Ordered Data

Jorge Navarro[1]
Universidad de Murcia, Murcia, Spain.

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References

The talk is based on the following references:

- Navarro J, Arevalillo JM (2023). On connections between skewed, weighted and distorted distributions: applications to model extreme value distributions. Test 32, 1307–1335.
- Navarro J., Cramer E., Balakrishnan N. (2025). Inference under conditionally ordered heterogeneous exponential data. Submitted.
- Lagos G., Navarro J., Olivero H. (2025). Repair policies decomposition for monotone systems with a Lévy frailty Marshall-Olkin Process. In preparation.

Outline

Theoretical results
- Bivariate case
- Multivariate case
- Generalized Order Statistics

Inferencial results
- Case 1: First data (series system)
- Case 2: Second data (parallel system)
- Properties for systems under MO-model.

References

Theoretical results

Notation

- X and Y independent random variables (lifetimes).

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- $\bar{F}_X(t) = 1 - F_X(t) = \Pr(X > t)$ and $\bar{F}_Y(t) = \Pr(Y > t)$ survival (or reliability) functions (SF).

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- $\bar{F}_X(t) = 1 - F_X(t) = \Pr(X > t)$ and $\bar{F}_Y(t) = \Pr(Y > t)$ survival (or reliability) functions (SF).
- We assume that X and Y have absolutely continuous distributions with probability density functions (PDF) $f_X = F'_X$ and $f_Y = F'_Y$.

Ordered data-Bivariate case

- If they are independent, the joint PDF of (X, Y) is

$$f(x,y) = f_X(x)f_Y(y).$$

For results with dependency see Navarro et al. (2022a).

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For results with dependency see Navarro et al. (2022a).
- If we assume that $X<Y$, the joint PDF of $(X,Y|X<Y)$ is
$$g(x,y) = c\ f_X(x)f_Y(y),\ x \le y,$$
where $c = 1/\Pr(X<Y)$.

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where $c = 1/\Pr(X < Y)$.

- The first marginal PDF of $(X, Y|X < Y)$ is

$$g_1(x) = \int_{-\infty}^{\infty} g(x, y) dy = \int_{x}^{\infty} c\ f_X(x) f_Y(y) dy = c\ f_X(x) \bar{F}_Y(x).$$

Ordered data-Bivariate case

- Analogously, the second marginal PDF of $(X, Y|X < Y)$ is

$$g_2(y) = \int_{-\infty}^{\infty} g(x, y)dx = \int_{-\infty}^{y} c\, f_X(x)f_Y(y)dx = c\, f_Y(y)F_X(y).$$

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- The PDF $g_2=c\,f_YF_X$ is the **right skewed distribution** associated to f_Y and F_X, as defined in Navarro and Arevalillo (Test, 2023).

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- These marginal PDF are skew versions of f_X and f_Y.
- The PDF $g_2 = c\, f_Y F_X$ is the **right skewed distribution** associated to f_Y and F_X, as defined in Navarro and Arevalillo (Test, 2023).
- The PDF $g_1 = c\, f_X \bar{F}_Y$ is the **left skewed distribution** associated to f_X and $\bar{F}_Y$, as defined in Navarro and Arevalillo (Test, 2023).

Ordered data-Particular cases: IID

- If they are identically distributed $f_X = f_Y = f$, then $c = 2$,

$$g_1(x) = 2\ f(x)\bar{F}(x).$$

and

$$g_2(y) = 2\ f(y)F(y).$$

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- Therefore, $\bar{G}_1(x) = \bar{F}^2(x)$, which is the survival function of $\min(X, Y)$ and $G_2(x) = F^2(x)$, which is the CDF of $\max(X, Y)$.

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- Therefore, $\bar{G}_1(x) = \bar{F}^2(x)$, which is the survival function of $\min(X, Y)$ and $G_2(x) = F^2(x)$, which is the CDF of $\max(X, Y)$.
- Note that both are distorted distributions, that is, $G_i = q_i(F)$.
- Similar representations are obtained for series and parallel systems with dependent components in Navarro and Arevalillo (Test, 2023) and for general coherent systems in

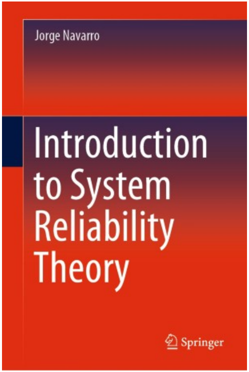

Figure: Publicity of my book on System Reliability Theory.

