Uniqueness of measure extensions in Banach spaces

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Banach space theory: classical topics and new directions

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The question above has *affirmative answer* if μ and ν are **Radon**.

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Corollary (Talagrand, 1984)

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In general:

$$\text{PIP} \implies \text{UMEP} \implies (X, w) \text{ realcompact.}$$

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So the completion $\tilde{\vartheta}$ of each Radon measure ϑ on $\mathrm{Borel}(X, w^*)$ can be restricted to $\mathrm{Baire}(X, w)$.

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(iii) If X has the PIP, then every measure on $\mathrm{Baire}(X, w^*)$ can be extended in a unique way to a measure on $\mathrm{Baire}(X, w)$.

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where the red inclusion is due to Haydon (1976).

So the completion $\tilde{\vartheta}$ of each Radon measure ϑ on $\mathrm{Borel}(X, w^*)$ can be restricted to $\mathrm{Baire}(X, w)$. Set $\vartheta^0 = \tilde{\vartheta}|_{\mathrm{Baire}(X,w)}$.

Theorem

(i) A measure μ on $\mathrm{Baire}(X, w)$ is convexly τ-additive if and only if $\mu = \vartheta^0$ for some Radon measure ϑ on $\mathrm{Borel}(X, w^*)$.

(ii) Every measure on $\mathrm{Baire}(X, w^*)$ can be extended in a unique way to a convexly τ-additive measure on $\mathrm{Baire}(X, w)$.

(iii) If X has the PIP, then every measure on $\mathrm{Baire}(X, w^*)$ can be extended in a unique way to a measure on $\mathrm{Baire}(X, w)$.

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This is the case if X has the property (C) of Corson.

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