Variational Compactness

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First Meeting in Topology and Functional Analysis.
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Supported by
A birthday Theorem

- P. Kenderov 2003
- J. Lindenstrauss 2006
- M. Valdivia 2010
- W. Schachermayer 2010
- J. Borwein 2011
- I. Karatzas 2012
- F. Delbaen 2012
- A. Defant 2013
- P. Kenderov 2013
- J. Kakol 2013
- More coming 2014....
$E \neq UA_n$

Let $E$ be a Banach space. Does there exist a closed, separable subspace of $E$ that is isomorphic to $E$? If so, find such a subspace.
The coauthors

- M. Ruiz Galán and J.O. *A coercive and nonlinear James’s weak compactness theorem* Nonlinear Analysis 75 (2012) 598-611.
Contents

- Compactness and Optimization.
- Variational problems and reflexivity.
- One-side James’ Theorem.
- Conic Godefroy’s Theorem.
- Dual variational problems.
One-Perturbation Variational Principle

Compact domain $\Rightarrow$ lsc functions attain their minimum

**Theorem (Borwein-Fabian-Revalski)**

Let $X$ be a Hausdorff topological space and $\alpha : X \to (-\infty, +\infty]$ proper, lsc map s.t. $\{\alpha \leq c\}$ is compact for all $c \in \mathbb{R}$. Then for any proper lsc map $f : X \to (-\infty, +\infty]$ bounded from below, the function $\alpha + f$ attains its minimum.

**Theorem (Borwein-Fabian-Revalski)**

If $X$ is metrizable and $\alpha : X \to (-\infty, +\infty]$ is a proper function such that for all bounded continuous function $f : X \to (-\infty, +\infty]$, the function $\alpha + f$ attains its minimum, then $\alpha$ is a lsc map, bounded from below, whose sublevel sets $\{\alpha \leq c\}$ are all compact.
In a metric space $X$, the conditions imposed on the unique perturbation $\varphi$ in Theorem 6.5.1 are also necessary.

**Theorem 6.5.2** Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function on a metric space $X$. Suppose that for every bounded continuous function $f : X \to \mathbb{R}$, the function $f + \varphi$ attains its minimum. Then $\varphi$ is a lsc function, bounded from below, whose sublevel sets are all compact.
Theorem

A Banach space is reflexive if, and only if, each continuous linear functional attains its supremum on the unit ball

Theorem

A bounded and weakly closed subset $K$ of a Banach space is weakly compact if, and only if, each continuous linear functional attains its supremum on $K$

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A Banach space is reflexive if, and only if, each continuous linear functional attains its supremum on the unit ball.

A bounded and weakly closed subset $K$ of a Banach space is weakly compact if, and only if, each continuous linear functional attains its supremum on $K$.

Let us fix a Banach space $E$ with dual $E^*$.

$K$ is a closed convex set in the Banach space $E$.

- $\nu_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise.
- $x^* \in E^*$ attains its supremum on $K$ at $x_0 \in K$ if and only if $\nu_K(y) - \nu_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$.

The minimization problem

$$\min \{ \nu_K(\cdot) - x^*(\cdot) \}$$

on $E$ for every $x^* \in E^*$ has always solution if and only if the set $K$ is weakly compact.

When the minimization problem

$$\min \{ \alpha(\cdot) + x^*(\cdot) \}$$

on $E$ has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \to (-\infty, +\infty]$?
The Theorem of James as a minimization problem

- Let us fix a Banach space $E$ with dual $E^*$
- $K$ is a closed convex set in the Banach space $E$
- $\nu_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on $K$ at $x_0 \in K \iff \nu_K(y) - \nu_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem
  \[
  \min \{ \nu_K(\cdot) - x^*(\cdot) \}
  \]
  on $E$ for every $x^* \in E^*$ has always solution if and only if the set $K$ is weakly compact
- When the minimization problem
  \[
  \min \{ \alpha(\cdot) + x^*(\cdot) \}
  \]
  on $E$ has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \to (-\infty, +\infty]$?
The Theorem of James as a minimization problem

- Let us fix a Banach space $E$ with dual $E^*$
- $K$ is a closed convex set in the Banach space $E$
- $\iota_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise
- $x^* \in E^*$ attains its supremum on $K$ at $x_0 \in K$ if $\iota_K(y) - \iota_K(x_0) \geq x^*(y - x_0)$ for all $y \in E$
- The minimization problem

$$\min \{ \iota_K(\cdot) - x^*(\cdot) \}$$

on $E$ for every $x^* \in E^*$ has always solution if and only if the set $K$ is weakly compact

- When the minimization problem

$$\min \{ \alpha(\cdot) + x^*(\cdot) \}$$

on $E$ has solution for all $x^* \in E^*$ and a fixed proper function $\alpha : E \rightarrow (-\infty, +\infty]$?
Theorem (M. Ruiz and J. O.)

