)] = [\bar{F}^{-1}(0.75\bar{F}(x_1)), \bar{F}^{-1}(0.25\bar{F}(x_1))]$$

  and

$$[\overline{\mathbf{G}}_{2|1}^{-1}(0.95|x_1), \overline{\mathbf{G}}_{2|1}^{-1}(0.05|x_1)] = [\bar{F}^{-1}(0.95\bar{F}(x_1)), \bar{F}^{-1}(0.05\bar{F}(x_1))].$$

Case $i = n$ and $j = n + 1$

Proposition

The joint survival function $\overline{\mathbf{G}}_{n,n+1}$ of (R_n, R_{n+1}) can be written as

$$\overline{\mathbf{G}}_{n,n+1}(x_n, x_{n+1}) = \widehat{D}_{n,n+1}(\bar{F}(x_n), \bar{F}(x_{n+1}))$$

for $x_n \le x_{n+1}$, where

$$\widehat{D}_{n,n+1}(u, v) = -\frac{1}{n!} v(-\log u)^n + \bar{\gamma}_{n+1}(-\log v),\ 1 > u \ge v > 0,$$

$$\bar{\gamma}_{n+1}(z) = \frac{1}{n!} \int_z^\infty x^n e^{-x} dx \tag{2.9}$$

is the survival function of a gamma distribution with scale parameter equal to one and shape parameter equal to $n + 1$.

Case $i = n$ and $j = n + 1$

- Therefore, we get

$$\partial_1 \widehat{D}_{n,n+1}(u,v) = \frac{1}{(n-1)!}(-\log u)^{n-1}\frac{v}{u}$$

for $1 > u \geq v > 0$ and the conditional survival function is

$$\overline{\mathbf{G}}_{n+1|n}(x_{n+1}|x_n) = (n-1)!\frac{\partial_1 \widehat{D}_{n,n+1}(\bar{F}(x_n), \bar{F}(x_{n+1}))}{(-\log \bar{F}(x_n))^{n-1}} = \frac{\bar{F}(x_{n+1})}{\bar{F}(x_n)}$$

for $x_{n+1} \geq x_n$ and x_n such that $\bar{F}(x_n) > 0$ and $f(x_n) > 0$.

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- This expression is also a very well known result (see, e.g., Nevzorov, 2001, p. 68).

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- This expression is also a very well known result (see, e.g., Nevzorov, 2001, p. 68).
- Even more, the record values form a Markov chain, that is,

$$\Pr(R_{n+1} > x_{n+1}|R_1 = x_1, \ldots, R_n = x_n) = \Pr(R_{n+1} > x_{n+1}|R_n = x_n)$$

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$$\Pr(R_{n+1} > x_{n+1}|R_1 = x_1, \dots, R_n = x_n) = \Pr(R_{n+1} > x_{n+1}|R_n = x_n)$$
- Therefore the median regression curve to predict R_{n+1} from R_n is the same as that in the preceding case.

Case $i = m$ and $j = n$

Proposition

The joint survival function $\overline{\mathbf{G}}_{m,n}$ *of* (R_m, R_n) *for* $1 \le m < n$ *can be written as*

$$\overline{\mathbf{G}}_{m,n}(x_m, x_n) = \widehat{D}_{m,n}(\bar{F}(x_m), \bar{F}(x_n))$$

for $x_n \le x_{n+1}$, *where*

$$\widehat{D}_{m,n}(u,v) = \bar{\gamma}_m(-\log v) + \frac{1}{(m-1)!}\int_{-\log u}^{-\log v} \frac{z^{m-1}}{e^z}\bar{\gamma}_{n-m}(-z-\log v)dz \tag{2.10}$$

for $1 > u \ge v > 0$ *and* $\bar{\gamma}_k$ *is the survival function in* (2.9).

Case $i = m$ and $j = n$

- From (2.10), we obtain

$$\partial_1 \widehat{D}_{m,n}(u,v) = \frac{1}{(m-1)!}(-\log u)^{m-1}\bar{\gamma}_{n-m}\left(-\log \frac{v}{u}\right)$$

for $1 > u > v > 0$ and, from (2.5),

$$\overline{\mathbf{G}}_{n|m}(x_n|x_m) = (m-1)!\frac{\partial_1 \widehat{D}_{m,n}(\bar{F}(x_m), \bar{F}(x_n))}{(-\log \bar{F}(x_m))^{m-1}} = \bar{\gamma}_{n-m}\left(-\log \frac{\bar{F}(x_n)}{\bar{F}(x_m)}\right)$$

for $x_n \geq x_m$ since $\lim_{v \to 0^+} \partial_1 \widehat{D}_{m,n}(u,v) = 0$ for all $0 < u < 1$.

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for $x_n \geq x_m$ since $\lim_{v\to 0^+} \partial_1 \widehat{D}_{m,n}(u,v) = 0$ for all $0 < u < 1$.

