Locally uniformly rotund ($F$)-norms

J. Orihuela\textsuperscript{1}

\textsuperscript{1}Department of Mathematics
University of Murcia

Seminario de Análisis Matemático. Universidad de Valencia.
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The coauthors

- J.O. *On $T_p$ locally uniformly rotund norms* Set Valued and Variational Analysis, 2013
Metrizable topological vector spaces and \((F)\)-norms.
Kadec norm implies LUR renorming with \((F)\)-norm
Descriptive Bananch spaces
Bing-Nagata-Smirnov-Stone meets Kadec
Construction of LUR \((F)\)-norms in descriptive Bananch spaces
Metrizable topological vector spaces

S. Banach “Theory of linear operations”
V. L. Klee
G. Köthe
H. Jarchow
N. Kalton

Linear metric space

uniformly

Hilbert space

 uniformly

Aharoni - Maurey - Misiagin

Is there an equivalent translation invariant metric on $L^0(S, S, P)$ such that $1\tilde{x}_1 := d(\tilde{x}, 0)$ vanishes
\[
\lim_{n \to \infty} [2|\tilde{x}_n|^2 + 2|\tilde{x}|^2 - |\tilde{x} + \tilde{x}_n|^2] = 0 \Rightarrow (\tilde{x}_n) \overset{w}{\to} \tilde{x}?
\]
A function \( \| \cdot \| : X \to [0, +\infty) \) is called \((F)\)-norm on the vector space \( X \) if the following properties are satisfied:

- \( x = 0 \) if, and only if, \( \|x\| = 0 \);
- \( \|\lambda x\| \leq \|x\|, \text{ if } |\lambda| \leq 1 \) and \( x \in X \);
- \( \|x + y\| \leq \|x\| + \|y\| \) for every \( x, y \in X \);
- \( \lim_{n} \|\lambda x_{n}\| = 0, \text{ if } \lim_{n} \|x_{n}\| = 0 \) for every \( (x_{n})_{n \in \mathbb{N}} \subseteq X \) and \( \lambda \in \mathbb{R} \);
- \( \lim_{n} \|\lambda_{n} x\| = 0, \text{ if } \lim_{n} \lambda_{n} = 0 \) for every \( (\lambda_{n})_{n \in \mathbb{N}} \) and \( x \in X \).
Theorem

If a normed space \((X, \| \cdot \|)\) has a Kadec norm there is an equivalent Kadec and locally uniformly rotund \((F)\)-norm \(\| \cdot \|_1\) on \(X\), i.e. an \((F)\)-norm \(\| \cdot \|_1\) such that the topology determined by the \((F)\)-norm \(\| \cdot \|_1\) on \(X\) coincides with the norm topology and moreover we have:

1. the weak and norm topologies coincide on every sphere \(\{x \in X : \|x\| \leq \rho\}\) for \(\rho > 0\).

2. For every \((x_n)_{n \in \mathbb{N}} \subseteq X\) and \(x \in X\) we have \(\lim_{n \to \infty} \|x_n - x\| = 0\) whenever

\[
\lim_{n \to \infty} (2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2) = 0
\]
The lexicographic product $K = [0,1]^\omega$ gives $C(K)$ with a Kadec renorming, but $C(K)$ does not have equivalent LUR norm.

"Continuous functions on totally ordered spaces that are compact in their order topologies" J. Funct. Anal. 178, 23-63 (2000)

A. Motto
S. Troyanski
M. Valdivia
3.0

Theorem

Let \((X, \| \cdot \|)\) be a normed space with a norming subspace \(Z\) in \(X^*\). TFAE:

1. There is a norm-equivalent and \(\sigma(X, Z)\)-lower semicontinuous \((F)\)-norm \(\| \cdot \|_0\) on \(X\) such that \(\sigma(X, Z)\) and norm topologies coincide on the unit sphere

\[
\{ x \in X : \| x \|_0 = 1 \}
\]

2. There are isolated families \(\mathcal{B}_n\) for the \(\sigma(X, Z)\)-topology, \(n = 1, 2, \cdots\) such that for every \(x \in X\) and every \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) and some set \(B \in \mathcal{B}_n\) with the property that \(x \in B\) and

\[
\| \cdot \| - \text{diam}(B) < \varepsilon
\]
Let \((X, \| \cdot \|)\) be a normed space with a norming subspace \(Z\) in \(X^*\). TFAE:

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Descriptive Banach spaces

Fragmentability:

$\varepsilon$-fragmented if $\forall F \subset A \exists W$ w-open $W \cap F \neq \emptyset$ and $\lim_{n \to \infty} \text{diam} (W \cap F) < \varepsilon$

$\varepsilon$-$\sigma$-fragmented $A = \bigcup_{n=1}^{\infty} A_{n,\varepsilon}$ s.t.