Ordered data-Particular cases: Skew normal

- If $f_X = \phi$ is the standard normal PDF and

$$\bar{F}_Y(x) = 1 - \Phi(\alpha x) = \Phi(-\alpha x),$$

where Φ is the standard normal CDF and $\alpha > 0$, then $c = 2$ and

$$g_1(x) = 2\ \phi(x)\Phi(-\alpha x)$$

is a skew normal distribution, see Azzalini and Capitanio (2014).

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- In this case, G_1 is the PDF of $\min(X, Y)$ where (X, Y) has a bivariate normal distribution with standarized marginals, correlation ρ and $\alpha = \sqrt{(1-\rho)/(1+\rho)}$, see Theorem 3.2 in Loperfido, Navarro, Ruiz, and Sandoval (2007).

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- If $f_Y = \phi$ is the standard normal PDF and

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where Φ is the standard normal CDF and $\alpha > 0$, then $c = 2$ and

$$g_2(y) = 2\ \phi(y)\Phi(\alpha y)$$

is a skew normal distribution, see Azzalini and Capitanio (2014).

- In this case, G_2 is the PDF of $\max(X, Y)$ where (X, Y) has a bivariate normal distribution with standarized marginals, correlation ρ and $\alpha = \sqrt{(1-\rho)/(1+\rho)}$, see Roberts (1966).

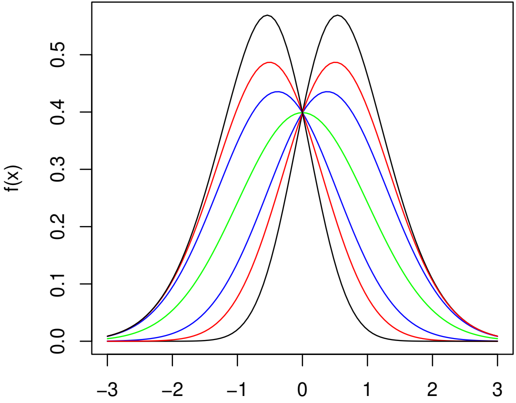

Figure: PDF for $\max(X, Y)$ and $\min(X, Y)$ for $\rho = -0.5$ (black), 0 (red) and 0.5 (blue). The green line represents the standard normal PDF $\rho = 1, \alpha = 0$.

Ordered data-General result under dependency

Proposition (Navarro and Arevalillo, 2023)

Let (X, Y) be an EXC random vector with absolutely continuous copula C and common marginal CDF F and PDF f. Then the PDF of $\max(X, Y)$ *can be written as*

$$f_{2:2}(x) = c_R f(x) G(x)$$

where $G(x) = \partial_1 C(F(x), F(x))$ *and* $c_R = 2$.

Ordered data-General result under dependency

- A similar result holds for $\min(X, Y)$.

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- A similar result holds for $\min(X, Y)$.
- G is not always a CDF.

Ordered data-General result under dependency

- A similar result holds for $\min(X, Y)$.
- G is not always a CDF.
- These results can be extended to the multivariate case, see Navarro and Arevalillo (2023) and Loperfido, Navarro, Ruiz, and Sandoval (2007).

Ordered data-Particular cases: PHR

- If X and Y are independent, $\bar{F}_X = \bar{F}^{\alpha_1}$ and $\bar{F}_Y = \bar{F}^{\alpha_2}$ for $\alpha_1, \alpha_2 > 0$, i.e., they satisfy the Proportional Hazard Rate (PHR) model, then the PDF of $(X|X < Y)$ is

$$g_1(x) = c\ f_X(x)\bar{F}_Y(x) = (\alpha_1 + \alpha_2)\ f(x)\bar{F}^{\alpha_1+\alpha_2-1}(x)$$

with $c = (\alpha_1 + \alpha_2)/\alpha_1$.

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  with $c = (\alpha_1 + \alpha_2)/\alpha_1$.
- It is both a left skew distribution from f and a distortion of $\bar{F}$ since

$$\bar{G}_1(x) = \bar{F}^{\alpha_1+\alpha_2}(x).$$

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- It is both a left skew distribution from f and a distortion of $\bar{F}$ since

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- Note that $(X|X < Y)$ belongs to the same PHR model.

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$$\bar{G}_1(x) = \bar{F}^{\alpha_1+\alpha_2}(x).$$

- Note that $(X|X < Y)$ belongs to the same PHR model.
- Even more $(X|X < Y)$, $(Y|X > Y)$ and $\min(X, Y)$ have the same distribution!!

