Compactness, Optimization and Risk

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The coauthors

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- M. Ruiz Galán and J.O. Lebesgue Property for Convex Risk Measurers on Orlicz Spaces
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Weak Compactness Theorem of R.C. James

**Theorem**

A *Banach space is reflexive if and only if each continuous linear functional attains its supremum on the unit ball*

**Theorem**

A bounded and weakly closed subset $K$ of a Banach space is weakly compact if and only if each continuous linear functional attains its supremum on $K$

Weak Compactness Theorem of R.C. James

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Let us fix a Banach space $E$ with dual $E^*$

- $K$ is a closed convex set in the Banach space $E$
- $ι_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on $K$ at $x_0 \in K \iff ι_K(y) - ι_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min \{ι_K(\cdot) - x^*(\cdot)\}$$

on $E$ for every $x^* \in E^*$ has always solution if and only if the set $K$ is weakly compact
The Theorem of James as a minimization problem

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A monetary utility function is a concave non-decreasing map

\[ U : \mathbb{L}^{\infty}(\Omega, \mathcal{F} \mathcal{P}) \rightarrow [-\infty, +\infty) \]

with \( \text{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset \) and

\[ U(X + c) = U(X) + c, \text{ for } X \in \mathbb{L}^{\infty}, c \in \mathbb{R} \]

Defining \( \rho(X) = -U(X) \) the above definition of monetary utility function yields the definition of a convex risk measure.

The space of financial positions \( \mathcal{X} \) verifies \( \mathbb{L}^{\infty} \subseteq \mathcal{X} \subseteq \mathcal{L}^{0} \) and monetary risk measures \( \rho \) are defined on \( \mathcal{X} \).
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Risk measures

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Convex Monetary Risk Measure:

\[ B(t) \]

\[ \text{TODAY} \rightarrow \text{TIME HORIZON} \]

\[ \left\{ \begin{array}{l}
\alpha) \rho(X) \leq \rho(\bar{X}) \\
\beta) \rho(\bar{X} + (1-\bar{X})Y) \\
\gamma) \rho(\bar{X} + \mu) = \rho(\bar{X} - \mu)
\end{array} \right. \]
Theorem (Jouini-Schachermayer-Touzi)

Let $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a monetary utility function with the Fatou property and $U^* : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ its Fenchel-Legendre transform. They are equivalent:

1. $\{U^* \leq c\}$ is $\sigma(L^1, L^\infty)$-compact subset for all $c \in \mathbb{R}$
2. For every $X \in L^\infty$ the infimum in the equality

$$U(X) = \inf_{Y \in L^1} \{U^*(Y) + \mathbb{E}[XY]\},$$

is attained
3. For every uniformly bounded sequence $(X_n)$ tending a.s. to $X$ we have

$$\lim_{n \to \infty} U(X_n) = U(X).$$
Convex Monetary Risk Measure:

\[ B(t) \]

\[ \begin{align*}
\alpha) \rho(X) & \leq \rho(Y) \\
\beta) \rho(\sum \lambda_i X_i) & = \sum \lambda_i \rho(X_i) \\
\gamma) \rho(X + m) & = \rho(X) - m
\end{align*} \]

Today has Fatou if \( X_n \downarrow X \Rightarrow \rho(X_n) \uparrow \rho(X) \iff \sigma(L^\infty, L^1) \) lower semicontinuous.

Order sequentially continuous \( \iff |X_n| \leq X \iff X_n \xrightarrow{a.s.} X \)

\( \lim_{n \to \infty} \rho(X_n) = \rho(X) \)

has Lebesgue property.
Let \( U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \) be a monetary utility function with the Fatou property and \( U^* : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty] \) its Fenchel-Legendre transform. They are equivalent:

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   \[ \lim_{n \to \infty} U(X_n) = U(X). \]
Minimizing $\{V(Y) + E(X \cdot Y) : Y \in L^1\}$

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Tools for the proof

- The proof in [JST] is for separable $L^1(\Omega, \mathcal{F}, \mathbb{P})$. The separability is needed to show $2) \Rightarrow 1)$ with a variant of the separable James’ compactness Theorem.

