Integration in Hilbert generated Banach spaces

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Summary of the talk

1. Meeting the integrals
   - The Pettis integral
   - The McShane integral
2. Pettis versus McShane in non-separable Banach spaces
   - First equivalence results
   - A unified approach
   - McShane integrability of scalarly null functions

The general framework

Integration of functions

$$[0,1] \xrightarrow{f} X$$

where:

- the unit interval $[0,1]$ is equipped with the Lebesgue measure,
- X is a Banach space.

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- Bochner integrable $\Longrightarrow$ Pettis integrable.
- Bochner $\equiv$ Pettis $\iff$ $\dim(X) < \infty$.

McShane's approach to the Lebesgue integral

Theorem (McShane, 1969)

$f : [0,1] \longrightarrow \mathbb{R}$ is **(Lebesgue) integrable** if and only if there exists $I \in \mathbb{R}$ with the following property:

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for each $\varepsilon > 0$ there is a function $\delta : [0,1] \to \mathbb{R}^+$ such that

$$\left| \sum_{k=1}^{n} (b_k - b_{k-1}) f(t_k) - I \right| < \varepsilon$$

for every partition $0 = b_0 < b_1 < \cdots < b_n = 1$ and every choice of points $t_1, \ldots, t_n \in [0,1]$ such that

$$\left[b_{k-1}, b_k\right] \subset \left(t_k - \delta(t_k), t_k + \delta(t_k)\right).$$

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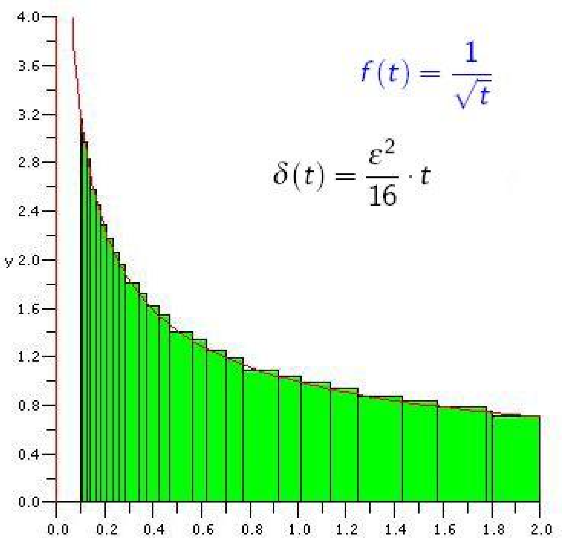
- In this case, $I=\int_{[0,1]}f$.

An example



$$f(t) = \frac{1}{\sqrt{t}}$$

$$\delta(t) = \frac{\varepsilon^2}{16} \cdot t$$

The McShane integral for vector-valued functions

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Contributors: Di Piazza, Fremlin, Gordon, Mendoza, Musial, Preiss, . . .

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- Key: strong measurability $\equiv$ scalar measurability if X is separable.

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Indeed:

- $\ell^1(\mathfrak{c})$ RNP $\Longrightarrow$ f is scalarly equivalent to a strongly measurable $g : [0,1] \to \ell^1(\mathfrak{c})$.
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$f, g : [0,1] \to X$
are **scalarly equivalent** iff ...

... for each $x^* \in X^*$, we have:

$$x^* f = x^* g \text{ a.e.}$$

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Suppose any of the following conditions holds:

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Their techniques . . .

▶ Projectional resolutions of the identity (PRI) on X.
▶ Reduction to the case of **scalarly null** functions (i.e. scalarly equivalent to 0).

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Theorem (Edgar, 1977)

If X is weakly Lindelöf, then every scalarly measurable $f : [0,1] \to X$ is scalarly equivalent to a strongly measurable $g : [0,1] \to X$.

McShane $\equiv$ Pettis in non-trivial cases II

Problem (Di Piazza-Preiss, 2003)

Are McShane and Pettis integrability equivalent for functions with values in arbitrary **WCG** spaces ?????

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Are McShane and Pettis integrability equivalent for functions with values in arbitrary **WCG** spaces ?????

Another partial answer ...

Theorem (R., 2008)

Let μ be a probability measure defined on a σ-algebra. Then for any $f : [0,1] \to L^1(\mu)$ we have:

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Hilbert generated spaces and their subspaces

Definition

X is called **Hilbert generated** iff there exist a Hilbert space H and an operator $T : H \to X$ with dense range.

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X is a subspace of a Hilbert generated space

$\Updownarrow$

X admits an equivalent uniformly Gâteaux differentiable norm

$\Updownarrow$

B_{X^*} is a uniform Eberlein compact

A unified approach

Theorem (Deville-R., Israel J. Math. 2010)

If X is a **subspace of a Hilbert generated space**, then for any $f : [0,1] \to X$ we have

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Some ideas of the proof ...

- Reduction to the case of scalarly null functions.
- Markushevich basis $(x_i, x_i^*)_{i \in I}$ of X.
- *Particular case:* $f([0,1]) \subset \{\lambda x_i\}_{i \in I, \lambda \in \mathbb{R}}$.
- *General case:* using a strong Markushevich basis, i.e.

$$\boxed{x \in \overline{\text{span}}\{x_i^*(x) x_i\}_{i \in I}} \qquad \forall x \in X.$$

McShane integrability of scalarly null functions

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Scalarly null $\Longrightarrow$ McShane integrable ?????

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In general, the answer is **"NO"**:

Under CH there exist...

- ...a scalarly null $f : [0, 1] \to \ell^\infty(\mathfrak{c})$ which is not McShane integrable (Di Piazza-Preiss, 2003),
- ...a WLD space X and a scalarly null $f : [0, 1] \to X$ which is not McShane integrable (R., 2008).

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ZFC Example (Deville-R., 2010) based on Fremlin (1987)

There exist a Radon probability space Ω and
a scalarly null $f : \Omega \to \ell^1(\mathfrak{c}^+)$ which is not McShane integrable.