Ordered data-Particular cases: PHR

- If X and Y are independent, $\bar{F}_X = \bar{F}^{\alpha_1}$ and $\bar{F}_Y = \bar{F}^{\alpha_2}$ for $\alpha_1, \alpha_2 > 0$, then the PDF of $(Y|X < Y)$ is

$$g_2(y) = c\ f_Y(y)F_X(y) = c\ \alpha_2 f(y)\bar{F}^{\alpha_2-1}(y)(1-\bar{F}^{\alpha_1}(y))$$

where $c = (\alpha_1 + \alpha_2)/\alpha_1$, that is,

$$g_2(y) = c\ \alpha_2 f(y)\bar{F}^{\alpha_2-1}(y) - c\ \alpha_2 f(y)\bar{F}^{\alpha_1+\alpha_2-1}(y)$$

and

$$\bar{G}_2(y) = \frac{\alpha_1+\alpha_2}{\alpha_1}\bar{F}^{\alpha_2}(y) - \frac{\alpha_2}{\alpha_1}\bar{F}^{\alpha_1+\alpha_2}(y).$$

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- If X and Y are independent, $\bar{F}_X = \bar{F}^{\alpha_1}$ and $\bar{F}_Y = \bar{F}^{\alpha_2}$ for $\alpha_1, \alpha_2 > 0$, then the PDF of $(Y|X < Y)$ is

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- It is negative mixture of members of the PHR model.

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- It is negative mixture of members of the PHR model.
- Note that $(Y|X < Y)$, $(X|X > Y)$ and $\max(X, Y)$ have different distributions!!

Ordered data-Particular cases: exponential model

- If X and Y are independent, $\bar{F}_X(t) = e^{-\alpha_1 t}$ and $\bar{F}_Y(t) = e^{-\alpha_2 t}$ for $t \geq 0$ and $\alpha_1, \alpha_2 > 0$, then $c = (\alpha_1 + \alpha_2)/\alpha_1$, the PDF of $(X|X < Y)$ is

$$g_1(t) = c\ f_X(t)\bar{F}_Y(t) = (\alpha_1 + \alpha_2)\ e^{-(\alpha_1+\alpha_2)t},\ \ t \geq 0.$$

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$$g_1(t) = c\ f_X(t)\bar{F}_Y(t) = (\alpha_1 + \alpha_2)\ e^{-(\alpha_1+\alpha_2)t},\ \ t \geq 0.$$

- Analogously, the survival function of $(Y|X < Y)$ is

$$\bar{G}_2(t) = \frac{\alpha_1 + \alpha_2}{\alpha_1} e^{-\alpha_2 t} - \frac{\alpha_2}{\alpha_1} e^{-(\alpha_1+\alpha_2)t},\ \ t \geq 0.$$

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- The first one is an exponential distribution and the second a negative mixture of exponential distributions.

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- The first one is an exponential distribution and the second a negative mixture of exponential distributions.
- It is a obtained as a convolution of two independent exponential distributions $(X|X < Y)$ and $(Y - X|X < Y)$ with different parameters.

Multivariate case

Proposition (Navarro, Cramer and Balakrishnan (2025))

Let $X_1, \dots, X_n$ be independent random variables with reliability functions $\overline{F}_i = \overline{F}^{\alpha_i}$ for $\alpha_i > 0$ and $i = 1, \dots, n$. Then, the reliability function $\bar{G}_k$ of $(X_k | X_1 < \cdots < X_n)$ is given by the reliability function of the k-th generalized order statistic based on parameters $\gamma_\ell = \sum_{j=\ell}^{n} \alpha_j$, $1 \le \ell \le n$, that is,

$$\bar{G}_k(t) = \sum_{i=1}^{k} \left(\prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{\gamma_j}{\gamma_j - \gamma_i} \right) \bar{F}^{\gamma_i}(t), \quad t \ge 0. \tag{1.1}$$

If $X_1, \dots, X_n$ are exponentially distributed then, conditionally on $X_1 < \cdots < X_n$, the spacings $X_{k:n} - X_{k-1:n}$ are independent exponentially distributed with scale parameters γ_k for $k = 1, \dots, n$.