- The median regression curve to predict R_n from $R_m = x_m$ is

$$m_{n|m}(x_m) = \overline{\mathbf{G}}^{-1}_{n|m}(0.5|x_m) = \bar{F}^{-1}(c_{n-m}(0.5)\bar{F}(x_m)), \quad (2.11)$$

where $c_k(y) = \exp(-\gamma_k^{-1}(y))$ and γ_k^{-1} is the quantile function of a gamma distribution with shape parameter k and scale parameter equal to one.

Case $i = m$ and $j = n$

- Analogously, the 50% and 90% quantile prediction bands for R_n are

$$[\bar{F}^{-1}(c_{n-m}(0.25)\bar{F}(x_m)), \bar{F}^{-1}(c_{n-m}(0.75)\bar{F}(x_m))]$$

$$[\bar{F}^{-1}(c_{n-m}(0.05)\bar{F}(x_m)), \bar{F}^{-1}(c_{n-m}(0.95)\bar{F}(x_m))].$$

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- Hence, if $\bar{F}$ is known, then we have common prediction bands for the sequence of paired records $(R_1, R_{1+k}), (R_2, R_{2+k}), \dots$
- This is not the case if we estimate $\bar{F}$ (or a parameter in $\bar{F}$) at each R_m for $m = 1, 2, \dots$

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- They are plotted in the following figure jointly with $m_{n+1|n}$ (black) and the sequence of the first paired records (R_1, R_2), $\dots$, (R_8, R_9). The sequence obtained by simulation is

$$0.319, 0.784, 0.8729, 0.9018, 0.9504, 0.98365, 0.98411, 0.99982, 0.99996.$$

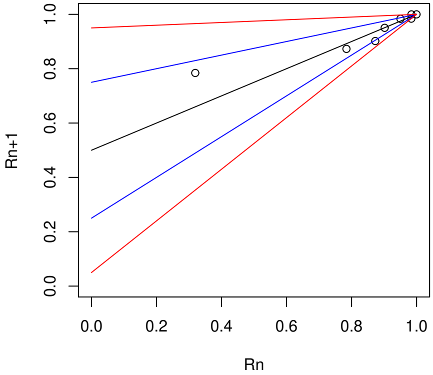

Figure: Plots of the paired records (R_n, R_{n+1}) from a standard uniform distribution jointly with the median regression curve (black) and the limits for the 50% (blue) and 90% (red) centered prediction bands.

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- The median regression curve to predict R_{n+k} from $R_n = x_n$ is

$$m_{n+k|n}(x_n) = \bar{F}^{-1}(c_k(0.5)\bar{F}(x_n)) = 1 - c_k(0.5) + c_k(0.5)x_n,$$

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- They are plotted in the following figure for $k = 2$.

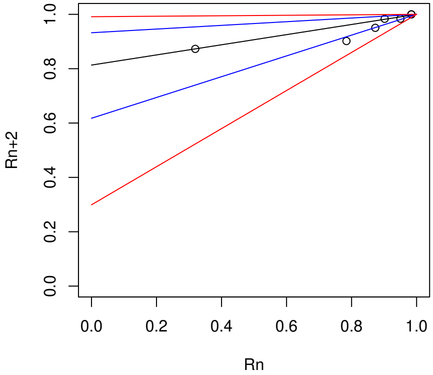

Figure: Plots of the paired records (R_n, R_{n+2}) from a standard uniform distribution jointly with the median regression curve (black) and the limits for the 50% (blue) and 90% (red) centered prediction bands.

The PHR model

- If F_θ has a known parametric form with an unknown parameter θ and $R_1 = x_1, \dots, R_n = x_n$ are known, then we can use

$$\ell(\theta) = h_\theta(x_1) \dots h_\theta(x_n) \bar{F}_\theta(x_n),$$

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- Hence, the MLE of θ is

$$\widehat{\theta}_n = -\frac{n}{\log \bar{F}(x_n)}. \tag{2.12}$$

The PHR model

- The exact median regression curve for the PHR model is

$$m_{n+1|n}(x_n) = \bar{F}_\theta^{-1}(0.5\bar{F}_\theta(x_n)) = \bar{F}^{-1}(0.5^{1/\theta}\bar{F}(x_n)).$$

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- If we replace θ with the MLE $\widehat{\theta}_n$ given in (2.12), we obtain the *estimated median regression curve* (EMRC) as

$$\widehat{m}_{n+1|n}(x_n) = \bar{F}^{-1}(0.5^{1/\widehat{\theta}_n}\bar{F}(x_n)) = \bar{F}^{-1}\left(\bar{F}(x_n)0.5^{-\frac{1}{n}\log \bar{F}(x_n)}\right).$$

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- The *estimated quantile prediction bands* (EQPB) are obtained in a similar way.

The PHR model

- If we want to predict R_{n+k} from R_n for $k>0$, the EMRC is

$$m_{n+k|n}(x_n)=\bar{F}^{-1}_{\widehat{\theta}_n}(c_k(0.5)\bar{F}_{\widehat{\theta}_n}(x_n))=\bar{F}^{-1}\left(\bar{F}(x_n)(c_k(0.5))^{-\frac{1}{n}\log\bar{F}(x_n)}\right).$$

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- The 90% and 50% EQPB are obtained in a similar way by replacing 0.5 with $0.05, 0.25, 0.75, 0.95$.

