Pettis versus McShane in vector integration

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The general framework

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$$f : \Omega \to X$$

where

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A few names

Pioneers (30's): Birkhoff, Bochner, Dunford, Gelfand, Pettis, ...

Maturity (70-80's): Edgar, Fremlin, Musial, Stegall, Talagrand ...

Recently:
Cascales, Di Piazza, Fremlin, Kadets, Mendoza, Musial, Preiss, ...

A funny story (according to J. Diestel)

The quote is from a 1975 lecture of B.J. Pettis at the Univ. of Pittsburgh on "Recent advances in the study of the Pettis integral". Pettis led off the meeting with the plea:

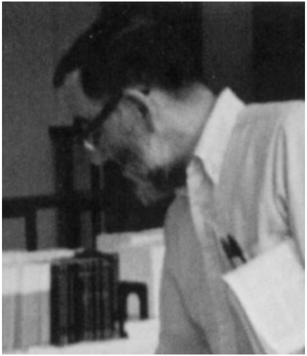
We have plenty of integrals already, so please,
no new integrals!

The next three speakers started off their talks with three "new integrals" apiece. Pettis wept.

According to Fremlin and Mendoza (1994) . . .

The ordinary functional analyst is naturally impatient with the multiplicity of definitions of 'integral' which have been proposed for vector-valued functions, and would prefer to have a single canonical one for general use.

Our stars



B.J. Pettis
(1913-1979)

E.J. McShane
(1904-1989)

Summary of the talk

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$$\boxed{x^*\left(\int_A f \, d\mu\right) = \int_A x^* f \, d\mu} \quad \forall x^* \in X^*.$$

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- Bochner integrable $\Longrightarrow$ Pettis integrable.
- Bochner $\equiv$ Pettis $\iff$ $\dim(X) < \infty$.

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for each $\varepsilon > 0$ *there is a function* $\delta : [0,1] \to \mathbb{R}^+$ *such that*

$$\left| \sum_{k=1}^{n} (b_k - b_{k-1}) f(t_k) - I \right| < \varepsilon$$

for every partition $0 = b_0 < b_1 < \cdots < b_n = 1$ *and every choice of points* $t_1, \ldots, t_n \in [0,1]$ *such that*

$$\big[b_{k-1}, b_k \big] \subset \big(t_k - \delta(t_k), t_k + \delta(t_k) \big).$$

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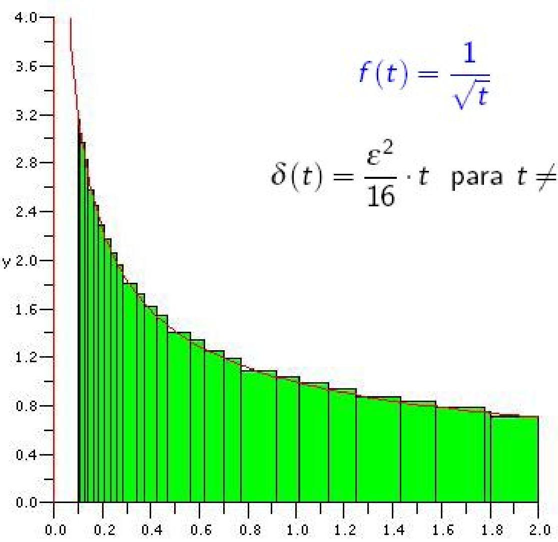
In this case, $I = \int f(t)\, dt$.

An example



$$f(t) = \frac{1}{\sqrt{t}}$$

$$\delta(t) = \frac{\varepsilon^2}{16} \cdot t \quad \text{para } t \neq 0$$

The McShane integral for vector-valued functions

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Contributors: Di Piazza, Fremlin, Gordon, Mendoza, Musial, Preiss, . . .

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Pettis $\not\Longrightarrow$ McShane (Fremlin-Mendoza, 1994)

$f : [0,1] \to \ell^\infty$ given by

$$f(t) = (\chi_{A_1}(t), \chi_{A_2}(t), \dots)$$

where $A_1, A_2, \dots$ is an independent sequence of measurable subsets of $[0,1]$ with $\lambda(A_n) = 1/n$.

