Prediction of future data in multivariate constant conditional hazard rate models

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References

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Outline

Preliminary results

Hazard rate functions

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- The jth **hazard (or failure) rate** function is

$$\lambda_j(t) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}(X_j \leq t + \Delta t \,|X_j > t) = \frac{f_j(t)}{\bar{F}_j(t)}.$$

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- The condition $\lambda_j(t) = c_j$ for $t \geq 0$ leads to the exponential distribution with $\bar{F}_j(t) = \exp(-c_j t)$ for $t \geq 0$ and $c_i > 0$.

Hazard rate functions

- For $j \in [n]$ and $i_1, \dots, i_k \in [n]$ with $j \notin I = \{i_1, \dots, i_k\}$, and $0 \le t_1 \le \dots \le t_k$, the jth **multivariate conditional hazard rate** (MCHR) function $\lambda_j(t|i_1, \dots, i_k; t_1, \dots, t_k)$ is defined as:

$$\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}\left(X_j \le t + \Delta t \,\middle|\, X_{i_1} = t_1, \dots, X_{i_k} = t_k, \min_{h \notin I} X_h > t \right).$$

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- We use the following notation for the MCHR functions with no failures (also called risk-specific or initial hazard rate)

$$\lambda_j(t|\emptyset) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}(X_j \le t + \Delta t \,|X_{1:n} > t),$$

where $X_{1:n} = \min(X_1, \dots, X_n)$.

Particular cases

- If $X_1,\dots,X_n$ are independent, then, for all $j\notin\{i_1,\dots,i_k\}$,

$$\lambda_j(t|i_1,\dots,i_k;t_1,\dots,t_k)=\lambda_j(t) \text{ for all } t>0.$$

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- If $X_1, \dots, X_n$ are exchangeable, i.e.,

$$(X_1, \dots, X_n) =_{ST} (X_{\pi(1)}, \dots, X_{\pi(n)}) \text{ for any permutation } \pi,$$

then the MCHR functions do not depend on j and $i_1, \dots, i_k$ but only on k and the failure times $t_1, \dots, t_k$, that is,

$$\lambda_j(t|i_1, \dots, i_k; t_1, \dots, t_k) = \lambda^{(k)}(t|t_1, \dots, t_k)$$

and

$$\lambda_j(t|\emptyset) = \lambda^{(0)}(t),$$

for all $k \in \{1, 2, \dots, n-1\}$ and all $0 \le t_1 \le t_2 \le \dots \le t_k \le t$.

Inversion formula

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$$f(t_1, \ldots, t_n) = \lambda_1(t_1|\emptyset) \exp\left[-\sum_{j=1}^{n} \int_0^{t_1} \lambda_j(u|\emptyset)du\right]$$

$$\lambda_2(t_2|1; t_1) \exp\left[-\sum_{j=2}^{n} \int_{t_1}^{t_2} \lambda_j(u|1; t_1)du\right] \ldots$$

$$\lambda_n(t_n|1, \ldots, n-1; t_1, \ldots, t_{n-1}) \exp\left[-\int_{t_{n-1}}^{t_n} \lambda_n(u|1, \ldots, n-1; t_1, \ldots, t_{n-1})du\right]$$

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- For the proof see Shaked and Shanthikumar (1988).

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- This model is a natural generalization of the joint distribution of a vector of independent and exponentially distributed random variables.
- For a review on these models see Spizzichino (2018).

The models

Definition

$(X_1,\dots,X_n)$ is distributed according to a load-sharing (LS) model if, for any $i_1,\dots,i_k\in[n]$ and $j\notin I=\{i_1,\dots,i_k\}$, there exist functions $\mu_j(t|I)$ such that, for all $0\le t_1\le\dots\le t_k\le t$,

$$\lambda_j(t|i_1,\dots,i_k;t_1,\dots,t_k)=\mu_j(t|I).$$

Furthermore, a load-sharing model is time-homogeneous (THLS) when there exist non-negative numbers $\mu_j(I)$ and $\mu_j(\emptyset)$ such that, for any $t>0$ and any $j\notin I$,

$$\mu_j(t|I)=\mu_j(I),$$
$$\lambda_j(t|\emptyset)=\mu_j(\emptyset).$$

The models

In this paper, we will consider a more general model.