Let \( E \) be a Banach space, \( \alpha : E \to (-\infty, +\infty] \) proper, (lower semicontinuous) function with

\[
\lim_{\|x\| \to \infty} \frac{\alpha(x)}{\|x\|} = +\infty
\]

Suppose that there is \( c \in \mathbb{R} \) such that the level set \( \{\alpha \leq c\} \) fails to be (relatively) weakly compact. Then there is \( x^* \in E^* \) such that the infimum

\[
\inf_{x \in E} \{\langle x, x^* \rangle + \alpha(x)\}
\]

is not attained.
Minimizing \( \{\alpha(x) + x^*(x) : x \in E\} \)

**Theorem (M. Ruiz and J. O.)**

Let \( E \) be a Banach space, \( \alpha : E \to (-\infty, +\infty] \) proper, (lower semicontinuous) function with

\[
\lim_{\|x\| \to \infty} \frac{\alpha(x)}{\|x\|} = +\infty
\]

Suppose that there is \( c \in \mathbb{R} \) such that the level set \( \{\alpha \leq c\} \) fails to be (relatively) weakly compact. Then there is \( x^* \in E^* \) such that, the infimum

\[
\inf_{x \in E} \{\langle x, x^* \rangle + \alpha(x)\}
\]

is not attained.
\[ \partial \alpha(x_0) = \{ x^* \in E^* : x^*(x-x_0) \leq \alpha(x) - \alpha(x_0) \forall x \} \]

\[ \alpha(x_0) - x^*(x_0) \leq \alpha(x) - x^*(x) \quad \forall x \in E \]
Lemma

Let $A$ be a bounded but not relatively weakly compact subset of the Banach space $E$. If $(a_n) \subset A$ is a sequence without weak cluster point in $E$, then there is $(x^*_n) \subset B_{E^*}$, $g_0 = \sum_{n=1}^{\infty} \lambda_n x^*_n$ with $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ such that: for every $h \in l^\infty(A)$, with

$$\lim \inf_{n} x^*_n(a) \leq h(a) \leq \lim \sup_{n} x^*_n(a)$$

for all $a \in A$, we will have that $g_0 + h$ does not attain its minimum on $A$. 

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Lemma

Let $A$ be a bounded but not relatively weakly compact subset of the Banach space $E$. If $(a_n) \subset A$ is a sequence without weak cluster point in $E$, then there is $(x^*_n) \subset B_{E^*}$, $g_0 = \sum_{n=1}^{\infty} \lambda_n x^*_n$ with $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ such that: for every $h \in l^\infty(A)$, with

$$\liminf_{n} x^*_n(a) \leq h(a) \leq \limsup_{n} x^*_n(a)$$

for all $a \in A$, we will have that $g_0 + h$ does not attain its minimum on $A$.
Let $E$ be a Banach space, $\alpha : E \to (-\infty, +\infty]$ proper, lower semicontinuous function, then we have:

- If $\partial \alpha(E) = E^*$ then the level sets $\{\alpha \leq c\}$ are weakly compact for all $c \in \mathbb{R}$.

- If $\alpha$ has weakly compact level sets and the Fenchel-Legendre conjugate $\alpha^*$ is finite, i.e. $\sup\{x^*(x) - \alpha(x) : x \in E\} < +\infty$ for all $x^* \in E^*$, then $\partial \alpha(E) = E^*$
A monetary utility function is a concave non-decreasing map

\[ U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to [-\infty, +\infty) \]

with \( \text{dom}(U) = \{X : U(X) \in \mathbb{R}\} \neq \emptyset \) and

\[ U(X + c) = U(X) + c, \text{ for } X \in L^\infty, c \in \mathbb{R} \]