$A_{n,\varepsilon}$ is $\varepsilon$-fragmented

$\varepsilon$-descriptive $A = \bigcup_{n=1}^{\infty} A_{n,\varepsilon}$ s.t.

$\forall x \in A_{n,\varepsilon} \exists W$ w-open, $x \in W$ and $\lim_{n \to \infty} \text{diam} (W \cap A_{n,\varepsilon}) < \varepsilon$

$\varepsilon$-descriptive $\iff$ There are families $\mathcal{O}_n$, relatively discrete for the weak topology, s.t. $\forall x \in X, \forall \varepsilon \in \mathcal{O}_n \exists W$, $x \in B \in \mathcal{O}_n$, $\lim_{n \to \infty} \text{diam} (B_n) < \varepsilon$

Kadec norm $\Rightarrow$ descriptive $\Rightarrow \sigma$-fragmented

$\omega^*$-descriptive $\iff$ dual LUR (M. Raja)

J. Orihuela
Theorem (Kadec metrization)

Let \((X, \| \cdot \|)\) be a normed space with a norming subspace \(Z\) in \(X^*\). Then the following conditions are equivalent:

1. The normed space \(X\) is \(\sigma(X, Z)\)-descriptive; i.e there are isolated families \(\mathcal{B}_n\) for the \(\sigma(X, Z)\)-topology, \(n = 1, 2, \ldots\) such that for every \(x \in X\) and every \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) and some set \(B \in \mathcal{B}_n\) with the property that \(x \in B\) and 
\[\| \cdot \| - \text{diam}(B) < \varepsilon\]

2. The norm topology admits a basis \(\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n\) such that each one of the families \(\mathcal{B}_n\) is norm discrete and \(\sigma(X, Z)\)-isolated
Theorem (Kadec metrization)

Let $(X, \| \cdot \|)$ be a normed space with a norming subspace $Z$ in $X^*$. Then the following conditions are equivalent:

1. The normed space $X$ is $\sigma(X, Z)$-descriptive; i.e. there are isolated families $B_n$ for the $\sigma(X, Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in B_n$ with the property that $x \in B$ and $\| \cdot \| - \text{diam}(B) < \varepsilon$

2. The norm topology admits a basis $\mathcal{B} = \bigcup_{n \in \mathbb{N}} B_n$ such that each one of the families $B_n$ is norm discrete and $\sigma(X, Z)$-isolated
LUR \((F)\)-renorming

**Theorem**

Let \((X, \| \cdot \|)\) be a normed space with a norming subspace \(Z\) in \(X^*\). TFAE:

1. **The normed space** \(X\) **is** \(\sigma(X, Z)\)-**descriptive**; i.e. there are isolated families \(\mathcal{B}_n\) for the \(\sigma(X, Z)\)-topology, \(n = 1, 2, \cdots\) such that for every \(x \in X\) and every \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) and some set \(B \in \mathcal{B}_n\) with the property that \(x \in B\) and 
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2. **There is a norm-equivalent,** \(\sigma(X, Z)\)-**lower semicontinuous and LUR \((F)\)-norm** \(\| \cdot \|_0\) **on** \(X\); i.e. such that for every \((x_n)_{n \in \mathbb{N}} \subset X\) and \(x \in X\) we have 
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\lim_{n \to +\infty} \| x_n - x \| = 0
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Method of proof

Definition ($p$-convex set and hull)

Let $A$ be a subset of a vector space $X$ and $p \in (0, 1]$. $A$ is said to be $p$-convex if for every $x, y \in A$ and $\tau, \mu \in [0, 1]$ such that $\tau^p + \mu^p = 1$ we have $\tau x + \mu y \in A$.

If $A$ is $p$-convex and absorbent, its $p$-Minkowski functional is

$$p_A(x) := \inf\{\lambda^p : \lambda > 0, x \in \lambda A\}$$

$p_A$ is a $p$-seminorm, i.e we have

- $p_A(\lambda x) = |\lambda|^p p_A(x)$
- $p_A(x + y) \leq p_A(x) + p_A(y)$.