- Delbaen has given a proof for general non separable spaces using an homogenisation trick. He shows how to apply directly the non separable James’ compactness Theorem in the duality $\langle L^1(\Omega, \mathcal{F}, \mathbb{P}), L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rangle$. 
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Minimizing \( \{ V(y) + x^*(y) : y \in E \} \)

**Theorem (J. Orihuela)**

*Let \( E \) be a separable Banach space,\n
\[ \alpha : E \to \mathbb{R} \cup \{ \infty \} \]

proper, convex l.s.c. with \( \text{dom}(\alpha) = \{ \alpha < \infty \} \) a bounded subset of \( E \). Suppose that there is \( c \in \mathbb{R} \) such that the level set \( \{ \alpha \leq c \} \) fails to be weakly compact. Then there is \( x^* \in E^* \) such that, the infimum

\[ \inf_{x \in E} \{ \langle x, x^* \rangle + \alpha(x) \} \]

is not attained.*
Minimizing \( \{ V(y) + x^*(y) : y \in E \} \)

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Lemma (Simons)

Let $\Gamma$ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If $\Lambda$ is a subset of $\Gamma$ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup \left\{ \sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma \right\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then

$$\sup_{b \in \Lambda} \limsup_{n \to \infty} z_n(b) \geq \inf_{\Gamma} \sup_{w} w \in \text{co}\{z_n : n \in \mathbb{N}\}.$$
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then

$$\sup \{ \limsup_{n \to \infty} z_n(b) \} \geq \inf_{b \in \Lambda} \sup_{n \to \infty} \{ w \in \text{co}\{z_n : n \in \mathbb{N}\} : w \in \Gamma \}.$$
Weak Compactness through inequalities

**Theorem**

Let $E$ be a separable Banach space and $K \subseteq E$ a closed convex and bounded subset. They are equivalent:

1. $K$ is weakly compact.
2. For every sequence $(x_n^*) \subseteq B_{E^*}$ we have

$$\sup_{k \in K} \limsup_{n \to \infty} x_n^*(k) \geq \inf \left\{ \sup_{\kappa \in K^{w^*}} w(\kappa) : w \in \text{co}\{x_n^* : n \in \mathbb{N}\} \right\}$$
Simons inequality \(\Rightarrow\) Compactness

- If (2) happens and \(K\) is not weakly compact there is 
  \[ x_{0}^{**} \in \overline{K}^{w*} \subset E^{**} \text{ with } x_{0}^{**} \notin E \]

- The Hahn Banach Theorem provide us \(x^{***} \in B_{E^{***}} \cap E^\perp\) with 
  \(x^{***}(x_{0}^{**}) = \alpha > 0\)

- The separability of \(E\), Ascoli’s and Bipolar Theorems permit to construct a sequence \((x_{n}^{*}) \subset B_{E^{*}}\) such that:
  - \(\lim_{n \to \infty} x_{n}^{*}(x) = 0\) for all \(x \in E\)
  - \(x_{n}^{*}(x_{0}^{**}) > \alpha/2\) for all \(n \in \mathbb{N}\)

- Then

\[
0 = \sup\{ \lim_{n \to \infty} x_{n}^{*}(k) \} = \sup\{ \limsup_{n \to \infty} x_{n}^{*}(k) \} \geq \inf \{ \sup_{\kappa \in \overline{K}^{w*}} w(\kappa) : w \in \text{co}\{x_{n}^{*} : n \in \mathbb{N}\} \} \geq \alpha/2 > 0
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- If (2) happens and \(K\) is not weakly compact there is \(x^{**}_0 \in K \subset E^{**}\) with \(x^{**}_0 \notin E\).

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Simons inequality $\Rightarrow$ Compactness

- If (2) happens and $K$ is not weakly compact there is $x_{0}^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_{0}^{**} \notin E$
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- The separability of $E$, Ascoli’s and Bipolar Theorems permit to construct a sequence $(x_{n}^{*}) \subset B_{E^*}$ such that:
  1. $\lim_{n \to \infty} x_{n}^{*}(x) = 0$ for all $x \in E$
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If (2) happens and $K$ is not weakly compact there is $x_{0}^{**} \in \overline{K}^{w*} \subset E^{**}$ with $x_{0}^{**} \notin E$.

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The separability of $E$, Ascoli’s and Bipolar Theorems permit to construct a sequence $(x^{*}_{n}) \subset B_{E^{*}}$ such that:

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Theorem (Fonf and Lindenstrauss)

Let $E$ be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. $K$ is weakly compact.
2. For any covering $K \subset \bigcup_{n=1}^{\infty} D_n$ by an increasing sequence of closed convex subsets $D_n \subset K$, we have

$$
\bigcup_n D_n \overset{w^*}{\longrightarrow} \| \cdot \| = K^{w^*}.
$$

The proof uses Krein Milman and Bishop Phelps theorems.
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$$\bigcup_{n} D_n^{w^*} \| \cdot \| = K^{w^*}.$$ 

The proof uses Krein Milman and Bishop Phelps theorems.
I-generation $\Rightarrow$ Weak Compactness

- Take $\{x_n : n \in \mathbb{N}\}$ norm dense in $K$
- $B_m := \text{co}(\{x_n : n \leq m\})$ is finite dimensional closed compact set
- $D_m := B_m + \delta B_{E^{**}}$ for $\delta > 0$ fixed
- Since $K \subset \bigcup_{m=1}^{\infty} D_m$, the I-generation says that
  \[
  \bigcup_{m}^{\infty} D_m^{w^*} = K^{w^*}.
  \]

- So $(\bigcup_{m}^{\infty} B_m) + 2\delta B_{E^{**}} \supseteq \overline{K}^{w^*}$.