Definition

$(X_1, \ldots, X_n)$ is distributed according to an order dependent load-sharing (ODLS) model if, for any $i_1, \ldots, i_k \in [n]$ and $j \notin I = \{i_1, \ldots, i_k\}$, there exist functions $\mu_j(t|i_1, \ldots, i_k)$ such that, for all $0 \leq t_1 \leq \cdots \leq t_k \leq t$,

$$\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \mu_j(t|i_1, \ldots, i_k).$$

Furthermore, an ODLS model is time-homogeneous (ODTHLS) when there exist non-negative numbers $\mu_j(i_1, \ldots, i_k)$ and $\mu_j(\emptyset)$ such that, for any $t > 0$ and any $j \notin I$,

$$\mu_j(t|i_1, \ldots, i_k) = \mu_j(i_1, \ldots, i_k),$$
$$\lambda_j(t|\emptyset) = \mu_j(\emptyset).$$

The models

- If for any non-empty set $I \subset [n]$ and any $j \notin I$, the function $\mu_j(t|i_1,\ldots,i_k)$ is invariant under permutations of $i_1,\ldots,i_k$, then the ODLS model reduces to a LS model.

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- In the same way, if for any non-empty set $I \subset [n]$ and any $j \notin I$ the number $\mu_j(i_1, \ldots, i_k)$ is invariant under permutations of $i_1, \ldots, i_k$, then the ODTHLS model reduces to a THLS model.

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- Note that the LS model includes a kind of weak exchangeability property since the MCHR functions just depend on the set of broken components $I = \{i_1,\dots,i_k\}$ instead of the vector of ordered failures $(i_1,\dots,i_k)$ used in the ODLS model.

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- The same holds for the THLS and the ODTHLS models.

Inversion formula for the ODTHLS model

Proposition

The PDF of $(X_1,\dots,X_n)$ under the ODTHLS model can be obtained for $0 \le t_1 \le t_2 \le \cdots \le t_n$ as

$$f(t_1,\dots,t_n) = \mu_1(\emptyset)\exp\left[-t_1\sum_{j=1}^{n}\mu_j(\emptyset)\right]$$

$$\mu_2(1)\exp\left[-(t_2-t_1)\sum_{j=2}^{n}\mu_j(1)\right]\dots$$

$$\mu_n(1,\dots,n-1)\exp\left[-(t_n-t_{n-1})\mu_n(1,\dots,n-1)\right].$$

Similar expressions hold when $t_1,\dots,t_n$ are such that $t_{\pi(1)} \le \cdots \le t_{\pi(n)}$ for some permutation π.

Properties

- Under the ODTHLS model we use

$$M(i_1,\dots,i_k)=\sum_{h\notin\{i_1,\dots,i_k\}}\mu_h(i_1,\dots,i_k); \quad (1)$$

$$\rho_j(i_1,\dots,i_k)=\frac{\mu_j(i_1,\dots,i_k)}{M(i_1,\dots,i_k)}. \quad (2)$$

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- Then if π is a fixed permutation,

$$\mathbb{P}(X_{1:n}=X_{\pi(1)},\dots,X_{r:n}=X_{\pi(r)})=\rho_{\pi(1)}(\emptyset)\rho_{\pi(2)}(\pi(1))\,\rho_{\pi(3)}(\pi(1),\pi(2))\dots\rho_{\pi(r)}(\pi(1),\dots,\pi(r-1)) \quad (3)$$

for $1\le r<n$ and

$$\mathbb{P}(X_{1:n}=X_{\pi(1)},\dots,X_{n:n}=X_{\pi(n)})=\mathbb{P}(X_{1:n}=X_{\pi(1)},\dots,X_{n-1:n-1}=X_{\pi(n-1)})$$

Properties

- For $\Lambda^{(r)} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}_+^r$, $\overline{G}_{\Lambda^{(r)}}(t)$ is the survival function of $\sum_{s=1}^r \Gamma_s$, where $\Gamma_1, \ldots, \Gamma_r$ are independent r. v. with exponential distributions of parameters $\lambda_1, \ldots, \lambda_r$.