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J. Orihuela
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p-convex sets

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**Definition**

A real function $\phi : X \rightarrow \mathbb{R}$ is said to be $p$-convex for $p \in (0, 1]$ if

$$
\phi(\tau x + \mu y) \leq \tau \phi(x) + \mu \phi(y)
$$

whenever $\tau \geq 0$, $\mu \geq 0$ and $\tau^p + \mu^p = 1$.

- the epigraph of $\phi$ is $p$-convex if and only if $\phi$ is $p$-convex;
- if $\phi$ is convex and $\phi(0) = 0$, then $\phi$ is $p$-convex for every $p \in (0, 1]$;
- if $\phi_p$ is $p$-convex, $\phi_q$ is $q$-convex, with $0 < p \leq q < 1$ and both of them are non-negative, then $\phi_p + \phi_q$ is $p$-convex;
- if $\phi : X \rightarrow \mathbb{R}$ is $p$-convex for some $0 < p \leq 1$ and bounded from above in a neighbourhood of $x \in X$, then $\phi$ is locally Lipschitz at $x$
- $\tau^p \mu^p (\phi(x) - \phi(y))^2 \leq \tau^p \phi(x)^2 + \mu^p \phi(y)^2 - \phi(\tau x + \mu y)^2$

whenever $\tau^p + \mu^p = 1$ and $\tau \geq 0$, $\mu \geq 0$. 

J. Orihuela
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- if $\phi : X \rightarrow \mathbb{R}$ is p-convex for some $0 < p \leq 1$ and bounded from above in a neighbourhood of $x \in X$, then $\phi$ is locally Lipschitz at $x$.
- $$\tau^p \mu^p (\phi(x) - \phi(y))^2 \leq \tau^p \phi(x)^2 + \mu^p \phi(y)^2 - \phi(\tau x + \mu y)^2$$
  whenever $\tau^p + \mu^p = 1$ and $\tau \geq 0$, $\mu \geq 0$.  

J. Orihuela
Definition (p-distance)

Let $C$ be a $w^*$-compact and $p$-convex subset of $X^{**}$, $0 < p \leq 1$,

$$\varphi(x) := \inf_{c^{**} \in C} \left\{ \sup \{ | \langle x - c^{**}, z^* \rangle | : z^* \in B_{X^*} \cap Z \} \right\}$$

$\varphi$ is a $p$-convex, $\sigma(X, Z)$-lower semicontinuous and $1$-Lipschitz map from $X$ to $[0, +\infty)$.

Definition

A family $\mathcal{B} := \{ B_i : i \in I \}$ of subsets in the normed space $X$ is said to be $p$-isolated for the $\sigma(X, Z)$-topology if for every $i \in I$

$$B_i \cap \overline{\text{co}_p \{ B_j : j \neq i, j \in I \}}^{\sigma(X,Z)} = \emptyset.$$
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B_i \cap \text{co}_p \{B_j : j \neq i, j \in I\}^{\sigma(X,Z)} = \emptyset.
$$
Orthogonal p-convex sets and functions

Theorem

Let $\mathcal{B} := \{ B_i : i \in I \}$ be an uniformly bounded family of subsets of $X$. The following are equivalent:

1. The family $\mathcal{B}$ is p-isolated for the $\sigma(X, Z)$-topology; i.e.
   $$ B_i \cap \overline{\operatorname{co}_p\{ B_j : j \neq i, j \in I \}}^{\sigma(X, Z)} = \emptyset. $$

   for every $i \in I$

2. There exists a family $\mathcal{L} := \{ \varphi_i : X \rightarrow [0, +\infty) : i \in I \}$ of p-convex and $\sigma(X, Z)$-lower semicontinuous functions such that for every $i \in I$

   $$ \{ x \in X : \varphi_i(x) > 0 \} \cap \bigcup_{j \in I} B_j = B_i. $$

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**Theorem**

Let $\mathcal{B} := \{B_i \mid i \in I\}$ be an uniformly bounded family of subsets of $X$. The following are equivalent:

1. The family $\mathcal{B}$ is $p$-isolated for the $\sigma(X, Z)$-topology; i.e.
   \[
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   \[
   \{x \in X : \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i.
   \]
Lemma (Decomposition lemma)

Let $\mathcal{B}$ be a uniformly bounded and isolated family of sets for the $\sigma(X, Z)$ topology. Then for every $B \in \mathcal{B}$ we can write

$$B = \bigcup_{n=1}^{\infty} B_n$$

in such a way that, for every $n \in \mathbb{N}$ fixed, the family

$$\{ B_n : B \in \mathcal{B} \}$$

is $\sigma(X, Z)$-$q$-isolated whenever $q < \frac{\log 2}{\log 4n}$. 