The PHR model: Exponential distribution.

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$$\widehat{m}_{n+1|n}(x_n) = x_n - \log(0.5)\frac{1}{\widehat{\theta}_n} = x_n - \frac{\log(0.5)}{n}x_n. \quad (2.13)$$

The PHR model: Exponential distribution.

- The exponential model with survival function $\bar{F}_\theta(x) = \exp(-\theta x)$ for $x \geq 0$ satisfies the PHR model with $h(x) = 1$ and $\bar{F}(x) = \exp(-x)$.
- Then, the MLE for θ is $\widehat{\theta}_n = n/x_n$. This is a well known result (see, e.g., Awad and Raqab, 2000).
- If θ is known, the MRC is

$$m_{n+1|n}(x_n) = \bar{F}^{-1}(0.5^{1/\theta}\bar{F}(x_n)) = x_n - \log(0.5)\frac{1}{\theta}.$$

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- The *estimated (mean) regression curve (ERC)* is

$$\tilde{m}_{n+1|n}(x_n) = x_n + \frac{1}{\widehat{\theta}_n} = x_n + \frac{1}{n}x_n. \qquad (2.14)$$

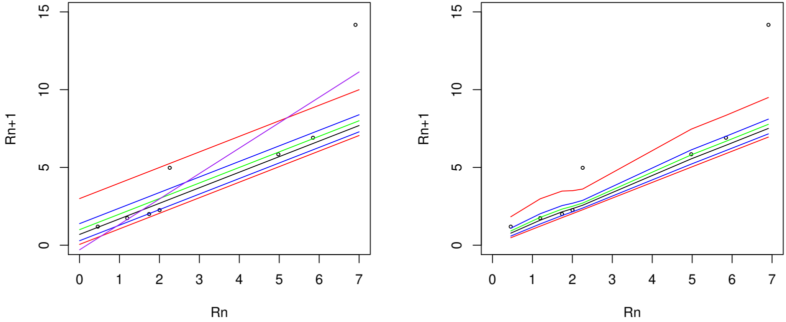

Figure: Plot of the paired records (R_n, R_{n+1}) from a standard exponential distribution jointly with the median regression curve (black), the theoretical and sample regression curves (green, purple) and the limits for the 50% (blue) and 90% (red) exact prediction bands (left). The same is done in the right plot by assuming that θ is unknown.

Table: Predicted values $\widehat{R}_{n+1} = \widehat{m}_{n+1|n}(R_n)$ and $\tilde{R}_{n+1} = \tilde{m}_{n+1|n}(R_n)$ and centered prediction intervals $[l_n, u_n]$ (50%) and $[L_n, U_n]$ (90%) for the first nine records from a standard exponential distribution when θ is unknown. R_{n+1} represents the exact values for $n = 1, \ldots, 8$.

n	L_n	l_n	$\widehat{R}_{n+1}$	$\tilde{R}_{n+1}$	R_{n+1}	u_n	U_n
1	0.48045	0.58849	0.77379	0.91402	1.19403	1.09056	1.82609
2	1.22465	1.36578	1.60785	1.79105	1.74177	2.02167	2.98250
3	1.77155	1.90879	2.14420	2.32236	2.00398	2.54664	3.48106
4	2.02968	2.14811	2.35124	2.50497	2.25833	2.69850	3.50482
5	2.28149	2.38826	2.57140	2.70999	4.97619	2.88447	3.61139
6	5.01873	5.21478	5.55106	5.80556	5.84512	6.12594	7.46075
7	5.88795	6.08534	6.42391	6.68014	6.90868	7.00269	8.34661
8	6.95297	7.15712	7.50727	7.77226	14.1657	8.10586	9.49575

Additional results in Navarro (2021)

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- Comparisons with other estimators.
- A case study in reliability by using lower record values (which come first when we study lifetimes).

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- These representations are very useful!!

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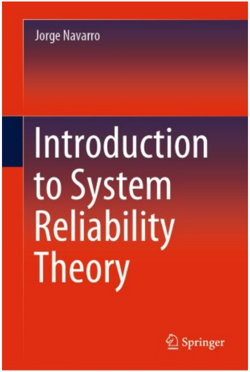
Final slide

- Publicity of my new book on System Reliability Theory.



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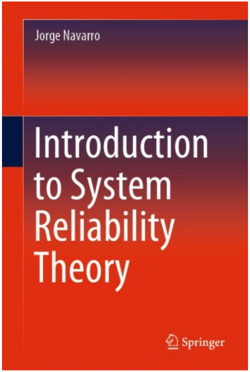

- That's all. Thank you for your attention!!

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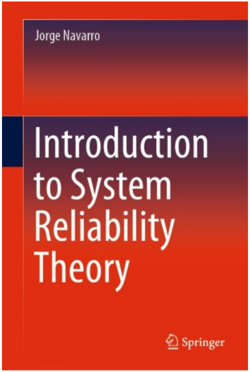

- That's all. Thank you for your attention!!
- Questions?