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- Moreover, for a permutation π of $[n]$ and $r \in [n]$, we place

$$\Lambda^{(r)}(\pi) = (M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(r-1))).$$

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- Moreover, for a permutation π of $[n]$ and $r \in [n]$, we place

$$\Lambda^{(r)}(\pi) = (M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(r-1))).$$

- In the ODTHLS model, for any $j \in [n]$ we have

$$\mathbb{P}(X_{1:n} > t | X_{1:n} = X_j) = \exp(-tM(\emptyset))$$

and for any permutation π of $[n]$ and $k \in \{2, \ldots, n\}$,

$$\mathbb{P}(X_{k:n} > t | X_{1:n} = X_{\pi(1)}, \ldots, X_{k:n} = X_{\pi(k)}) = \overline{G}_{\Lambda^{(k)}(\pi)}(t).$$

Properties.

- In Spizzichino (2018) it is observed that conditioning on the event $(X_{1:n} = X_{\pi(1)}, \ldots, X_{k:n} = X_{\pi(k)})$, the interarrival times $X_{1:n}, X_{2:n} - X_{1:n}, \ldots, X_{k:n} - X_{k-1:n}$ are independent random variables exponentially distributed with parameters $M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(k-1))$, respectively.

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- In particular, the events $(X_{1:n} > t)$ and $(X_{1:n} = X_j)$ are independent.
- Hence, under this conditioning event, the distribution of $X_{k:n}$ is a convolution of k independent exponential distributions.

Predictions

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- We analyze different scenarios given by different levels of knowledge.
- We start by giving the prediction of $X_{k+1:n}$ from the observed history

$$\mathcal{H}_k = \{X_{1:n} = X_{\pi(1)} = t_1, \ldots, X_{k:n} = X_{\pi(k)} = t_k\}$$

for $k < n$, where π is a permutation of $[n]$.

Predictions

Proposition

Let $(X_1,\dots,X_n)$ follow an ODTHLS model. Given the history $\mathcal{H}_k$ for $k<n$, the median and the mean predictions of $X_{k+1:n}$ are

$$\widehat{X}_{k+1:n}=\mathfrak{m}(t_k)=t_k+\frac{\log 2}{M(\pi(1),\dots,\pi(k))}, \tag{4}$$

and

$$\tilde{X}_{k+1:n}=t_k+\frac{1}{M(\pi(1),\dots,\pi(k))}.$$

Moreover, a prediction band of size $\gamma=\beta-\alpha$, with $\alpha,\beta,\gamma\in(0,1)$, is given by $[t_k+q_\alpha, t_k+q_\beta]$, where q_α and q_β are the quantiles of the exponential distribution with parameter $M(\pi(1),\dots,\pi(k))$.

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- For example, in the above proposition, the centered 90% prediction band is obtained with $\beta = 0.95$ and $\alpha = 0.05$ as

$$C_{90} = \left[t_k - \frac{\log(0.95)}{M(\pi(1),\dots,\pi(k))}, t_k - \frac{\log(0.05)}{M(\pi(1),\dots,\pi(k))} \right].$$

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- Here, we prefer to use the predictions given by the median $\mathfrak{m}(t_k)$, instead of the ones based on the mean, since they are obtained by using quantiles as well as the prediction bands.
- Let us denote by $m_c = \frac{\log 2}{c}$ the median of an exponential distribution with parameter c.

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- Then the median and the mean predictions for the next failure time $X_{k+1:n}$ are respectively given by

$$\begin{aligned}\widehat{X}_{k+1:n} &= m_{M(\emptyset)} + m_{M(\pi(1))} + \dots + m_{M(\pi(1),\dots,\pi(k))},\\ \tilde{X}_{k+1:n} &= \frac{1}{M(\emptyset)} + \frac{1}{M(\pi(1))} + \dots + \frac{1}{M(\pi(1),\dots,\pi(k))}.\end{aligned}$$