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Theorem

Let $\mathcal{B} = \{B_i : i \in I\}$ be a uniformly bounded and $p$-isolated family of subsets of $X$ for the $\sigma(X, Z)$ topology. Then there is a norm-equivalent $\sigma(X, Z)$-lower semicontinuous $p$-norm $q_\mathcal{B}(\cdot)$ on $X$ such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ the condition

$$\lim_{n \to +\infty} [2q_\mathcal{B}^2(x_n) + 2q_\mathcal{B}^2(x) - q_\mathcal{B}^2(x + x_n)] = 0,$$

implies that:

1. there exists $n_0 \in \mathbb{N}$ such that
   $$x_n, \frac{x_n + x}{2^{1/p}} \notin \text{co}_p \{B_i : i \neq i_0, \ i \in I\}^{\sigma(X, Z)} \text{ for every } n \geq n_0;$$

2. for every positive $\delta$ there is $n_\delta \in \mathbb{N}$ such that
   $$x_n \in \text{co}(B_{i_0} \cup \{0\}) + B(0, \delta)^{\sigma(X, Z)} \text{ whenever } n \geq n_\delta.$$
Let $\mathcal{B} = \{B_i : i \in I\}$ be a uniformly bounded and $p$-isolated family of subsets of $X$ for the $\sigma(X, Z)$ topology. Then there is a norm-equivalent $\sigma(X, Z)$-lower semicontinuous $p$-norm $q_\mathcal{B}(\cdot)$ on $X$ such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ the condition

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Let $\mathcal{B} = \{B_i : i \in I\}$ be a uniformly bounded and $p$-isolated family of subsets of $X$ for the $\sigma(X,Z)$ topology. Then there is a norm-equivalent $\sigma(X,Z)$-lower semicontinuous $p$-norm $q_B(\cdot)$ on $X$ such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ the condition

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   for every $n \geq n_0$;

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2. for every positive $\delta$ there is $n_\delta \in \mathbb{N}$ such that $x_n \in \text{co}(B_{i_0} \cup \{0\}) + B(0, \delta)^{\sigma(X, Z)}$ whenever $n \geq n_\delta$. 
Fix isolated families $B_n$ for the $\sigma(X, Z)$-topology such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in B_n$ with $x \in B$ and $\| \cdot \| - \text{diam}(B) < \epsilon$.

$\{B_n\}_{n \in \mathbb{N}}$ are assumed to be $p_n$-isolated for some sequence $p_n \in (0, 1]$ by decomposition lemma.

Consider the $p_n$-norms $q_{B_n}(\cdot)$ constructed using the $p$-Localization Theorem

$$
F^2_{B}(x) := \|x\|^2 + \sum_{n=1}^{\infty} \frac{1}{\zeta_{p_n}^2 2^n} q^2_{B_n}(x) \quad \text{where} \quad q_{B_n}(x) \leq \zeta_{p_n}^p \|x\|^{p_n} \leq \zeta_{p_n}^p \max\{1, \|x\|\}.
$$

If $\lim_{n \to +\infty} [2F^2_{B}(x_n) + 2F^2_{B}(x) - F^2_{B}(x + x_n)] = 0$ then $\lim_{n \to +\infty} [2q^2_{B_m}(x_n) + 2q^2_{B_m}(x) - q^2_{B_m}(x + x_n)] = 0$ for all $m$.

If $\epsilon > 0$, $m \in \mathbb{N}$ and $B_0 \in B_m$ with $x \in B_0 \subseteq x + \frac{\epsilon}{2} B_X$ there exists $n_{\frac{\epsilon}{2}}$ such that $x_n \in \text{co}(B_0 \cup \{0\}) + B(0, \frac{\epsilon}{2})^\sigma(X, Z)$ whenever $n \geq n_{\frac{\epsilon}{2}}$. 

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\(\| \cdot \| \text{dist}(x_n, l_x) \leq \varepsilon \text{ for } n \geq n_{\varepsilon}^2\)

- there is \(r_{(n,\varepsilon)} \in [0, 1]\) such that \(\|x_n - r_{(n,\varepsilon)}x\| \leq \varepsilon\) for \(n \geq n_{\varepsilon}^2\).