Defining \( \rho(X) = -U(X) \) the above definition of monetary utility function yields the definition of a convex risk measure. Both \( U, \rho \) are called coherent if \( U(0) = 0, U(\lambda X) = \lambda U(X) \) for all \( \lambda > 0, X \in L^\infty \)
TODAY has Fatou if \( \sum_n \alpha_n \Rightarrow \rho(\alpha_n) \Rightarrow \sigma(\mathbb{L}^\infty, \mathbb{L}^1) \) lower semivariation.

is order sequentially continuous \( \Leftrightarrow \sum_n |\alpha_n| \leq \beta \quad \forall \beta \in \mathbb{R} \quad \sum_n \alpha_n \Rightarrow \alpha \frac{\beta \Rightarrow \beta^+}{\beta \Rightarrow \beta^-}

\( \lim_{n \to \infty} \rho(\alpha_n) = \rho(\alpha) \)
Representing risk measures

**Theorem**

A convex (resp. coherent) risk measure $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ admits a representation

$$\rho(X) = \sup\{\mu(-X) - \alpha(\mu) : \mu \in \text{ba}, \mu \geq 0, \mu(\Omega) = 1\}$$

(resp.

$$\rho(X) = \sup\{\mu(-X) : \mu \in \mathcal{S} \subseteq \{\mu \in \text{ba}, \mu \geq 0, \mu(\Omega) = 1\}\})$$

If in addition $\rho$ is $\sigma(L^\infty, L^1)$-lower semicontinuous we have:

$$\rho(X) = \sup\{\mathbb{E}_Q(-X) - \alpha(Q) : Q \ll \mathbb{P} \text{ and } \mathbb{E}_\mathbb{P}(dQ/d\mathbb{P}) = 1\}$$

(resp.

$$\rho(X) = \sup\{\mathbb{E}_Q(-X) : Q \in \{Q \ll \mathbb{P} \text{ and } \mathbb{E}_\mathbb{P}(dQ/d\mathbb{P}) = 1\}\}$$

Theorem (Jouini-Schachermayer-Touzi)

Let \( U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \) be a monetary utility function with the Fatou property and \( U^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty] \) its Fenchel-Legendre transform. They are equivalent:

1. \( \{ U^* \leq c \} \) is \( \sigma(\mathbb{L}^1, \mathbb{L}^\infty) \)-compact subset for all \( c \in \mathbb{R} \)
2. For every \( X \in \mathbb{L}^\infty \) the infimum in the equality

\[
U(X) = \inf_{Y \in \mathbb{L}^1} \{ U^*(Y) + \mathbb{E}[XY] \},
\]

is attained

3. For every uniformly bounded sequence \( (X_n) \) tending a.s. to \( X \) we have

\[
\lim_{n \to \infty} U(X_n) = U(X).
\]
Theorem (Lebesgue Risk Measures on Orlicz spaces)

Let \( \rho(X) = \sup_{Y \in \mathcal{M}^\psi} \{ \mathbb{E}_P[-XY] - \alpha(Y) \} \) be a finite convex risk measure on \( L^\psi \) with \( \alpha : (L^\psi(\Omega, \mathcal{F}, \mathbb{P})^*) \rightarrow (-\infty, +\infty) \) a penalty function \( w^* \)-lower semicontinuous. T.F.A.E.:

(i) For all \( c \in \mathbb{R} \), \( \alpha^{-1}((-\infty, c]) \) is a relatively weakly compact subset of \( \mathcal{M}^\psi(\Omega, \mathcal{F}, \mathbb{P}) \).

(ii) For every \( X \in L^\psi(\Omega, \mathcal{F}, \mathbb{P}) \), the supremum in the equality

\[
\rho(X) = \sup_{Y \in \mathcal{M}^\psi} \{ \mathbb{E}_P[-XY] - \alpha(Y) \}
\]

is attained.

(iii) \( \rho \) is sequentially order continuous
Theorem (Reflexivity frame)

Let $E$ be a real Banach space and

$$\alpha : E \rightarrow \mathbb{R} \cup \{+\infty\}$$

a a function such that $\text{dom}(\alpha)$ has nonempty interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with

$$\alpha(x_0) + x^*(x_0) = \inf_{x \in E} \{\alpha(x) + x^*(x)\}$$

Then $E$ is reflexive.
[\partial \alpha(E) = E^* \Rightarrow E = E^{**}]