Multivariate case

- For $k = 1$, the reliability function of $(X_1|X_1 < \cdots < X_n)$ is

$$\bar{G}_1(t) = \bar{F}^{\alpha_1+\cdots+\alpha_n}(t), \quad t \geq 0 \tag{1.2}$$

and we have

$$\Pr(X_1 < \cdots < X_n) = \frac{\alpha_{n-1}\ldots\alpha_1}{(\alpha_{n-1}+\alpha_n)\ldots(\alpha_1+\cdots+\alpha_n)}. \tag{1.3}$$

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- The distribution of $(X_1|X_1 < \cdots < X_n)$ also belongs to the same PHR model with parameter $\alpha_1 + \cdots + \alpha_n$ and coincides with that of $X_{1:n} = \min(X_1, \ldots, X_n)$.

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- Hence, the distribution of $(X_i|X_i < X_{\sigma(1)} < \cdots < X_{\sigma(n)})$ is the same for any permutation σ of these indices.

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- The distribution of $(X_1|X_1 < \cdots < X_n)$ also belongs to the same PHR model with parameter $\alpha_1 + \cdots + \alpha_n$ and coincides with that of $X_{1:n} = \min(X_1, \ldots, X_n)$.
- Hence, the distribution of $(X_i|X_i < X_{\sigma(1)} < \cdots < X_{\sigma(n)})$ is the same for any permutation σ of these indices.
- This is a quite surprising property since the distributions of $X_1, \ldots, X_n$ are heterogeneous.

Multivariate case

- In particular, for the exponential model, we obtain

$$
\begin{aligned}
E(X_1|X_1 < \cdots < X_n) &= E(X_i|X_i < X_{\sigma(1)} < \cdots < X_{\sigma(n)}) \\
&= E(X_{1:n}) = \frac{1}{\alpha_1 + \cdots + \alpha_n}.
\end{aligned}
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- In the case $k = 2$ the reliability function of $(X_2|X_1 < \cdots < X_n)$ is

$$\bar{G}_2(t) = \frac{\alpha_1 + \cdots + \alpha_n}{\alpha_1} \bar{F}^{\alpha_2 + \cdots + \alpha_n}(t) - \frac{\alpha_2 + \cdots + \alpha_n}{\alpha_1} \bar{F}^{\alpha_1 + \cdots + \alpha_n}(t), \tag{1.4}$$

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$$= E(X_{1:n}) = \frac{1}{\alpha_1 + \cdots + \alpha_n}.$$

- In the case $k = 2$ the reliability function of $(X_2|X_1 < \cdots < X_n)$ is

$$\bar{G}_2(t) = \frac{\alpha_1 + \cdots + \alpha_n}{\alpha_1} \bar{F}^{\alpha_2 + \cdots + \alpha_n}(t) - \frac{\alpha_2 + \cdots + \alpha_n}{\alpha_1} \bar{F}^{\alpha_1 + \cdots + \alpha_n}(t), \tag{1.4}$$

- For properties of negative mixtures see Navarro and Sarabia (2024).

Exponential case

- If $X_1, \dots, X_n$ are independent and have exponential distributions of parameters $\alpha_1, \dots, \alpha_n$, then

$$E(X_k | X_1 < \cdots < X_n) = \sum_{i=1}^{k} \frac{1}{\gamma_i} = \frac{1}{\alpha_1 + \cdots + \alpha_n} + \cdots + \frac{1}{\alpha_k + \cdots + \alpha_n},$$

$$Var(X_k | X_1 < \cdots < X_n) = \sum_{i=1}^{k} \frac{1}{\gamma_i^2} = \frac{1}{(\alpha_1 + \cdots + \alpha_n)^2} + \cdots + \frac{1}{(\alpha_k + \cdots + \alpha_n)^2}$$

and

$$E(X_k^2 | X_1 < \cdots < X_n) = \sum_{i=1}^{k} \frac{1}{\gamma_i^2} + \left(\sum_{i=1}^{k} \frac{1}{\gamma_i} \right)^2$$

with $\gamma_i = \sum_{j=i}^{n} \alpha_j$, $1 \le i \le n$.

Generalized Order Statistics (GOS)

Theorem

Let $X_1, \ldots, X_n$ be independent random variables with strictly increasing continuous reliability functions $\overline{F}_i = \overline{F}^{\alpha_i}$ for $\alpha_i > 0$ and $i = 1, \ldots, n$. Then,

$$(X_1, \ldots, X_n | X_1 < \cdots < X_n) \stackrel{d}{=} (Y_1, \ldots, Y_n),$$

where $(Y_1, \ldots, Y_n)$ are the GOS based on F and parameters $\gamma_1, \ldots, \gamma_n$ with $\gamma_\ell = \sum_{j=\ell}^{n} \alpha_j$, $1 \leq \ell \leq n$.