- By induction we select integers \(n_1 < n_2 < \cdots < n_k < \cdots\)
  such that \(\|x_{n_k} - r_{(n_k,1/k)}x\| \leq \frac{1}{k}\).

- By compactness there is a sequence of integers \(k_1 < k_2 < \cdots < k_j < \cdots\)
  such that \(\lim_{j \to +\infty} r_{(n_k,1/k_j)} = r \in [0, 1]\) and \(\| \cdot \| - \lim_{j \to +\infty} x_{n_{k_j}} = rx\).

- If \(\|x\|_Z = 1\) we also have \(\lim_n \|x_n\|_Z = \|x\|_Z = 1\) and it
  follows that \(r = 1\), so we have found a subsequence \((x_{n_j})\)
  of the given sequence \((x_n)\) which norm converges to \(x\).

- Since the reasoning is valid for every subsequence too, the
  proof is over
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- there is \( r_{(n, \epsilon)} \in [0, 1] \) such that \( \| x_n - r_{(n, \epsilon)}x \| \leq \epsilon \) for \( n \geq n_{\epsilon} \).

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  If \( \| x \|_Z = 1 \) we also have \( \lim_n \| x_n \|_Z = \| x \|_Z = 1 \) and it follows that \( r = 1 \), so we have found a subsequence \( (x_{n_j}) \)
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there is \( r_{(n,\epsilon)} \in [0, 1] \) such that \( \| x_n - r_{(n,\epsilon)} x \| \leq \epsilon \text{ for } n \geq n_{\frac{\epsilon}{2}}. \)

By induction we select integers \( n_1 < n_2 < \cdots < n_k < \cdots \)

such that \( \| x_{n_k} - r_{(n_k,1/k)} x \| \leq \frac{1}{k}. \)

By compactness there is a sequence of integers

\[ k_1 < k_2 < \cdots < k_j < \cdots \text{ such that} \]

\[ \lim_{j \to +\infty} r_{(n_{k_j},1/k_j)} = r \in [0, 1] \text{ and } \| \cdot \| - \lim_{j \to +\infty} x_{n_{k_j}} = r x \]

If \( \| x \|_Z = 1 \) we also have \( \lim_n \| x_n \|_Z = \| x \|_Z = 1 \) and it follows that \( r = 1 \), so we have found a subsequence \((x_{n_j})\)

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there is \( r(n,\epsilon) \in [0, 1] \) such that \( \| x_n - r(n,\epsilon)x \| \leq \epsilon \text{ for } n \geq n_{\frac{\epsilon}{2}} \).

By induction we select integers \( n_1 < n_2 < \cdots < n_k < \cdots \) such that \( \| x_{n_k} - r(n_k,1/k)x \| \leq \frac{1}{k} \).

By compactness there is a sequence of integers \( k_1 < k_2 < \cdots < k_j < \cdots \) such that
\[ \lim_{j \to +\infty} r(n_{k_j},1/k_j) = r \in [0, 1] \text{ and } \| \cdot \| - \lim_{j \to +\infty} x_{n_{k_j}} = rx \]

If \( \| x \|_Z = 1 \) we also have \( \lim_{n} \| x_n \|_Z = \| x \|_Z = 1 \) and it follows that \( r = 1 \), so we have found a subsequence \((x_{n_j})\) of the given sequence \((x_n)\) which norm converges to \( x \).

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By compactness there is a sequence of integers \( k_1 < k_2 < \cdots < k_j < \cdots \) such that

\[ \lim_{j \to +\infty} r_{(n_{k_j},1/k_j)} = r \in [0, 1] \text{ and } \| \cdot \| - \lim_{j \to +\infty} x_{n_{k_j}} = r x \]

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Since the reasoning is valid for every subsequence too, the proof is over.
Lemma

Let $X$ be a topological space, $S$ be a set and $\varphi_s, \psi_s : X \to [0, +\infty)$ lower semicontinuous functions such that $\sup_{s \in S}(\varphi_s(x) + \psi_s(x)) < +\infty$ for every $x \in X$. Define

$$\varphi(x) = \sup_{s \in S} \varphi_s(x), \quad \theta_m(x) = \sup_{s \in S} (\varphi_s(x) + 2^{-m} \psi_s(x)),$$