Predictions, Scenario 2

- Let $(X_1,\dots,X_n)$ follow an ODTHLS model.
- Let us suppose to know the history $X_{1:n}=X_{\pi(1)}, X_{2:n}=X_{\pi(2)},\dots,X_{k:n}=X_{\pi(k)}$, for $k<n$.
- Then the median and the mean predictions for the next failure time $X_{k+1:n}$ are respectively given by

$$\begin{aligned}\widehat{X}_{k+1:n} &= m_{M(\emptyset)}+m_{M(\pi(1))}+\dots+m_{M(\pi(1),\dots,\pi(k))},\\ \tilde{X}_{k+1:n} &= \frac{1}{M(\emptyset)}+\frac{1}{M(\pi(1))}+\dots+\frac{1}{M(\pi(1),\dots,\pi(k))}.\end{aligned}$$

- The prediction can also be obtained from the median of the convolution of $k+1$ independent exponential distributions with parameters $M(\emptyset)$, $M(\pi(1)),\dots,M(\pi(1),\dots,\pi(k))$.

Predictions, Scenario 3

Proposition

Let $(X_1, \ldots, X_n)$ follow an ODTHLS model. Given the history $\mathcal{H}_k$ for $k < n-1$, the prediction of $X_{k+2:n}$ is given by

$$\widehat{X}_{k+2:n} = \widehat{X}_{k+1:n} + \sum_{j \notin \{\pi(1),\ldots,\pi(k)\}} \rho_j(\pi(1), \ldots, \pi(k)) \frac{\log 2}{M(\pi(1), \ldots, \pi(k), j)}$$

where $\widehat{X}_{k+1:n}$ is the median prediction of $X_{k+1:n}$ obtained before.

Predictions, Scenario 3

Proposition

Let $(X_1,\dots,X_n)$ follow an ODTHLS model. Let π be a fixed permutation of $[n]$ and $k<n-1$. Then,

$$\mathbb{P}(X_{k+2:n}-t_k>t|\mathcal{H}_k)=\sum_{j\notin\{\pi(1),\dots,\pi(k)\}}\rho_j(\pi(1),\dots,\pi(k))\overline{G}_{\Upsilon_j^{(k)}(\pi)}(t),$$

where $\mathcal{H}_k$ is the history defined above, $\overline{G}_{\Upsilon_j^{(k)}(\pi)}(t)$ is the survival function of $Y=Y_1+Y_2$, where Y_1 and Y_2 are independent random variables with exponential distributions of parameters $M(\pi(1),\dots,\pi(k))$ and $M(\pi(1),\dots,\pi(k),j)$.

Predictions, Scenario 3

- Conditioning on the observed history, the interarrival time $X_{k+2:n} - X_{k:n}$ is a mixture of $n-k$ distributions which are sums of two independent exponential distributions.

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- It is necessary to distinguish between the case in which the exponential distributions have the same parameter or not.
- If they have parameters λ and μ with $\lambda \neq \mu$, then

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- In the case $\lambda = \mu$, then

$$\bar{F}_Y(t) = (1 + \lambda t) e^{-\lambda t}, \ t \geq 0. \tag{6}$$

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- Then, if we want to get a confidence $\gamma = \beta - \alpha$, where $\alpha, \beta, \gamma \in (0,1)$ and q_α and q_β are the respective quantiles of the distribution given in the preceding proposition, we use that

$$\mathbb{P}\left(t_k + q_\alpha \leq X_{k+2:n} \leq t_k + q_\beta | \mathcal{H}_k\right) = \gamma.$$

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- In this way, it is possible to predict $X_{s:n}$ for $s > k$.
- With the increase of s there will be more terms in the convolutions.

Simulations

- The preceding results can be used to get simulated data from an ODTHLS (or THLS) model.

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- The preceding results can be used to get simulated data from an ODTHLS (or THLS) model.
- The algorithm can be summarized as follows:

Step 1. Choose π according to the probabilities given in (3).
Step 2. Simulate n independent exponential distributions $Z_1, \ldots, Z_n$ with parameters $M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(n-1))$.
Step 3. Compute $X_{k:n} = Z_1 + \cdots + Z_k$, for $k = 1, \ldots, n$.
Step 4. Compute $X_{\pi(k)} = X_{k:n}$, for $k = 1, \ldots, n$.