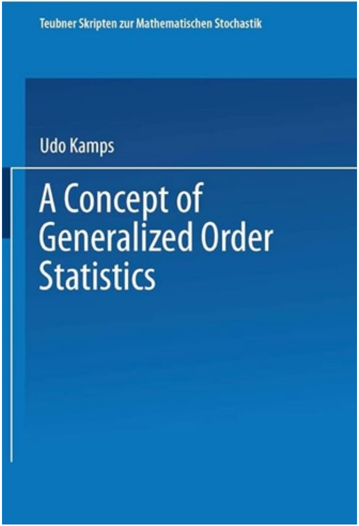

Figure: A tribute to Udo.

Inferencial results

Ordered data

- Suppose that we observe m independent samples of the random variables $X_1, \ldots, X_n$ represented by $X_1^{(j)}, \ldots, X_n^{(j)}$ for $j = 1, \ldots, m$.

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- For illustration, we consider below some typical cases.
- Other cases can be managed in a similar way.

Case 1: First data (series system)

- Suppose that we observe only the minimum in each sample

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- In the literature, this model is called 'first failure censoring'.
- From Balakrishnan and Cramer (2014), p. 529, this scheme can be seen as a standard progressive censoring scheme with a modified censoring plan $\mathcal{S} = n\mathcal{R} + (n-1)(1, \ldots, 1)$.

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- In particular, if $X_2 = t_0$ for some $t_0 > 0$ then we have an experiment with truncated data in the fixed period of time $[0, t_0]$.
- This scheme can also represent data from stress-strength models where a data is observed (fails) if and only if the strength $X_1^{(j)}$ is below the stress $X_2^{(j)}$.

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- Thus, assuming exponentially distributed lifetimes X_i with

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we get

$$E\left(\bar{X}_{1:n}\right) = \mu_{1:n} = E(X_{1:n}) = \frac{1}{\alpha_1+\cdots+\alpha_n}$$

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- The same problem occurs for

$$\bar{X}_{1|1<} = \frac{1}{m_1} \sum_{j=1}^{m} I_1^{(j)} X_{1:n}^{(j)}, \quad \text{where } m_1 = \sum_{j=1}^{m} I_1^{(j)}.$$

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and the property $m_1 \sim Binom(m, p_1)$ holds, i.e.,

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- Therefore, α_1 can be estimated by

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- Analogously, α_2 can be estimated with $\widehat{\alpha}_2 = (1 - \widehat{p}_1)/\bar{X}_{1|1<2}$.

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- In this case, it is easy to get distributional properties for
$$\widehat{\mu}_1 = \frac{1}{\widehat{\alpha}_1} = \frac{\bar{X}_{1:2}}{\widehat{p}_1} = \frac{1}{m_1}\sum_{j=1}^{m} X_{1:2}^{(j)} \geq \bar{X}_{1:2} = \frac{1}{m}\sum_{j=1}^{m} X_{1:2}^{(j)}.$$

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- Thus, from the Central Limit Theorem (CLT) we know that

$$\frac{\bar{X}_{1:2} - \mu_{1:2}}{\sigma_{1:2}/\sqrt{m}} \xrightarrow{d} Z,$$

where $\mu_{1:2} = E(X_{1:2}) = 1/(\alpha_1 + \alpha_2)$,
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$$\frac{\sqrt{m}}{\mu_1}\left(\widehat{\mu}_1-\frac{\mu_{1:2}}{\widehat{p}_1}\right)=\frac{\sqrt{m}\left(\bar{X}_{1:2}-\mu_{1:2}\right)/\sigma_{1:2}}{\widehat{p}_1/p_1} \xrightarrow{d} Z, \qquad (2.1)$$

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$$\bar{X}_{1:2} = 0.3402184 \approx \frac{1}{\alpha_1 + \alpha_2} = \frac{1}{3}.$$

- As mentioned above, this information is not enough to estimate the unknown parameters α_1 and α_2.