- Finally $\bigcap_{\delta > 0} (\bigcup_{m}^{\infty} B_m) + 2\delta B_{E^{**}} = \overline{K}^{\|\cdot\|} = K = \overline{K}^{w^*}$.
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I-generation ⇒ Weak Compactness

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Since \( K \subset \bigcup_{m=1}^{\infty} D_m \), the I-generation says that

\[ \bigcup_{m}^{\infty} D_m^{w^*} = K^{w^*}. \]

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So $(\bigcup_{m} B_m) + 2\delta B_{E^{**}} \supseteq K^{w^*}$.

Finally $\bigcap_{\delta > 0} (\bigcup_{m} B_m) + 2\delta B_{E^{**}} = K^\|\cdot\| = K = K^{w^*}$. 
Let $E$ be a Banach space, $K \subset E^*$ be $w^*$-compact convex, $B \subset K$, TFAE:

1. For any covering $B \subset \bigcup_{n=1}^{\infty} D_n$ by an increasing sequence of convex subsets $D_n \subset K$, we have
   \[
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Simons versus Fonf-Lindenstrauss

Theorem (Cascales, Fonf, Troyanski and Orihuela, J.F.A.-2010)

Let $E$ be a Banach space, $K \subset E^*$ be $w^*$-compact convex, $B \subset K$, TFAE:

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J. Orihuela
Compactness, Optimization and Risk
Theorem (Inf-liminf Theorem in $\mathbb{R}^\Gamma$)

Let $\{\Phi_k\}_{k \geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^\Gamma$. We set $\Lambda \subseteq \Gamma$ satisfying the following boundary condition:

For all $\Phi = \sum_{i=1}^{\infty} \lambda_i \Phi_i$, $\sum_{i=1}^{\infty} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$, there exists $\lambda_0 \in \Lambda$ with $\Phi(\lambda_0) = \inf \{ \Phi(\gamma) : \gamma \in \Gamma \}$

Then

$$\inf \left\{ \lambda \in \Lambda \right\} \left( \lim_{k \geq 1} \inf \Phi_k(\lambda) \right) = \inf \left\{ \gamma \in \Gamma \right\} \left( \lim_{k \geq 1} \inf \Phi_k(\gamma) \right).$$
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A Nonlinear James Theorem

**Theorem**

Let $E$ be a Banach space with $B_{E^*}$ convex-block compact for $\sigma(E^*, E)$. If

$$\alpha : E \to \mathbb{R} \cup \{+\infty\}$$

is a proper map such that for every $x^* \in E^*$ the minimization problem

$$\inf \{\alpha(y) + x^*(y) : y \in E\}$$

is attained at some point of $E$, then the level sets

$$\{y \in E : \alpha(y) \leq c\}$$

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Questions we answer

- The former Theorem applies to arbitrary $L^1(\Omega)$ including Delbaen-JST Theorem.
- The former Theorem extends the separable case of S. Calvert-Fitzpatrick’s work for arbitrary maps.
- B. Calvert and S. Fitzpatrick proved, in a 1985 paper:

**Theorem (Calvert, Fitzpatrick)**

*If the subdifferential of a proper, convex and lower semicontinuous map $f : E \to \mathbb{R} \cup \{\infty\}$, with $\text{dom}(f) \neq \emptyset$, is such that $\partial f(E) = E^*$, then the Banach space $E$ must be reflexive.*
S. Simons showed omissions in their proof and the authors presented an Erratum in 2000. The paper reduce its generality assuming coercitivity everywhere. It become more difficult to read since referenced lemmas must be adjusted too.

**Conjecture:** The Nonlinear James Theorem is true in arbitrary Banach spaces without any control on the sequential compactness of the dual unit ball.
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Namioka-Klee Theorem

**Theorem**

Any linear and positive functional \( \varphi : \mathcal{X} \to \mathbb{R} \) on a Fréchet lattice \( \mathcal{X} \) is continuous.

