Integration in Banach spaces: new trends and open problems

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Plan of the talk

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INTRODUCTION

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Recently:
Cascales, Di Piazza, Fremlin, Mendoza, Musial, Preiss, R. ...

Fremlin-Mendoza (1994)

The ordinary functional analyst is naturally impatient with the multiplicity of definitions of 'integral' which have been proposed for vector-valued functions, and would prefer to have a single canonical one for general use.

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The converse does not hold in general.

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Corollary

Strong and scalar measurability are equivalent for functions taking values in a separable Banach space.

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THE BIRKHOFF INTEGRAL

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f is Lebesgue integrable if and only if the intersection

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Fréchet (1915)

This way of presenting the theory of integration due to M. Lebesgue has the advantage, over the way M. Lebesgue presented his theory himself, that is very much close to the views of Riemann-Darboux to which many students are familiar with.

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In this case, the Birkhoff integral of f is the only point in the intersection

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Suppose X contains no isomorphic copy of ℓ^1.
Then every μ-continuous X^*-valued measure with σ-finite variation admits a **Birkhoff** integrable "Radon-Nikodým derivative".

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Problem

When do the classical convergence theorems (e.g. Lebesgue, Vitali) hold true for the Birkhoff integral?

Convergence theorems for the Birkhoff integral II

Theorem (R., 2006)

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In this case,

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[Fremlin, Gordon, Mendoza (90's), Solodov (2005)]

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Theorem (Di Piazza-Preiss, 2003)

Suppose either:

- $X = c_0(\Gamma)$ (for some non-empty set Γ), or
- X admits a uniformly convex equivalent norm.

Then a function $f : [0,1] \to X$ is McShane integrable if and only if it is Pettis integrable.

Techniques used by Di Piazza and Preiss . . .

- Projectional resolutions of the identity (PRIs).
- Reduction to the case of scalarly null functions.

Theorem (Lewis 1970 and Edgar 1977)

Suppose X is WCG. Let $f : [0,1] \to X$ be a *scalarly* measurable function. Then there is a *strongly* measurable function $g : [0,1] \to X$ such that $f - g$ is scalarly null.

McShane vs Pettis II

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Problem (Di Piazza-Preiss, 2003)

Are the McShane and Pettis integrals equivalent for functions taking values in arbitrary WCG spaces?

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...there exist scalarly null functions $f : [0,1] \to \ell^\infty([0,1])$ which are **not** McShane integrable (Di Piazza-Preiss, 2003).

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Example (R., 2008)

Under CH, there exist a weakly Lindelöf determined Banach space X and a scalarly null function $f : [0,1] \to X$ which is not McShane integrable.

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Theorem (R., 2008)

Let μ be a finite, non-negative and countably additive measure defined on a σ-algebra.

McShane vs Pettis III

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THANKS FOR YOUR ATTENTION !!

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