Series and integrals in infinite-dimensional spaces

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I. Series

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Both notions of convergence are equivalent.

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If $\sum_{n=1}^{\infty} x_n$ converges but is not unconditionally convergent, then $\forall x \in \mathbb{R}$ there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges with sum x.

Series of vectors in $\mathbb{R}^k$

Consider $\mathbb{R}^k$ equipped with the euclidean norm $\|\cdot\|_2$, i.e.

$$\|(x(1),\dots,x(k))\|_2 = \sqrt{x(1)^2+\cdots+x(k)^2},$$

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We can work coordinate by coordinate!

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Any Hilbert space, for instance

$$\ell^2(\mathbb{N}) = \left\{x = (x(i))_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \ \|x\|_2 = \left(\sum_{i=1}^{\infty} |x(i)|^2\right)^{\frac{1}{2}} < \infty\right\},$$

equipped with the norm $\|\cdot\|_2$.

Series in Banach spaces: the Dvoretzki-Rogers theorem

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Absolutely convergent $\Rightarrow$ Unconditionally convergent

Theorem (Dvoretzki-Rogers, 1950)

If X is infinite-dimensional, then there is a sequence (x_n) in X such that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent but **not** absolutely convergent.

Series in Hilbert spaces

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Example

$\forall n \in \mathbb{N}$, let $x_n \in \ell^2(\mathbb{N})$ be defined by $x_n(i) = \delta_{n,i}/n$. Then $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent but not absolutely convergent.

Series in Hilbert spaces

Theorem (Orlicz, 1930)

Let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series in $\ell^2(\mathbb{N})$. Then

$$\sum_{n=1}^{\infty} \|x_n\|_2^{2} < \infty.$$

II. Integrals

Fremlin-Mendoza (1994)

The ordinary functional analyst is naturally impatient with the multiplicity of definitions of 'integral' which have been proposed for vector-valued functions, and would prefer to have a single canonical one for general use.

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The Riemann integral of real-valued functions

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Theorem (Lebesgue)

A bounded function $f : [0,1] \to \mathbb{R}$ is Riemann integrable if and only if the set of points of discontinuity of f has measure 0.

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But f is discontinuous at every point!

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Theorem (Frechet, 1915)

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For a *non necessarily bounded* function... we need to consider countable partitions,

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A *bounded* function $f : [0,1] \to \mathbb{R}$ is **Lebesgue** integrable, with integral $\alpha \in \mathbb{R}$, if and only if for each $\varepsilon > 0$ there is a partition of $[0,1]$ into *finitely* many measurable subsets $A_1, \dots, A_n$ such that

$$\left| \sum_{i=1}^{n} \operatorname{meas}(A_i) f(t_i) - \alpha \right| \le \varepsilon$$

for every choice of points $t_i \in A_i$.

For a *non necessarily bounded* function... we need to consider countable partitions, requiring that the series

$$\sum_{i} \operatorname{meas}(A_i) f(t_i)$$

are unconditionally convergent.

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This way of presenting the theory of integration due to Mr. Lebesgue has the advantage, over the way Mr. Lebesgue presented his theory himself, that is very much close to the views of Riemann-Darboux to which many students are familiar with.

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In this case, the Birkhoff integral of f is

$$\left(\int_0^1 f_i(t)\ dt\right)_{i=1}^{\infty} \in \ell^2(\mathbb{N}).$$

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