**Theorem (S.Biagini and M.Fritelli 2009)**

Any proper convex monotone increasing functional \( U : \mathcal{X} \to (-\infty, +\infty] \) on a Frechet lattice \( (\mathcal{X}, T) \) is continuous and subdifferentiable on the interior of its domain. Moreover, it admits a dual representation as

\[
U(x) = \max_{y' \in \mathcal{X}_+} \{ y'(x) - U^*(y') \}
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for all \( x \in \text{int}(\text{Dom}(U)) \).
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Theorem (S. Biagini and M. Fritelli 2009)

Let \((X, \mathcal{T})\) be an order continuous Frechet lattice. Any convex monotone increasing functional \(U : X \rightarrow \mathbb{R}\) is order continuous and it admits a dual representation as

\[
U(x) = \max_{y' \in (X^*)^+} \{ y'(x) - U^*(y') \}
\]

for all \(x \in X\)
C-Property

Definition

A linear topology $\mathcal{T}$ on a Riesz space $\mathcal{X}$ has the $C$-property if for every $A \subset X$ and every $x \in \overline{A}$ there is a sequence $(x_n) \in A$ together with $z_n \in \text{co}\{x_p : p \geq n\}$ such that $(z_n)$ is order convergent to $x$.

Theorem (S. Biagini and M. Fritelli 2009)

Let $(\mathcal{X}, \mathcal{T})$ a locally convex Frechet lattice and $U : \mathcal{X} \rightarrow (-\infty, +\infty]$ proper and convex. If $\sigma(\mathcal{X}, X^\sim)$ has the $C$-property then $U$ is order lower semicontinuous if, and only if

$$U(x) = \sup_{y' \in (X^\sim_n)} \{y'(x) - U^*(y')\}$$

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Orlicz spaces

An even, convex function $\psi : E \to \mathbb{R} \cup \{\infty\}$ such that:

1. $\psi(0) = 0$
2. $\lim_{x \to \infty} \psi(x) = +\infty$
3. $\psi < +\infty$ in a neighbourhood of 0

is called a Young function

1. $L^\psi(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0 : \exists \alpha > 0, \mathbb{E}_\mathbb{P}[\psi(\alpha X)] < +\infty\}$
2. $N_\psi(X) := \inf\{c > 0 : \mathbb{E}_\mathbb{P}[\psi(\frac{1}{c}X)] \leq 1\}$
3. $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset L^\psi(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$
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Order Continuity of Risk Measures

Theorem (Lebesgue Risk Measures)

Let \( \rho(X) = \sup_{Y \in \mathcal{M}^{\Psi^*}} \left\{ \mathbb{E}[XY] - \alpha(Y) \right\} \) be a strong convex risk measure on \( L^\Psi \) with \( \alpha : (L^\Psi(\Omega, \mathcal{F}, \mathbb{P})^* \to \mathbb{R} \cup \{+\infty\} \):

(i) For all \( c \in \mathbb{R} \), \( \alpha^{-1}((-\infty, c]) \) is a relatively weakly compact subset of \( \mathcal{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P}) \).

(ii) For every \( X \in L^\Psi(\Omega, \mathcal{F}, \mathbb{P}) \), the supremum in the equality

\[
\rho(X) = \sup_{Y \in \mathcal{M}^{\Psi^*}} \left\{ \mathbb{E}[XY] - \alpha(Y) \right\}
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is attained.

(iii) \( \rho \) is sequentially order continuous.
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Variational problems

**Theorem (Nonlinear James Theorem)**

Let $E$ be a real Banach space,

$$f : E \rightarrow \mathbb{R} \cup \{+\infty\}$$

a proper, coercive and weakly lower semicontinuous function. Then

$$\partial V(E) = E^*$$

if, and only if,

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Theorem

Let $A$ be a weakly closed subset of a real Banach space and let

$$
\psi : A \longrightarrow \mathbb{R}
$$

be a bounded function such that for all $x^* \in E^*$ the function

$$
x^* - \psi,
$$

when restricted to $A$, attains its supremum.

Then $A$ is weakly compact.
Nonlinear Variational Problems

**Theorem (Reflexivity frame)**

Let $E$ be a real Banach space and

$$f : E \rightarrow \mathbb{R} \cup \{+\infty\}$$

a coercive function such that $\text{dom}(f)$ has nonempty interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with

$$f(x_0) + x^*(x_0) = \inf_{x \in E} \{f(x) + x^*(x)\}$$

Then $E$ is reflexive.

Moreover, if the dual ball $B_{E^*}$ is a $w^*$-convex-block compact no coercive assumption is needed for $f$.
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**Corollary**

A real Banach space $E$ is reflexive, provided there exists a monotone, coercive, symmetric and surjective operator

$$\phi : E \longrightarrow E^*$$

**Corollary**

A real Banach space with dual ball $w^*$-convex-block compact, for instance without copies of $l^1$, is reflexive whenever there exists a monotone, symmetric and surjective operator

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