- Fix an open ball \( B \subseteq \text{dom}(\alpha) \)
  \[
  B = \bigcup_{p=1}^{+\infty} B \cap \alpha^{-1}((-\infty, p])
  \]
- Baire Category Theorem \( \Rightarrow \) there is \( q \in \mathbb{N} : \)
  \[
  B \cap \alpha^{-1}((-\infty, q])
  \]
  has non void interior relative to \( B \)
- There is \( G \) open in \( E \) such that
  \[
  \emptyset \neq B \cap G \subset B \cap \alpha^{-1}((-\infty, q])
  \]
- \( \alpha^{-1}((-\inf, q]) \) weakly compact \( \Rightarrow \) \( G \) contains an open relatively weakly compact ball
- \( B_E \) is weakly compact
Fix an open ball $B \subseteq \text{dom}(\alpha)$. 

$B = \bigcup_{p=1}^{+\infty} B \cap \alpha^{-1}((-\infty, p])$ 

Baire Category Theorem $\Rightarrow$ there is $q \in \mathbb{N}$: 

$B \cap \alpha^{-1}((-\infty, q])$ 

has non void interior relative to $B$. 

There is $G$ open in $E$ such that 

$\emptyset \neq B \cap G \subset B \cap \alpha^{-1}((-\infty, q])$ 

$\alpha^{-1}((-\text{inf, } q])$ weakly compact $\Rightarrow$ $G$ contains an open relatively weakly compact ball 

$B_E$ is weakly compact
Fix an open ball $B \subseteq \text{dom}(\alpha)$

$$B = \bigcup_{p=1}^{+\infty} B \cap \alpha^{-1}((-\infty, p])$$

Baire Category Theorem $\Rightarrow$ there is $q \in \mathbb{N}$:

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$\alpha^{-1}((-\text{inf}, q)]$ weakly compact $\Rightarrow$ $G$ contains an open relatively weakly compact ball

$B_E$ is weakly compact
[\partial \alpha(E) = E^*] \Rightarrow E = E^{**}

- Fix an open ball $B \subseteq \text{dom}(\alpha)$

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- There is $G$ open in $E$ such that

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- $\alpha^{-1}((-\inf, q])$ weakly compact $\Rightarrow$ $G$ contains an open relatively weakly compact ball

- $B_E$ is weakly compact
\[ \partial \alpha(E) = E^* \Rightarrow E = E^{**} \]

- Fix an open ball \( B \subseteq \text{dom}(\alpha) \)
- \( B = \bigcup_{p=1}^{+\infty} B \cap \alpha^{-1}((-\infty,p]) \sigma(E,E^*) \)
- Baire Category Theorem \( \Rightarrow \) there is \( q \in \mathbb{N} : \)
  \[
  B \cap \alpha^{-1}((-\infty,q]) \sigma(E,E^*)
  \]
  has non void interior relative to \( B \)
- There is \( G \) open in \( E \) such that
  \[
  \emptyset \neq B \cap G \subset B \cap \alpha^{-1}((-\infty,q]) \sigma(E,E^*)
  \]
- \( \alpha^{-1}((-\infty,q]) \sigma(E,E^*) \) weakly compact \( \Rightarrow \) \( G \) contains an open relatively weakly compact ball
- \( B_E \) is weakly compact
[\partial \alpha(E) = E^*] \Rightarrow E = E^{**}

- Fix an open ball \( B \subseteq \text{dom}(\alpha) \)
- \( B = \bigcup_{p=1}^{+\infty} B \cap \alpha^{-1}((-\infty, p]) \)
- Baire Category Theorem \( \Rightarrow \) there is \( q \in \mathbb{N} \):
  \[ B \cap \alpha^{-1}((-\infty, q]) \]
  has non void interior relative to \( B \)
- There is \( G \) open in \( E \) such that
  \( \emptyset \neq B \cap G \subset B \cap \alpha^{-1}((-\infty, q]) \)
  \[ \alpha^{-1}((-\infty, q]) \]
  weakly compact \( \Rightarrow \) \( G \) contains an open relatively weakly compact ball
- \( B_E \) is weakly compact
Corollary 2.101 (Main Theorem on Monotone Operators). Let $X$ be a real, reflexive Banach space, and let $A : X \to X^*$ be a monotone, hemicontinuous, bounded, and coercive operator, and $b \in X^*$. Then a solution of the equation $Au = b$ exists.