Example 1

- Let (X_1, X_2, X_3) be distributed according to an ODTHLS model with parameters

$$\mu_1(\emptyset) = 1, \quad \mu_1(2) = 2, \quad \mu_1(3) = 1, \quad \mu_1(2,3) = \mu_1(3,2) = 3,$$
$$\mu_2(\emptyset) = 2, \quad \mu_2(1) = 1, \quad \mu_2(3) = 3, \quad \mu_2(1,3) = \mu_2(3,1) = 2,$$
$$\mu_3(\emptyset) = 2, \quad \mu_3(1) = 2, \quad \mu_3(2) = 1, \quad \mu_3(1,2) = \mu_3(2,1) = 2.$$

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- It is a THLS model since $\mu_i(j,k) = \mu_i(k,j)$ for all i, j and k.
- Hence, we have

$$M(\emptyset) = 5, \quad M(1) = 3, \quad M(2) = 3, \quad M(3) = 4,$$
$$M(1,2) = M(2,1) = 2, \quad M(1,3) = M(3,1) = 2,$$
$$M(2,3) = M(3,2) = 3.$$

Example 1

- Then

$$\rho_1(\emptyset) = \frac{1}{5}, \quad \rho_2(\emptyset) = \frac{2}{5}, \quad \rho_3(\emptyset) = \frac{2}{5},$$
$$\rho_2(1) = \frac{1}{3}, \quad \rho_3(1) = \frac{2}{3},$$
$$\rho_1(2) = \frac{2}{3}, \quad \rho_3(2) = \frac{1}{3},$$
$$\rho_1(3) = \frac{1}{4}, \quad \rho_2(3) = \frac{3}{4},$$

and, naturally,

$$\rho_1(2,3) = \rho_1(3,2) = \rho_2(1,3) = \rho_2(3,1) = \rho_3(1,2) = \rho_3(2,1) = 1.$$

Example 1

- For $n = 3$ there are six possible permutations with probabilities

$$\mathbb{P}(X_{1:3} = X_1, X_{2:3} = X_2, X_{3:3} = X_3) = \frac{1}{15},$$
$$\mathbb{P}(X_{1:3} = X_1, X_{2:3} = X_3, X_{3:3} = X_2) = \frac{2}{15},$$
$$\mathbb{P}(X_{1:3} = X_2, X_{2:3} = X_1, X_{3:3} = X_3) = \frac{4}{15},$$
$$\mathbb{P}(X_{1:3} = X_2, X_{2:3} = X_3, X_{3:3} = X_1) = \frac{2}{15},$$
$$\mathbb{P}(X_{1:3} = X_3, X_{2:3} = X_1, X_{3:3} = X_2) = \frac{1}{10},$$
$$\mathbb{P}(X_{1:3} = X_3, X_{2:3} = X_2, X_{3:3} = X_1) = \frac{3}{10}.$$

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- By generating a uniform number in $(0,1)$, the permutation $(2,1,3)$ is chosen.

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- In this way, the simulated interarrival times obtained are $0.17166,\ 0.14498,\ 0.25606$, respectively.

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- Hence, three exponential numbers are generated with parameters $M(\emptyset)=5$, $M(2)=3$, and $M(2,1)=2$.
- In this way, the simulated interarrival times obtained are $0.17166,\ 0.14498,\ 0.25606$, respectively.
- Then the simulated values of the order statistics are $X_{1:3}=0.17166$, $X_{2:3}=0.17166+0.14498=0.31663$ and $X_{3:3}=0.31663+0.25606=0.57270$.

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- Since we have chosen permutation $(2,1,3)$, the values $0.17166, 0.31663$ and 0.57270 represent a simulation of X_2, X_1 and X_3, respectively, i.e., the simulated data is $(0.31663, 0.17166, 0.57270)$.