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Pettis' measurability theorem (1938)

A function $f : \Omega \to X$ is strongly measurable if and only if

- f is scalarly measurable;
- there is $E \in \Sigma$ with $\mu(E) = 1$ such that $f(E)$ is separable.

The separable case: McShane $\equiv$ Pettis

Theorem (Gordon 1990, Fremlin-Mendoza 1994)

Suppose X is **separable**.
Then a function $f : [0,1] \to X$ is McShane integrable if and only if it is Pettis integrable.

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Idea of the proof:

- Pettis' measurability theorem $\Longrightarrow$

$$\Longrightarrow \quad \boxed{f = g + h} \quad \text{with } g \text{ Bochner integrable and } h = \sum_{n=1}^{\infty} x_n \chi_{A_n}$$

where $x_n \in X$ and $A_1, A_2, \ldots$ are disjoint measurable subsets of $[0,1]$.

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- Convergence theorem $\Longrightarrow$ h is McShane integrable.

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Definition

Two functions $f, g : \Omega \to X$ are **scalarly equivalent** iff, for each $x^* \in X^*$, we have $x^* f = x^* g$ μ-a.e.

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Their techniques ...

- Projectional resolutions of the identity (PRIs).
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Theorem (Edgar, 1977)

Suppose X is **weakly Lindelöf**. Then every scalarly measurable function $f : \Omega \to X$ is scalarly equivalent to a strongly measurable one $g : \Omega \to X$.

McShane ≡ Pettis in non-trivial cases II

Problem (Di Piazza-Preiss, 2003)

Are McShane and Pettis integrability equivalent for functions with values in arbitrary **WCG** spaces ??

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Another partial answer . . .

Theorem (R., 2008)

Let μ be a probability measure defined on a σ-algebra.
Then a function $f : [0,1] \to L^1(\mu)$ is McShane integrable if and only if it is Pettis integrable.

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Some ideas of the proof . . .

- Reduction to the case $\mu =$ usual product probability on $\{0,1\}^\kappa$.
- Approximation by $L^2(\mu)$-valued functions (using PRIs).

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X is a subspace of a Hilbert generated space

$\Updownarrow$

X admits an equivalent uniformly Gâteaux differentiable norm

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A unified approach

Theorem (Deville-R., 2008)

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In general, the answer is **"no"**:

Under CH ...

...there is a scalarly null function $f : [0, 1] \to \ell^\infty(\mathfrak{c})$ which is **not** McShane integrable (Di Piazza-Preiss, 2003).

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Theorem (Congxin-Xiaobo 1994, Di Piazza-Musial 2001)

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$$\boxed{\sum_{k=1}^{n}\left\|(b_k-b_{k-1})f(t_k)-\int_{b_{k-1}}^{b_k} f\,d\lambda\right\|<\varepsilon}$$

for every partition $0=b_0<b_1<\cdots<b_n=1$ and every choice of points $t_1,\dots,t_n\in[0,1]$ such that

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$$[b_{k-1}, b_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k)).$$

Application (Marraffa 2004, R. 2006)

If $u : X \to Y$ is an **absolutely summing** operator and $f : [0,1] \to X$ is McShane integrable, then $u \circ f : [0,1] \to Y$ is Bochner integrable.

Some consequences

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Example (R., 2008)

Under CH, there exist a weakly Lindelöf determined space X and a scalarly null function $f : [0,1] \to X$ which is not McShane integrable.

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Under CH, there exist a weakly Lindelöf determined space X and a scalarly null function $f : [0,1] \to X$ which is not McShane integrable.

Example (Deville-R., 2008)

There exist a Radon probability space (Ω, Σ, μ) and a scalarly null function

$$f : \Omega \to \ell^1(\mathfrak{c}^+)$$

which is not McShane integrable.

THANKS FOR YOUR ATTENTION !!

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