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- To do so we need to use

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- Analogously, for α_2 we get $\widehat{p}_2 = 1 - \widehat{p}_1$ and

$$\widehat{\alpha}_2 = \frac{\widehat{p}_2}{\bar{X}_{1:2}} = \frac{0.69}{0.3402184} = 2.028109 \approx \alpha_2 = 2.$$

Example 1

- If we prefer to estimate the means we get

$$\widehat{\mu}_1 = \frac{\bar{X}_{1:2}}{\widehat{p}_1} = \frac{0.3402184}{0.31} = 1.097479 \approx \mu_1 = E(X_1) = \frac{1}{\alpha_1} = 1$$

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- If we just have the 31 censored data obtained from X_1 at the first component failures, then

$$\widehat{\mu}_1 = \frac{1}{\widehat{\alpha}_1} = \frac{\bar{X}_{1|1<2}}{\widehat{p}_1} = \frac{0.4129864}{0.31} = 1.332214 \geq \bar{X}_{1|1<2} = 0.4129864.$$

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- The estimation is not very good since we just have 31 data and it is better to use the information of all the censored times as above getting the estimation 1.097479 for $\mu_1=1$.

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- Note that these values can also be used to estimate the means of the other variables obtaining

$$\widehat{\mu}_1 = \frac{\bar{X}_{2|2<1}}{\widehat{p}_1} = \frac{0.3075255}{0.31} = 0.9920176 \approx \mu_1 = 1$$

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$$E(X_2|X_1 < X_2, X_1 = x) = x + \frac{1}{\alpha_2} \approx m_2(x) = x + \frac{\bar{X}_{1:2}}{\widehat{p}_2} = x + 0.4930701$$

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- These regression lines are plotted in the right plot.

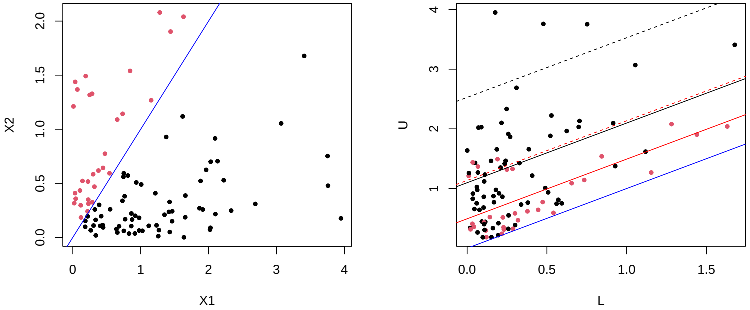

Figure: Censored data (left) for X_1 when $X_1 < X_2$ (red) and for X_2 when $X_2 < X_1$ (black). Data (right) from series $L = X_{1:2}$ and parallel $U = X_{2:2}$ systems. The regression lines are the red and black solid lines.

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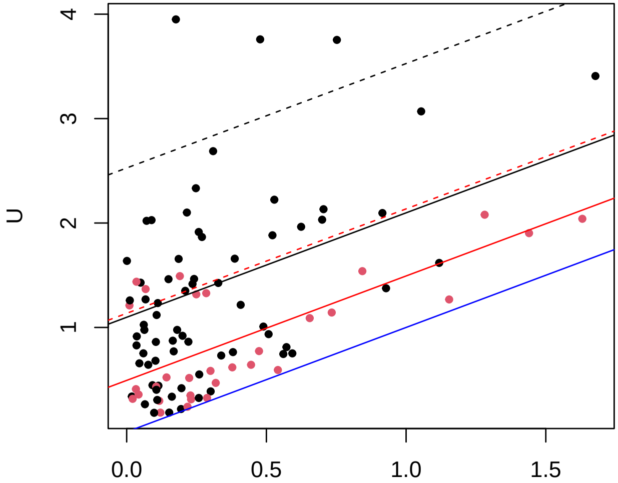
- The 90% lower confidence band for the black points is $(x, x + 2.527039)$ (black dashed line) and it contains 63-out-of-69 black points (i.e., it contains the 95.65217% of the black points).


U
4
3
2
1
0.0
0.5
1.0
1.5

Case 2: Second data with $n = 2$ (parallel system)

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- Again, we need extra information.

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- The other options can be seen in the paper.

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- Hence, as in case 1, we can estimate $p_1 = \alpha_1/(\alpha_1 + \alpha_2)$ with $\widehat{p}_1 = m_1/m$. Analogously, we can estimate $p_2 = \alpha_2/(\alpha_1 + \alpha_2)$ with
$$\widehat{p}_2 = 1 - \widehat{p}_1 = \frac{m_2}{m},$$
where $m_2 = m - m_1$.

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- By using that $\alpha_2 = \frac{1+p_2}{\mu_{2|1<2}}$. we can estimate $\mu_2 = 1/\alpha_2$ by

$$\widehat{\mu}_2 = \frac{\bar{X}_{2|1<2}}{1 + m_2/m} \leq \bar{X}_{2|1<2},$$

where we can use again Slutsky's theorem.