Example 1

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- Suppose now that we just know $X_{1:3} = X_2 = 0.17166$ and that our purpose is to predict $X_{2:3}$ and $X_{3:3}$.
- The mean and the median predictions of $X_{2:3} = 0.31663$ are

$$\tilde{X}_{2:3} = X_{1:3} + \frac{1}{M(2)} = 0.50499$$

and

$$\widehat{X}_{2:3} = \mathfrak{m}(X_{1:3}) = X_{1:3} + \frac{\log 2}{M(2)} = 0.40270,$$

Example 1

- Furthermore, the centered 90% and 50% prediction bands are

$$C_{90} = \left[X_{1:3} - \frac{\log(0.95)}{M(2)}, X_{1:3} - \frac{\log(0.05)}{M(2)}\right] = [0.18875, 1.17023]$$

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- Once $X_{2:3}$ has been predicted, also $X_{3:3}$ can be predicted.
- In this case the median prediction of $X_{3:3} = 0.57270$ is given by

$$\widehat{X}_{3:3} = \widehat{X}_{2:3} + \rho_1(2) \frac{\log 2}{M(2,1)} + \rho_3(2) \frac{\log 2}{M(2,3)} = 0.4027 + \frac{2}{3} \cdot \frac{\log 2}{2} + \frac{1}{3} \cdot \frac{\log 2}{3} = 0.710$$

Example 1

- We can get a different prediction for $X_{3:3}$ from

$$\bar{G}_{3|1}(t) = \mathbb{P}(X_{3:3} - X_{1:3} > t | X_{1:3} = X_2 = 0.17166)$$
$$= \rho_1(2)\overline{G}_{Y_{1,1}+Y_{1,2}}(t) + \rho_3(2)\overline{G}_{Y_{2,1}+Y_{2,2}}(t),$$

where $Y_{1,1}$, $Y_{1,2}$, $Y_{2,1}$ and $Y_{2,2}$ are independent and exponentially distributed with parameters $M(2) = 3, M(2,1) = 2, M(2) = 3$ and $M(2,3) = 3$.

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- Hence, we obtain

$$\bar{G}_{3|1}(t) = \rho_1(2)\frac{M(2)e^{-M(2,1)t} - M(2,1)e^{-M(2)t}}{M(2) - M(2,1)} + \rho_3(2)(1 + M(2)t)e^{-M(2)t}.$$

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where $Y_{1,1}$, $Y_{1,2}$, $Y_{2,1}$ and $Y_{2,2}$ are independent and exponentially distributed with parameters $M(2) = 3, M(2,1) = 2, M(2) = 3$ and $M(2,3) = 3$.

- Hence, we obtain

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- By resolving $\bar{G}_{3|1}(t) = 0.5$ we obtain a prediction for the difference $X_{3:3} - X_{1:3}$ that is 0.64409, from which

$$\widehat{X}_{3:3} = 0.17166 + 0.64409 = 0.81575.$$

Example 1

- By resolving $\bar{G}_{3|1}(t) = \alpha$, for $\alpha = 0.05, 0.25, 0.75, 0.95$, we obtain the 90% and 50% centered prediction bands as $C_{90} = [0.30639, 2.04858]$ and $C_{50} = [0.53811, 1.21520]$.

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- We observe that $X_{3:3} = 0.57270$ belongs to both regions.
- In the following figure we plot these predictions (red) for $X_{2:3}, X_{3:3}$ from $X_{1:3}$ jointly with the exact values (black points) and the prediction bands.

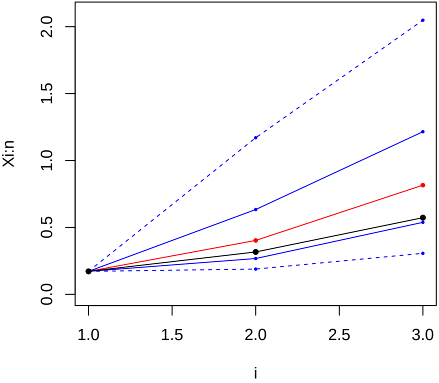

Figure: Predictions (red) for $X_{s:3}$ from $X_{1:3}$ for $s = 2, 3$ jointly with the exact values (black) for a simulated sample from an ODTHLS model.

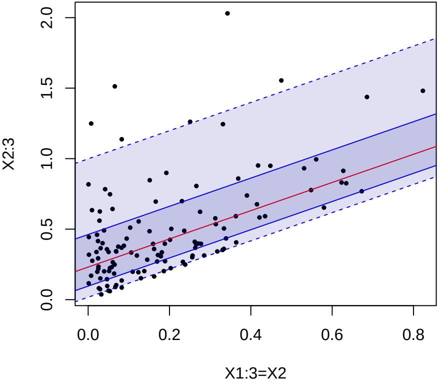

Figure: Scatterplots of 100 simulated samples from $(X_{1:3}, X_{2:3})$, for the case $X_{1:3} = X_2$ jointly with the median regression curves (red) and 50% (dark grey) and 90% (light grey) prediction bands.