\[
\begin{align*}
\text{variational equation} & \quad a(u, v) = f(v) \\
& \quad \text{for all } v \in V \\
\end{align*}
\]

\[
\begin{align*}
\text{variational problem} & \quad \min_{v \in V} J(v) \\
\end{align*}
\]

\[
\begin{align*}
\text{Galerkin method} & \quad a(u_h, v_h) = f(v_h) \\
& \quad \text{for all } v_h \in V_h \\
\end{align*}
\]

\[
\begin{align*}
\text{Ritz method} & \quad \min_{v_h \in V_h} J(v_h) \\
\end{align*}
\]

\[\iff \quad \text{by symmetry} \]

\[\iff \quad \text{by symmetry} \]
Given an operator \( \Phi : E \rightarrow E^* \) it is said to be *monotone* provided that

for all \( x, y \in E \), \( (\Phi x - \Phi y)(x - y) \geq 0 \),

and *symmetric* if for all \( x, y \in E \), \( \langle \Phi(x), y \rangle = \langle \Phi(y), x \rangle \).

**Corollary**

* A real Banach space \( E \) is reflexive whenever there exists a monotone, symmetric and surjective operator \( \Phi : E \rightarrow E^* \).

**Question**

Let \( E \) be a real Banach space and \( \Phi : E \rightarrow 2^{E^*} \) a monotone multivalued map with non void interior domain.

\[ \Phi(E) = E^* \Rightarrow E = E^{**} ? \]
Given an operator $\Phi : E \rightarrow E^*$ it is said to be *monotone* provided that

$$\text{for all } x, y \in E, \quad (\Phi x - \Phi y)(x - y) \geq 0,$$

and *symmetric* if for all $x, y \in E$, $\langle \Phi(x), y \rangle = \langle \Phi(y), x \rangle$.

**Corollary**

*A real Banach space $E$ is reflexive whenever there exists a monotone, symmetric and surjective operator $\Phi : E \rightarrow E^*$.*

**Question**

*Let $E$ be a real Banach space and $\Phi : E \rightarrow 2^{E^*}$ a monotone multivalued map with non void interior domain. Then $[\Phi(E) = E^*] \implies E = E^{**}$?*
Theorem (Simons)

Let $\Gamma$ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If $\Lambda$ is a subset of $\Gamma$ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup \left\{ \sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma \right\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then we have:

$$\sup_{\lambda \in \Lambda} \limsup_{k \to \infty} x_k(\lambda) = \sup_{\gamma \in \Gamma} \limsup_{k \to \infty} x_k(\gamma)$$
Let $\Gamma$ be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(\Gamma)$. If $\Lambda$ is a subset of $\Gamma$ such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ there exists $b \in \Lambda$ such that

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in \Gamma\} = \sum_{n=1}^{\infty} \lambda_n z_n(b),$$

then we have:

$$\sup_{\lambda \in \Lambda} \limsup_{k \to \infty} x_k(\lambda) = \sup_{\gamma \in \Gamma} \limsup_{k \to \infty} x_k(\gamma)$$
Let $E$ be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. $K$ is weakly compact.

2. For every sequence $(x_n^*) \subset B_{E^*}$ we have

$$\sup_{k \in K} \{ \limsup_{n \to \infty} x_n^*(k) \} = \sup_{\kappa \in \overline{K}^{w^*}} \{ \limsup_{n \to \infty} x_n^*(\kappa) \}$$
If \( K \) is not weakly compact there is \( x_{0}^{**} \in \overline{K}^{w*} \subset E^{**} \) with \( x_{0}^{**} \notin E \).

The Hahn Banach Theorem provide us \( x^{***} \in B_{E^{***}} \cap E^{\perp} \) with \( x^{***}(x_{0}^{**}) = \alpha > 0 \).

The separability of \( E \), Ascoli's and Bipolar Theorems permit to construct a sequence \( (x_{n}^{*}) \subset B_{E^{*}} \) such that:

1. \( \lim_{n \to \infty} x_{n}^{*}(x) = 0 \) for all \( x \in E \)
2. \( x_{n}^{*}(x_{0}^{**}) > \alpha/2 \) for all \( n \in \mathbb{N} \)

Then

\[
0 = \sup_{k \in K} \{ \lim_{n \to \infty} x_{n}^{*}(k) \} = \sup_{k \in K} \{ \limsup_{n \to \infty} x_{n}^{*}(k) \} \geq \sup_{v^{**} \in \overline{K}^{w*}} \{ \limsup_{n \to \infty} x_{n}^{*}(v^{**}) \} = \limsup_{n \to \infty} x_{n}^{*}(x_{0}^{**}) \geq \alpha/2 > 0
\]
Sup-limsup Theorem ⇒ Compactness