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- To this purpose we need to separate the data from X_1 and X_2.
- We also need m_1 and m_2.
- Thus, we obtain the following estimations

$$\widehat{\mu}_1 = \frac{\bar{X}_{1|2<1}}{1 + m_1/m} = \frac{1.305879}{1 + 31/100} = 0.9968541 \approx \mu_1 = 1$$

and

$$\widehat{\mu}_2 = \frac{\bar{X}_{2|1<2}}{1 + m_2/m} = \frac{0.8760269}{1 + 69/100} = 0.5183591 \approx \mu_2 = 0.5.$$

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$$\widehat{\mu}_2^* = \frac{\bar{X}_{1|2<1}}{m_2/m_1 + m_2/m} = \frac{1.305879}{69/31 + 69/100} = 0.447862 \approx \mu_2 = 0.5$$

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- Conversely, we can estimate μ_1 with the data from X_2 with

$$\widehat{\mu}_1^* = \frac{\bar{X}_{2|1<2}}{m_1/m_2 + m_1/m} = \frac{0.8760269}{(31/69 + 31/100)} = 1.153767 \approx \mu_1 = 1.$$

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- In the paper we also consider the case of a 2-out-of-3 system with lifetime $X_{2:3}$.
- The estimations are more complex in this case.
- Again, quantile regression techniques can be used to predict $X_{2:3}$ from $X_{1:3}$.

Properties for systems under the MO-model.

- The results are included in the paper:
- Lagos G., Navarro J., Olivero H. (2025). Repair policies decomposition for monotone systems with a Lévy frailty Marshall-Olkin Process. In preparation.

M-O model

- A random vector $(T_1, \ldots, T_n)$ have an exponential Marshall-Olkin (MO) distribution if

$$T_i = \min_{V \subseteq \{1,\ldots,n\} \,:\, i \in V} X_V, \qquad i = 1, \ldots, n, \tag{2.3}$$

where, for all $V \subseteq \{1, \ldots, n\}$, X_V is an exponential random variable with parameter $\lambda_V \geq 0$, and is independent of the other random variables.

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- It is exchangeable when $\lambda_U = \lambda_V$ whenever $|U| = |V|$.

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- It is not absolutely continuous (there are ties).
- All the series systems have exponential distributions.

M-O model

- The joint reliability function of $(T_1, \dots, T_n)$ is

$$\Pr(T_1 > t_1, \dots, T_n > t_n) = \exp\left(-\sum_{\emptyset \neq V \subset \{1,\dots,n\}} \lambda_V \max_{i \in V} t_i\right),$$

for $t_1, \dots, t_n \geq 0$.
- It is not absolutely continuous (there are ties).
- All the series systems have exponential distributions.
- All the coherent system distributions are negative mixtures of exponential distributions, Navarro (2022).

Cost of simple repair policies

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- Both types of repairs, i.e., of the system and of components, are performed instantaneously.

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- In a system operating under an r-out-of-n:R repair policy, denote by T_r^{sys} and T_r^{rep} the first time of system failure and repair, respectively.
- As the MO-model has the lack of memory property, after a repair the system has the same distribution as a new system.

Example 2

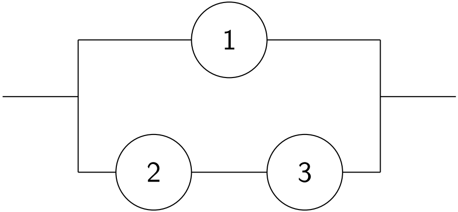

Figure: A coherent system with exchangeable components lifetimes (T_1, T_2, T_3) satisfying the exponential MO-model with $\lambda_1 := \lambda_{\{1\}} = 13/12$, $\lambda_2 := \lambda_{\{1,2\}} = 1/12$, and $\lambda_3 := \lambda_{\{1,2,3\}} = 1/4$.

Example 2: Non-repair case.