and $\theta(x) = \sum_{m \in \mathbb{N}} 2^{-m} \theta_m(x)$. Assume further that $\{x_\sigma : \sigma \in \Sigma\}$ is a net converging to $x \in X$ and $\theta(x_\sigma) \to \theta(x)$. Then there exists a finer net $\{x_\gamma\}_{\gamma \in \Gamma}$ and a net $\{i_\gamma\}_{\gamma \in \Gamma} \subseteq S$ such that

$$\lim_{\gamma \in \Gamma} \varphi_{i_\gamma}(x_\gamma) = \lim_{\gamma \in \Gamma} \varphi_{i_\gamma}(x) = \lim_{\gamma \in \Gamma} \varphi(x_\gamma) = \sup_{s \in S} \varphi_s(x)$$

and

$$\lim_{\gamma \in \Gamma} (\psi_{i_\gamma}(x_\gamma) - \psi_{i_\gamma}(x)) = 0.$$
Theorem

Let $\mathcal{B} := \{ B_i : i \in I \}$ be an uniformly bounded and p-isolated family of subsets of $X$ for the $\sigma(X, Z)$-topology and some $p \in (0, 1]$. Then there is an equivalent $\sigma(X, Z)$-lower semicontinuous quasinorm, with p-power a p-norm, $\| \cdot \|_\mathcal{B}$ on $X$ such that: for every net $\{ x_\alpha : \alpha \in A \}$ and $x$ in $X$ with $x \in B_{i_0}$ for $i_0 \in I$, the conditions $\sigma(X, Z) - \lim_\alpha x_\alpha = x$ and $\lim_\alpha \| x_\alpha \|_\mathcal{B} = \| x \|_\mathcal{B}$ imply that

1. there exists $\alpha_0 \in A$ such that $x_\alpha$ is not in $\overline{\text{co}_p \{ B_i : i \neq i_0, i \in I \}^{\sigma(X, Z)}}$ for $\alpha \geq \alpha_0$;

2. for every positive $\delta$ there exists $\alpha_\delta \in A$ such that

$$x, x_\alpha \in \overline{\text{co}( B_{i_0} \cup \{ 0 \} ) + B(0, \delta)^{\sigma(X, Z)}}$$

whenever $\alpha \geq \alpha_\delta$. 
We can construct norm-equivalent and $\sigma(X,Z)$-lower semicontinuous F-norms $F_1$ and $F_2$ such that $F_1$ has the LUR property and $F_2$ the Kadec property.

Then we define

$$\| \cdot \|_1(x)^2 := F_1(\cdot)^2 + F_2(\cdot)^2$$

which is an equivalent $\sigma(X,Z)$-lower semicontinuous F-norm which has both Kadec and the LUR property.

$$\lim_{n \to \infty} [2 \| x \|^2_1 + 2 \| x_n \|^2_1 - \| x + x_n \|^2_1] = 0$$

is equivalent to

$$\lim_{n \to \infty} [2F_i(x)^2 + 2F_i(x_n)^2 - F_i(x + x_n)^2] = 0$$

for $i = 1, 2$, and LUR property of $F_1$ is translated to $\| \cdot \|_1$.

If $\{x_\alpha : \alpha \in (A, \succ)\}$ is a net in $X$ which converges to $x$ in the topology $\sigma(X,Z)$ and $\lim_{\alpha \in A} \| x_\alpha \|_1 = \| x \|_1$ it follows that $\lim_{\alpha \in A} F_i^2(x_\alpha) = F_i^2(x)$ for $i = 1, 2$. Thus Kadec property of $F_2$ is translated to $\| \cdot \|_1$. 
We can construct norm-equivalent and \( \sigma(X, Z) \)-lower semicontinuous F-norms \( F_1 \) and \( F_2 \) such that \( F_1 \) has the LUR property and \( F_2 \) the Kadec property.

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\[
\lim_{n \to \infty} [2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2] = 0
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If \( \{x_\alpha : \alpha \in (A, \succ)\} \) is a net in \( X \) which converges to \( x \) in the topology \( \sigma(X, Z) \) and \( \lim_{\alpha \in A} \|x_\alpha\|_1 = \|x\|_1 \) it follows that \( \lim_{\alpha \in A} F_1^2(x_\alpha) = F_1^2(x) \) for \( i = 1, 2 \). Thus Kadec property of \( F_2 \) is translated to \( \| \cdot \|_1 \).
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$$\lim_{\alpha \in A} F_i^2(x_\alpha) = F_i^2(x)$$

for $i = 1, 2$. Thus Kadec property of $F_2$ is translated to $\| \cdot \|_1$. 
THANK YOU VERY MUCH !!!!!