- If $K$ is not weakly compact there is $x_0^{**} \in \overline{K}\,^w\,* \subset E^{**}$ with $x_0^{**} \notin E$

- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^\perp$ with $x^{***}(x_0^{**}) = \alpha > 0$

- The separability of $E$, Ascoli’s and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n \to \infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$

Then

$$0 = \sup_{k \in K} \{ \lim_{n \to \infty} x_n^*(k) \} = \sup_{k \in K} \{ \limsup_{n \to \infty} x_n^*(k) \} \geq \sup_{v^{**} \in \overline{K}\,^w\,*} \{ \limsup_{n \to \infty} x_n^*(v^{**}) \} = \limsup_{n \to \infty} x_n^*(x_0^{**}) \geq \alpha/2 > 0$$
If $K$ is not weakly compact there is $x_{0}^{**} \in \overline{K}^{W^*} \subset E^{**}$ with $x_{0}^{**} \notin E$.

The Hahn Banach Theorem provide us $x^{***} \in B_{E^{**}} \cap E^\perp$ with $x^{***} (x_{0}^{**}) = \alpha > 0$.

The separability of $E$, Ascoli’s and Bipolar Theorems permit to construct a sequence $(x_{n}^*) \subset B_{E^*}$ such that:

1. $\lim_{n \to \infty} x_{n}^*(x) = 0$ for all $x \in E$.
2. $x_{n}^*(x_{0}^{**}) > \alpha/2$ for all $n \in \mathbb{N}$.

Then

$$0 = \sup_{k \in K} \{ \lim_{n \to \infty} x_{n}^*(k) \} = \sup_{k \in K} \{ \limsup_{n \to \infty} x_{n}^*(k) \} \geq$$

$$= \sup_{v^{**} \in \overline{K}^{W^*}} \{ \limsup_{n \to \infty} x_{n}^*(v^{**}) \} = \limsup_{n \to \infty} x_{n}^*(x_{0}^{**}) \geq \alpha/2 > 0$$
Sup-limsup Theorem \Rightarrow \text{Compactness}

- If $K$ is not weakly compact there is $x_0^{**} \in \overline{K}^{w^*} \subset E^{**}$ with $x_0^{**} \notin E$
- The Hahn Banach Theorem provide us $x^{***} \in B_{E^{***}} \cap E^\perp$ with $x^{***}(x_0^{**}) = \alpha > 0$
- The separability of $E$, Ascoli’s and Bipolar Theorems permit to construct a sequence $(x_n^*) \subset B_{E^*}$ such that:
  1. $\lim_{n \to \infty} x_n^*(x) = 0$ for all $x \in E$
  2. $x_n^*(x_0^{**}) > \alpha/2$ for all $n \in \mathbb{N}$
- Then

\[
0 = \sup_{k \in K} \left\{ \lim_{n \to \infty} x_n^*(k) \right\} = \sup_{k \in K} \left\{ \limsup_{n \to \infty} x_n^*(k) \right\} \geq \\
= \sup_{v^{**} \in \overline{K}^{w^*}} \left\{ \limsup_{n \to \infty} x_n^*(v^{**}) \right\} = \limsup_{n \to \infty} x_n^*(x_0^{**}) \geq \alpha/2 > 0
\]
Theorem (Fonf and Lindenstrauss)

Let $E$ be a separable Banach space and $K \subset E$ a closed convex and bounded subset. They are equivalent:

1. $K$ is weakly compact.
2. For any covering $K \subset \bigcup_{n=1}^{\infty} D_n$ by an increasing sequence of closed convex subsets $D_n \subset K$, we have

$$\bigcup_{n} D_n^{w^*} \| \cdot \|_{w^*} = K^{w^*}.$$ 

The proof uses Krein Milman and Bishop Phelps theorems
Theorem (Cascales, Fonf, Troyanski and J.O., J.F.A.-2010)

Let $E$ be a Banach space, $K \subset E^*$ be $w^*$–compact convex, $B \subset K$, TFAE:

1. For any covering $B \subset \bigcup_{n=1}^{\infty} D_n$ by an increasing sequence of convex subsets $D_n \subset K$, we have

$$\bigcup_{n=1}^{\infty} \overline{D_n}^{w^*} = K.$$

2. $\sup_{f \in B} (\limsup_k f(x_k)) = \sup_{g \in K} (\limsup_k g(x_k))$

   for every sequence $\{x_k\} \subset B_X$.

