On the equivalence of McShane and Pettis integrability in non-separable Banach spaces

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Fremlin-Mendoza (1994)

The ordinary functional analyst is naturally impatient with the multiplicity of definitions of 'integral' which have been proposed for vector-valued functions, and would prefer to have a single canonical one for general use.

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In this case,

$$\alpha = \int_0^1 f \; d\lambda.$$

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where g is **Bochner** integrable, $x_n \in X$ and $A_1, A_2, \ldots$ are pairwise disjoint measurable subsets of $[0,1]$.

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Problem (Di Piazza-Preiss, 2003)

Are the McShane and Pettis integrals equivalent for functions taking values in arbitrary WCG spaces?

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...there exist scalarly null functions $f : [0,1] \to \ell^\infty(\mathfrak{c})$ which are **not** McShane integrable (Di Piazza-Preiss, 2003).

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- X is a WLD space such that (B_{X^*}, w^*) admits a Radon measure having **non-separable** support (Kalenda-Plebanek, 2002).

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