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- It does not coincide with the classical (structural) signature (see Navarro (2022)),
$$s = (0, 2/3, 1/3).$$

Case	Lifetimes	N	Probab.	Repair $r=1$	Repair $r=2$
1	$T_1 = T_2 = T_3$	3	$1/15$	No	No
2	$T_1 < T_2 = T_3$	3	$13/360$	**Yes**	No
3	$T_2 < T_1 = T_3$	3	$13/360$	**Yes**	No
4	$T_3 < T_1 = T_2$	3	$13/360$	**Yes**	No
5	$T_1 = T_2 < T_3$	2	$1/45$	No	No
6	$T_1 = T_3 < T_2$	2	$1/45$	No	No
7	$T_2 = T_3 < T_1$	3	$1/45$	**Yes**	**Yes**
8	$T_1 < T_2 < T_3$	2	$91/720$	**Yes**	No
9	$T_1 < T_3 < T_2$	2	$91/720$	**Yes**	No
10	$T_2 < T_1 < T_3$	2	$91/720$	**Yes**	No
11	$T_2 < T_3 < T_1$	3	$91/720$	**Yes**	**Yes**
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Table: Options and probabilities needed to compute the signature w of T.

Example 2: Non-repair case.

- The probability of case 1 is

$$p_1 = \Pr(T_1 = T_2 = T_3) = \Pr(X_{\{1,2,3\}} < Z)$$

where $X_{\{1,2,3\}} \sim Exp(\alpha_1 = \lambda_3 = 1/4)$ and

$Z = \min(X_{\{1\}}, X_{\{3\}}, X_{\{3\}}, X_{\{1,2\}}, X_{\{1,3\}}, X_{\{2,3\}}) \sim Exp(\alpha_2 = 7/2).$

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- As $X_{\{1,2,3\}}$ and Z are independent, then

$$p_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{1/4}{1/4 + 7/2} = \frac{1}{15}.$$

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- The other probabilities are computed in a similar way (by using the initial results) and thus we can obtain w_2 and w_3.

Example 2: Non-repair case.

- Hence, the expected cost in a cycle without repairs is

$$c(T) = c_{sys} + c_2 w_2 + c_3 w_3 = c_{sys} + c_2 \frac{11}{20} + c_3 \frac{9}{20}.$$

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- If we choose the values $c_{sys} = 1$ and $c_i = i$, $i = 1, 2, 3$, we get

$$c(T) = 1 + 2w_2 + 3w_3 = \frac{69}{20} = 3.45.$$

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- To measure the quality of the system we can consider the Mean Cost of a Cycle (MCC) defined as

$$MCC(T) = \frac{c(T)}{E(T)} = \frac{69/20}{31/40} = \frac{138}{31} = 4.451613.$$

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Table: Options needed to compute the probability of repair.

Example 2: Repair case with $r = 1$.

- For $r = 1$, the probability of repair is

$$p_R = 3\frac{13}{360} + \frac{1}{45} + 6\frac{91}{720} = \frac{8}{9} = 0.8888889.$$

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- The expected cost in a cycle with repair is

$$c(T) = c_2 w_2^R + c_3 w_3^R$$

where the conditional signature with repair is

$$\mathrm{w}^R = \left(\frac{39}{40}, \frac{1}{40}, 0\right).$$

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- The probability of "no repair" is $p_{\bar{R}} = 1/9$ and its conditional signature is

$$\mathrm{w}^{\bar{R}} = \left(0, \frac{2}{5}, \frac{3}{5}\right).$$

Example 2: Repair case with $r = 1$.

- Therefore, for $r = 1$, we get the expected cost

$$c(T_{(1)}) = c_{sys} + \frac{p_R}{1-p_R}\sum_{i=r}^{n-1} w_i^R c_i + \sum_{i=1}^{n} w_i^{\bar{R}} c_i = \frac{59}{5} = 11.8$$

and the MTTF

$$E(T_{(1)}) = \frac{p_R}{1-p_R} E(T_{(1)}|R) + E(T_{(1)}|\bar{R}) = \frac{12}{5} = 2.4.$$

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- Hence, the Mean Cost of a Cycle for $r = 1$ is

$$MCC(T_{(1)}) = \frac{c(T_{(1)})}{E(T_{(1)})} = \frac{59/5}{12/5} = \frac{59}{12} = 4.916667.$$

Example 2: Repair case with $r = 2$.

- Analogously, for $r = 2$, we get

$$MCC(T_{(2)}) = \frac{c(T_{(2)})}{E(T_{(2)})} = \frac{4}{0.816092} = 4.901408.$$

Example 2: Repair case with $r = 2$.

- Analogously, for $r = 2$, we get

$$MCC(T_{(2)}) = \frac{c(T_{(2)})}{E(T_{(2)})} = \frac{4}{0.816092} = 4.901408.$$

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$$MCC(T) = 4.451613 < MCC(T_{(2)}) = 4.901408 < MCC(T_{(1)}) = 4.916667.$$

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- Therefore, in this case, the best option is the one without repairs.

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- Questions?