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J. Orihuela
Let $C$ be a convex, bounded and closed, but not weakly compact subset of the Banach space $E$ with $0 \notin C$. The following problem has been posed by F. Delbaen motivated by risk measures theory:

**Question**

Is it possible to find a linear functional not attaining its minimum on $C$ and that stays strictly positive on $C$?

**Example (R. Haydon)**

In every non reflexive Banach space there is a closed, convex and bounded subset $C$ with non void interior and $0 \notin C$ such that every linear form $x^* \in E^*$ such that $x^*(C) > 0$ attains its minimum on $C$. 
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\[ g(x) = c_0 + (-c)(g(x) - y) \]
\[ \text{for } c_0 \in C \]

\[ g(x) > \frac{1}{M} \]

\( S_i: f(c_i) > 0 \Rightarrow f(A_{c_i}) > 0 \)

\( \forall c_i, c = x_0 + \mu(y - x_0), x, 0 \)

and \( y \in A_1, h^{-} \)

and \( f \) at \( x_0 \)

\[ g(x_0) = 1 \]

\[ A_x = \{ x: g(x) > 1, \mu > 0, x \in E \} \]

\[ f(A_1 > 0 \Rightarrow \| f - g \| \leq \frac{1}{M} \]

\[ A_{x, f} = \{ x: g(x) > 1, \mu > 0, x \in E \} \]
Positive results

Theorem (Birthday’s Theorem)
Let $E$ be a separable Banach space. Let $C$ be a closed, convex and bounded subset of $E \setminus \{0\}$, $D \subset C$ a relatively weakly compact set of directions such that, for every $x^* \in E^*$, we have that

$$\inf\{x^*(c) : c \in C\}$$

is attained at some point of $C$ whenever

$$x^*(d) > 0 \text{ for every } d \in D.$$ 

Then $C$ is weakly compact.
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Unbounded Simon’s inequality

**Theorem (Simons’s Theorem in \( \mathbb{R}^X \))**

Let \( X \) be a nonempty set, let \((f_n)\) be a pointwise bounded sequence in \( \mathbb{R}^X \) and let \( Y \) be a subset of \( X \) such that for every \( g \in \text{co}_{\sigma \rho}\{f_n: n \geq 1\} \) there exists \( y \in Y \) with

\[
g(y) = \sup\{g(x): x \in X\}.
\]

Then the following statements hold true:

\[
\inf\{\sup_{x \in X} g(x): g \in \text{co}_{\sigma \rho}\{f_n: n \geq 1\}\} \leq \sup_{y \in Y} (\limsup_n f_n(y)) \quad (1)
\]

and

\[
\sup_{n} (\limsup_{x \in X} f_n(x)) = \sup_{n} (\limsup_{y \in Y} f_n(y)) \quad (2)
\]
If $E$ is a Banach space, $B \subset C$ are nonempty subsets of $E^*$ and $(x_n)$ is a bounded sequence in $E$ such that for every $x \in \text{co}_\sigma\{x_n : n \geq 1\}$ there exists $b^* \in B$ with $\langle x, b^* \rangle = \sup\{\langle x, c^* \rangle : c^* \in C\}$, then

$$\sup_{b^* \in B} \left( \limsup_{n} \langle x_n, b^* \rangle \right) = \sup_{c^* \in C} \left( \limsup_{n} \langle x_n, c^* \rangle \right).$$

As a consequence

$$\sigma(E, B) - \lim_{n} x_n = 0 \Rightarrow \sigma(E, C) - \lim_{n} x_n = 0.$$
Theorem (Unbounded Godefroy’s Theorem)

Let $E$ a Banach space and $B$ a nonempty subset of $E^*$. Let us assume there is a relatively weakly compact subset $D \subset E^*$ such that:

1. $0 \notin \text{co}(B \cup D)^{\|\cdot\|}$

2. For every $x \in E$ with $x(d^*) < 0$ for all $d^* \in D$ we have $\sup\{x(c^*) : c^* \in B\} = x(b^*)$ for some $b^* \in B$.

