The weak topology on L^p of a vector measure

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for some **regular Borel measure** μ on K.

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Theorem (Curbera 1992)

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Theorem (Ferrando-R. 2007)

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In particular, the weak topology and $\sigma(Y, B)$ coincide on any bounded subset of Y.

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