4. Appendix.

Proof of Lemma 3.1 . From the definitions, we have

$$m(t) = \frac{G(t)}{R(t)} = \frac{pG_1(t) + (1-p)G_0(t)}{pR_1(t) + (1-p)R_0(t)},$$

where $R_i = 1 - F_i$ and $G_i = m_i R_i$, $i = 0, 1$, and by differentiating, we obtain

$$\frac{\partial}{\partial p} m(t) = \frac{R_1(t)R_0(t)}{R_0(t)}(m_1(t) - m_0(t)),$$

and hence the stated result. The proofs for the hr and st orderings are similar.

Proof of Lemma 3.2 . Let $\varepsilon > 0$ be such that $\liminf_{t\to\infty} m_1(t)/m_i(t) > 1 + \varepsilon$ for $i = 2, 3, ..., n$. Then there exists t_i such that

$$m_1(t)/m_i(t) > 1 + \varepsilon > 1$$

for all $t \geq t_i$ and for $i = 2, 3, ..., n$. So, we have

$$\begin{aligned} \frac{m_i(t)R_i(t)}{m_1(t)R_1(t)} &= \frac{m_i(0)}{m_1(0)} \exp\left\{ -\int_0^t \left(\frac{1}{m_i(x)} - \frac{1}{m_1(x)} \right) dx \right\} \\ &\leq \frac{m_i(0)}{m_1(0)} \exp\left\{ -\int_0^{t_i} \left(\frac{1}{m_i(x)} - \frac{1}{m_1(x)} \right) dx \right\} \exp\left\{ -\int_{t_i}^t \frac{\varepsilon}{m_1(x)} dx \right\} \end{aligned}$$

for $t \geq t_i$, $R_i = 1 - F_i$ and $i = 2, 3, ..., n$, where the first equality is obtained from the inversion formula for the MRL function. Letting $t \to \infty$, we obtain

$$\lim_{t\to\infty} \frac{m_i(t)R_i(t)}{m_1(t)R_1(t)} = 0$$

for $i = 2, 3, ..., n$.

By assumption (3.7), $m_1(t)/m_i(t)$ is bounded and hence

$$\lim_{t\to\infty} \frac{R_i(t)}{R_1(t)} = \lim_{t\to\infty} \frac{m_i(t)R_i(t)}{m_1(t)R_1(t)} \frac{m_1(t)}{m_i(t)} = 0$$

for $i = 2, 3, \ldots, n$.

Then, note that

$$\frac{m(t)R(t)}{m_1(t)R_1(t)} = \left(p_1 + \sum_{i=2}^n p_i \frac{m_i(t)R_i(t)}{m_1 R_1(t)} \right).$$

Therefore,

$$\lim_{t\to\infty} \frac{m(t)R(t)}{m_1(t)R_1(t)} = p_1 > 0. \tag{4.1}$$

Moreover, note that

$$\frac{m(t)}{m_1(t)} = \frac{m_1(t)R_1(t)}{m(t)R(t)}\left(p_1 + \sum_{i=2}^{n} p_i \frac{R_i(t)}{R_1(t)}\right). \tag{4.2}$$

Therefore, we have that $\lim_{t\to\infty} m(t)/m_1(t) = 1$.

Proof of Lemma 3.3 . From the definitions, we have

$$\frac{1}{m(t)} = \alpha(t)\frac{1}{m_1(t)} + (1-\alpha(t))\frac{1}{m_2(t)}.$$

Therefore,

$$\left(\frac{1}{m(t)}\right)' = \alpha(t)\left(\frac{1}{m_1(t)}\right)' + (1-\alpha(t))\left(\frac{1}{m_2(t)}\right)' + \alpha'(t)\left(\frac{1}{m_1(t)} - \frac{1}{m_2(t)}\right), \tag{4.3}$$

where

$$\alpha'(t) = -\alpha(t)(1-\alpha(t))\left(\frac{1}{m_1(t)} - \frac{1}{m_2(t)}\right).$$

The proof is completed by replacing the preceding expression in (4.3).

Proof of Lemma 3.4 . The function g can be written as

$$g(x) = cx\frac{x-a}{x-b},$$

where

$$a = \frac{(n-k)(k+1)}{(n-k-1)(k+2)},$$

$$b = \frac{(n-k+1)k}{(n-k)(k+1)}$$

and

$$c = \frac{(n-k-1)(k+2)}{(n-k)(k+1)} > 0.$$

Lengthy straightforward computation gives

$$a - b = \frac{n^2 - 2nk + 2k^2 + 2k}{(n-k-1)(k+2)(n-k)(k+1)} > 0.$$

Therefore, $a > b$ and hence $(x-a)/(x-b)$ is an increasing function and so is g for $x > 0$.