3. For every convex bounded subset $L \subset E$ and every $x^{**} \in \overline{\sigma}(E^{**}, B \cup D^w)$ there is a sequence $(x_n)$ in $L$ such that 
   \[ \langle x^{**}, z^* \rangle = \lim_n \langle x_n, z^* \rangle \text{ for every } z^* \in B \cup D^w \]

Then
\[ \text{co}(B)^{w^*} \subset \bigcup \{\text{co}(B)^{\|\cdot\|} + \lambda\text{co}(D)^{\|\cdot\|} : \lambda \in [0, +\infty)\} \]
Let $E$ a Banach space and $B$ a nonempty subset of $E^*$. Let us assume $0 \notin \text{co}(B)\|\cdot\|$ and fix $D \subset B$, a relatively weakly compact set so that:

1. For every $x \in E$ with $x(d^*) > 0$ for every $d^* \in D$, we have $\inf\{x(c^*) : c^* \in B\} = x(b^*) > 0$ for some $b^* \in B$.

2. For every convex bounded subset $L \subset E$, and every $x^{**} \in \overline{L}(E^{**}, B \cup D^w)$, there is a sequence $(x_n)$ in $L$ such that $\langle x^{**}, z^* \rangle = \lim_n \langle x_n, z^* \rangle$, for every $z^* \in B \cup \overline{D}^w$.

Then the norm closed convex truncated cone $C$ generated by $B$, i.e. $C := \bigcup\{\lambda \text{co}(B) : \lambda \in [1, +\infty)\} \|\cdot\|$, is $w^*$-closed.
Theorem

Let $E$ be a separable Banach space without copies of $\ell^1(\mathbb{N})$,

$$f : E^* \rightarrow \mathbb{R} \cup \{+\infty\}$$

norm lower semicontinuous, convex and proper map, such that

for all $x \in E$, $x - f$ attains its supremum on $E^*$.

Then the map $f$ is $\text{w}^*$-lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $f^{-1}((-\infty, \mu])$ is $\text{w}^*$-compact.
Theorem (Birhtday’s Theorem for Jerzcy)

Let $E$ be a Banach space,

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convex, proper and lower semicontinuous map with a weakly web-compact (for instance Lindelöf-$\Sigma$ ) epigraph, such that

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Let $E$ be a Banach space without copies of $\ell^1(\mathbb{N})$, 

$$f : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$$ 

convex, proper and norm lower semicontinuous map with

$w^*$-K-analytic epigraph, such that

for all $x \in E$, $x - f$ attains its supremum on $E^*$.

Then $f$ is $w^*$-lower semicontinuous and for every $\mu \in \mathbb{R}$, the

sublevel set $f^{-1}((-\infty, \mu])$ is $w^*$-compact.
Dual variational problems

- \( L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathcal{X} \subset L^0(\Omega, \mathcal{F}, \mathbb{P}) \) a solid vector subspace
- \( \mathcal{X}_n = \{ Z \in L^0 : XZ \in L^1 \} \) its order continuous dual such that \( \langle \mathcal{X}, \mathcal{X}_n \rangle \) is a dual pair
- \( f : \mathcal{X} \to \mathbb{R} \cup \{ +\infty \} \) proper convex with the Fatou property (i.e. order lower semicontinuity)
- **CONJECTURE**: \( f \) is \( \sigma(\mathcal{X}, \mathcal{X}_n) \) lower semicontinuous
- Biagini-Fritelli: yes if we have \( C \)-property, 2009
- \( C \)-property tool: \( x \in \overline{\sigma(\mathcal{X}, \mathcal{X}_n)} \) \( \Rightarrow \) there is a sequence \( (a_n) \subset A \) and \( z_p \in co(\{ a_m : m \geq p \}), p = 1, 2, \ldots \) such that \( (z_n) \) order converges to \( x \)
- Owari, 2013: There is a gap in Biagini-Fritelli and problem remains open at this level of generality
THANK YOU VERY MUCH FOR YOUR ATTENTION AND MY FAVORITE POEM FOR JERZCY:

- Cultivo la rosa blanca I grow the white rose
- tanto en julio como en enero, as much in July as January,
- para el amigo sincero for the real friend
- que me da su mano franca. who gives me his frank hand.
- Y para el cruel que arranca And for the cruel who drags
- el corazón con que vivo, the heart with I am living,
- cardo ni oruga cultivo neither thistle or larve I am growing
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