Compactness in L^1 of a vector measure

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Integration, Vector Measures and Related Topics VI
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Coauthors

J.M. Calabuig, J.R., E.A. Sánchez-Pérez,
On completely continuous integration operators of a vector measure,
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Contents

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2. $\sigma(L^1(\nu), \Gamma)$ is **angelic**.

Weak convergence vs $\sigma(L^1(\nu), \Gamma)$-convergence

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There exist a vector measure ν and a $\sigma(L^1(\nu),\Gamma)$-null sequence in $L^1(\nu)$ which is **equivalent to the ℓ^1-basis**.

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2. The converse implication fails in general.

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Theorem ($\overset{\star}{\Longrightarrow}$: Curbera 1992)

$$X \not\supset c_0 \overset{\star}{\Longrightarrow} L^1(\nu) \not\supset c_0 \iff L^1(\nu) \text{ WSC}$$

Positive Schur property

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A Banach lattice is said to have the **positive Schur property (PSP)** if every weakly null sequence of positive vectors is norm null.

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Consequence: **If $L^1(\nu)$ has the PSP, then it is SWCG.**

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The topology of norm convergence of the integrals

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Open Problem

What about arbitrary Banach spaces not containing ℓ^1 ?

A partial answer

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3. Every Banach space with **separable dual** embeds into a Banach space with **shrinking basis**. (Zippin 1988)

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An operator from a Banach lattice to a Banach space is **almost completely continuous** if it maps weakly null disjoint sequences to norm null ones.

Many thanks for your attention!

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