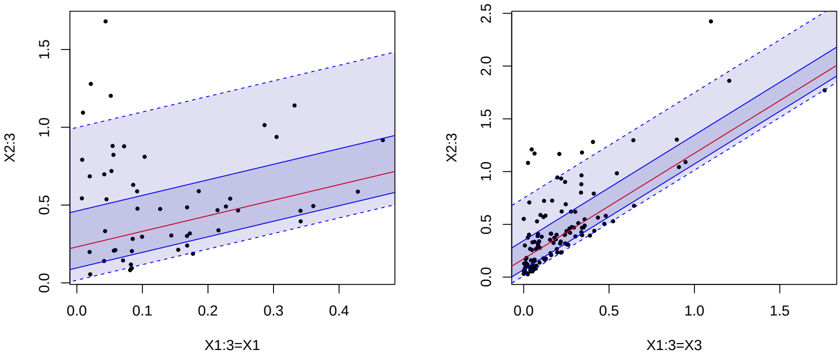

Figure: Scatterplots of 100 simulated sample from $(X_{1:3}, X_{2:3})$ for the ODTHLS model jointly with the median regression curves (red) and 50% (dark grey) and 90% (light grey) prediction bands for the cases $X_{1:3} = X_1$ (left) and $X_{1:3} = X_3$ (right).

Example 2

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- Then the predictions for the first and the second order statistics based on the median (left) and the mean (right) are

$$\widehat{X}_{1:3} = \frac{\log 2}{M(\emptyset)} = 0.13863, \qquad \tilde{X}_{1:3} = \frac{1}{M(\emptyset)} = 0.2,$$

$$\widehat{X}_{2:3} = \frac{\log 2}{M(\emptyset)} + \frac{\log 2}{M(2)} = 0.36968, \qquad \tilde{X}_{2:3} = \frac{1}{M(\emptyset)} + \frac{1}{M(2)} = 0.53333.$$

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- The prediction of $X_{2:3}$ can be obtained also by the median of the convolution $X_{1:3} + (X_{2:3} - X_{1:3})$.

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- The prediction of $X_{2:3}$ can be obtained also by the median of the convolution $X_{1:3} + (X_{2:3} - X_{1:3})$.
- Given that $X_{1:3} = X_2$, these interarrival times are independent and exponential with parameters $M(\emptyset) = 5$ and $M(2) = 3$.

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- The prediction of $X_{2:3}$ can be obtained also by the median of the convolution $X_{1:3} + (X_{2:3} - X_{1:3})$.
- Given that $X_{1:3} = X_2$, these interarrival times are independent and exponential with parameters $M(\emptyset) = 5$ and $M(2) = 3$.
- The median of such a distribution gives another prediction for $X_{2:3}$ as 0.44139.

Example 2

- If we know that the first and the second order statistics are assumed in X_2 and X_1, the maximum $X_{3:3}$ can be predicted by the median and the mean, respectively, as

$$\widehat{X}_{3:3} = \frac{\log 2}{M(\emptyset)} + \frac{\log 2}{M(2)} + \frac{\log 2}{M(2,1)} = 0.71625,$$

and

$$\tilde{X}_{3:3} = \frac{1}{M(\emptyset)} + \frac{1}{M(2)} + \frac{1}{M(2,1)} = 1.03333.$$

Example 2

- In addition, we can obtain the prediction of $X_{3:3}$ based on the convolution $Y = X_{1:3} + (X_{2:3} - X_{1:3}) + (X_{3:3} - X_{2:3})$, given that $X_{1:3} = X_2, X_{2:3} = X_1$.

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- The interarrival times are independent and have exponential distributions with parameters $M(\emptyset) = 5$, $M(2) = 3$ and $M(2,1) = 2$.

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- In practice, the parameters of the model should be estimated (see the paper).

Main references

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- Questions?