Compactness in L^1 of a vector measure

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Integration, Vector Measures and Related Topics VI Bedlewo, June 18, 2014

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J.M. CALABUIG, J.R., E.A. SÁNCHEZ-PÉREZ, On completely continuous integration operators of a vector measure, J. Convex. Anal. 21 (2014), no. 3.

J.M. CALABUIG, S. LAJARA, J.R., E.A. SÁNCHEZ-PÉREZ, Compactness in L^1 of a vector measure, in preparation.

- L^1 spaces of vector measures
- Q Equi-integrability and compactness
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Definition

A measurable function $f : \Omega \to \mathbb{R}$ is *v*-integrable if:

- f is x^*v -integrable $\forall x^* \in X^*$,
- for every $A \in \Sigma$ there is $\int_A f \, dv \in X$ such that

$$x^*\left(\int_A f\,dv\right) = \int_A f\,d(x^*v) \quad \forall x^* \in X^*.$$

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The space $L^1(v)$ of all (equivalence classes of) *v*-integrable functions is a **Banach lattice** with the ||v||-a.e. order and the norm $||\cdot||_{L^1(v)}$.

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$$L^{p}(\mu)$$
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Examples

 $L^p(\mu)$ (μ probability, $1 \le p < \infty$), spaces with unconditional basis, etc.

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$$f \in L^1(v)$$
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1 The identity $i: L^1(v) \to L^1(\mu)$ is

- injective,
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- **1** $L^1(v) \not\supseteq \ell^1$. (Curbera 1994, Okada 1993)
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Theorem (Lewis 1973, Okada 1993)

If $v(\Sigma)$ is relatively **norm** compact, then every $\sigma(L^1(v), \Gamma)$ -convergent **sequence** is weakly convergent.

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Example (Curbera 1994)

There exists a vector measure v with relatively norm compact range such that $\sigma(L^1(v), \Gamma) \neq$ weak topology on bounded sets.

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There exist a vector measure v and a $\sigma(L^1(v), \Gamma)$ -null sequence in $L^1(v)$ which is equivalent to the ℓ^1 -basis.

• L^1 spaces of vector measures

2 Equi-integrability and compactness

Ompletely continuous integration operators

Definition

A set $C \subset L^1(v)$ is **equi-integrable** if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(A) < \delta \implies \sup_{\mathbf{f} \in \mathbf{C}} \left\| \mathbf{f} \mathbf{1}_{\mathbf{A}} \right\|_{L^{1}(v)} < \varepsilon$$

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The classical Dunford-Pettis theorem

In the L^1 space of a scalar measure:

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Folk fact

 $C \subset L^1(v)$ is bounded and equi-integrable if and only if $\forall \varepsilon > 0 \ \exists \rho > 0$ such that

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2 The converse implication fails in general.

Strongly WCG spaces

A Banach space Y is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set $K \subset Y$ such that:

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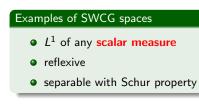
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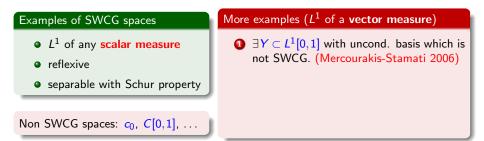
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Non SWCG spaces: c_0 , C[0,1], ...

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More examples (L^1 of a vector measure)

 ∃Y ⊂ L¹[0,1] with uncond. basis which is not SWCG. (Mercourakis-Stamati 2006)

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- ℓ²(L¹[0,1]) (= L¹(v) for an ℓ²-valued v) does not embed into any SWCG space.

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More examples (L¹ of a **vector <u>measure</u>)** Examples of SWCG spaces **1** $\exists Y \subset L^1[0,1]$ with uncond. basis which is • L¹ of any scalar measure not SWCG. (Mercourakis-Stamati 2006) reflexive 2 $\ell^2(\ell^1)$ does not embed into any SWCG separable with Schur property space. (Kampoukos-Mercourakis 2013) **3** $\ell^2(L^1[0,1]) (= L^1(v)$ for an ℓ^2 -valued v) Non SWCG spaces: c_0 , C[0,1], ... does not embed into any SWCG space. Theorem (Schlüchtermann-Wheeler 1988) Theorem ($\stackrel{\star}{\Longrightarrow}$: Curbera 1992) SWCG $X \not\supseteq c_0 \stackrel{\star}{\Longrightarrow} L^1(v) \not\supseteq c_0 \iff L^1(v)$ WSC weakly sequentially complete (WSC)

Positive Schur property

Folk fact (again)

 $C \subset L^1(v)$ is bounded and equi-integrable if and only if $\forall \varepsilon > 0 \ \exists \rho > 0$ such that

 $C \subset \rho B_{L^{\infty}(v)} + \varepsilon B_{L^{1}(v)}$



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Consequence: If $L^{1}(v)$ has the PSP, then it is SWCG.

Definition

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A bounded set $C \subset L^1(v)$ is weakly precompact if and only if

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The topology τ on $L^1(v)$ is defined by

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- L^1 spaces of vector measures
- 2 Equi-integrability and compactness
- **③** Completely continuous integration operators

The **integration operator** $I_v : L^1(v) \to X$ is defined by $I_v(f) := \int_{\Omega} f \, dv$.

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"Operator Ideal Principle" (Okada, Ricker, Rodríguez-Piazza 2011)

TFAE for an operator ideal \mathscr{A} :

(For every X-valued v) $I_v \in \mathscr{A} \implies v$ has finite variation.

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Theorem (Okada, Ricker, Rodríguez-Piazza 2011)

If X has an **unconditional basis** and $X \not\supseteq \ell^1$, then the ideal of **completely continuous** operators satisfies (1)-(2).

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Open Problem

What about arbitrary Banach spaces not containing ℓ^1 ?

If X is Asplund and I_V is completely continuous, then $L^1(v) = L^1(|v|)$.

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Ingredients of the proof

• We follow the approach of (Okada, Ricker, Rodríguez-Piazza 2011).

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② Simplification, using the Rådström embedding

{norm compact non-empty subsets of X} $\xrightarrow{\Phi} C(B_{X^*}, w^*)$

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Every Banach space with separable dual embeds into a Banach space with shrinking basis. (Zippin 1988)

Positive Schur property and integration operators

Theorem (Curbera 1992)

 I_v completely continuous $\implies L^1(v)$ WSC

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TFAE:

- () I_{v} is completely continuous if and only if $L^{1}(v)$ has the **PSP** and $v(\Sigma)$ is relatively **norm** compact.
- **2** I_v is almost completely continuous if and only if $L^1(v)$ has the **PSP**.

 I_v completely continuous $\implies L^1(v)$ WSC

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TFAE:

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- **2** I_v is almost completely continuous if and only if $L^1(v)$ has the **PSP**.

An operator from a Banach lattice to a Banach space is **almost completely continuous** if it maps weakly null disjoint sequences to norm null ones.

Many thanks for your attention!

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Inside a norm bounded subset of X:

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- $\sigma(X,B)$ -compact sets are weakly compact. (Pfitzner 2010)