The Birkhoff integral and the property of Bourgain

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Summary

We study Birkhoff integrability for functions defined on a complete probability space with values in a Banach space. This notion lies between Bochner and Pettis integrability and involves infinite Riemann-type sums.

- For real-valued functions Birkhoff and Lebesgue integrability coincide, [Fr15].
- For arbitrary Banach spaces we have

$$\textbf{Bochner} \implies \textbf{Birkhoff} \implies \textbf{Pettis}$$

- None of the reverse implications hold in general,[Bir35, Phi40].
- For separable Banach spaces: **Birkhoff = Pettis**, [Pet38].

Fréchet (1915)

Given $f : \Omega \longrightarrow \mathbb{R}$, for each partition Γ of Ω into countably many sets (A_n) of Σ consider a relative *upper* and *lower* integral by the expressions

$$J^*(f,\Gamma) = \sum_n \sup f(A_n)\, \mu(A_n) \quad \text{and} \quad J_*(f,\Gamma) = \sum_n \inf f(A_n)\, \mu(A_n),$$

(assuming both series are well defined and absolutely convergent).

Then the intersection of the "relative integral ranges"

$$J_*(f,\Gamma) \leq x \leq J^*(f,\Gamma),$$

for variable Γ is not empty. This intersection is a single point x if, and only if, f is Lebesgue integrable and $x = \int_\Omega f \; d\mu$.

Birkhoff (1935)

Let $f : \Omega \longrightarrow X$ be a function. If Γ is a partition of Ω into countably many sets (A_n) of Σ, the function f is called **summable** with respect to Γ if $f(A_n)$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f,\Gamma) = \left\{ \sum_n f(t_n)\mu(A_n) : \; t_n \in A_n \right\}$$

is made up of *unconditionally* convergent series.

The function f is said to be **Birkhoff integrable** if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of Ω in Σ for which f is summable and $\| \, \|$-diam $(J(f,\Gamma)) < \varepsilon$. In this case, the **Birkhoff integral** of f is the only point in the intersection

$$\bigcap \{ \overline{\mathrm{co}(J(f,\Gamma))} : \; f \text{ is summable with respect to } \Gamma \}.$$

Definition

We say that a family $\mathscr{H} \subset \mathbb{R}^\Omega$ has the **Birkhoff property** if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that

$$\left| \sum_{k=1}^{m} h(t_k)\mu(A_k) - \sum_{k=1}^{m} h(t'_k)\mu(A_k) \right| < \varepsilon$$

for all $t_k, t'_k \in A_k$, $k \in \mathbb{N}$, all $m \in \mathbb{N}$ and all $h \in \mathscr{H}$.

Definition ([RS85])

We say that a family $\mathscr{H} \subset \mathbb{R}^\Omega$ has the **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \ldots, A_n \subset A$, $A_i \in \Sigma$ with $\mu(A_i) > 0$, such that for every $h \in \mathscr{H}$

$$\inf_{1 \le i \le n} |\cdot|\text{-diam } (h(A_i)) < \varepsilon.$$

Lemma

Let $\mathscr{H} \subset \mathbb{R}^\Omega$ be a family of functions. Then:

(i) if $\mathscr{H}$ has the Birkhoff property, then $\mathscr{H}$ has the Bourgain property;

(ii) if $\mathscr{H}$ is *uniformly bounded* and has the Bourgain property, then $\mathscr{H}$ has the Birkhoff property.

Theorem

Let $f : \Omega \rightarrow X$ be a *bounded* function. The following conditions are equivalent:

(i) f is Birkhoff integrable;

(ii) the family $Z_f = \{\langle x^*, f\rangle : \; x^* \in X^*, \; \|x^*\| \leq 1\}$ has the Bourgain property.

Corollary

Let $f : \Omega \rightarrow X^*$ be a *bounded* function. The following conditions are equivalent:

(i) f is Birkhoff integrable;

(ii) the family $\{\langle f, x\rangle : \; x \in X, \; \|x\| \leq 1\}$ has the Bourgain property.

Lemma

Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:

(i) the family $Z_f = \{\langle x^*, f\rangle : \ x^* \in X^*, \ \|x^*\| \leq 1\}$ has the Bourgain property;

(ii) the family Z_f has the Birkhoff property.

In this case, there is a countable partition (A_n) of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.

Theorem

Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:

(i) f is Birkhoff integrable;

(ii) the family Z_f is a uniformly integrable subset of $\mathscr{L}^1(\mu)$ with the Bourgain property.

Corollary

Let $f : \Omega \longrightarrow X^*$ be a function. The following conditions are equivalent:

(i) f is Birkhoff integrable;

(ii) the family $\{\langle f, x\rangle : \ x \in X, \ \|x\| \leq 1\}$ is a uniformly integrable subset of $\mathscr{L}^1(\mu)$ with the Bourgain property.

Definition ([Mus79])

A Banach space X has the **weak Radon-Nikodým property (WRNP)** if for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X$ of σ-finite variation, there is a Pettis integrable function $f : \Omega \longrightarrow X$ such that

$$\nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \Sigma.$$

Theorem

Let X be a Banach space. The following conditions are equivalent:

(i) X^* has the WRNP;

(ii) X does not contain a copy of ℓ^1;

(iii) for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation, there is a *Birkhoff* integrable function $f : \Omega \longrightarrow X^*$ such that

$$\nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \Sigma.$$

References

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