Norming sets and integration with respect to vector measures

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Joint work with A. Fernández, F. Mayoral and F. Naranjo (Sevilla)

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 (Ω, Σ) measurable space, X Banach space with dual X^{*}, $v : \Sigma \to X$ countably additive (c.a.) vector measure

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 - weakly *v*-integrable if $f \in \mathscr{L}^1(x^*v)$ for all $x^* \in X^*$;

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Definition

A measurable function $f: \Omega \to \mathbb{R}$ is:

- weakly *v*-integrable if $f \in \mathscr{L}^1(x^*v)$ for all $x^* \in X^*$;
- Strongly *v*-integrable if it is weakly *v*-integrable and for each A ∈ Σ there is a vector ∫_A f dv ∈ X such that

$$x^*\left(\int_A f \, dv\right) = \int_A f \, d(x^*v)$$
 for all $x^* \in X^*$.

Here $x^*v : \Sigma \to \mathbb{R}$ denotes the composition of v and x^* .

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Question

What happens if we replace X^* by a smaller set of functionals in the previous definition ?????

Grothendieck (1952)

Let (f_n) be a bounded sequence in C(K) and $f \in C(K)$. If $f_n(t) \rightarrow f(t)$ for all $t \in K$, then $f_n \rightarrow f$ weakly.



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Let $B \subset B_{X^*}$ be a James boundary. A set function $\mu : \Sigma \to X$ is c.a. iff $x^*\mu$ is c.a. for all $x^* \in B$.

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Pfitzner (2008)

Let $B \subset B_{X^*}$ be a James boundary. Let $H \subset X$ be a bounded set. If H is $\sigma(X, B)$ -compact, then H is weakly compact.

Uniform Boundedness Theorem

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► $T \subset X^*$ is w^* -thick (Fonf, 1989) if, whenever we write $T = \bigcup T_n$ with $T_n \subset T_{n+1}$, there is *m* such that $\inf_{x \in S_X} \sup_{x^* \in T_m} |x^*(x)| > 0$.

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- A set $H \subset X$ is bounded iff $x^*(H)$ is bounded for all $x^* \in T$.
- 2 A series $\sum x_n$ in X is weakly unconditionally Cauchy iff

$$\sum |x^*(x_n)| < \infty \quad \text{for all } x^* \in \mathbf{T}.$$

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▶ Each of these properties characterizes *w*^{*}-thickness.

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 $N \subset X^*$

 $\mathscr{L}^1_{N}(v) := \{ f : \Omega \to \mathbb{R} \text{ measurable} : f \in \mathscr{L}^1(x^*v) \text{ for all } x^* \in N \}$

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Question

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Theorem

1 If *N* is
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-thick, then $\left| \mathscr{L}^1_w(v) = \mathscr{L}^1_N(v) \right|$

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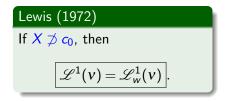
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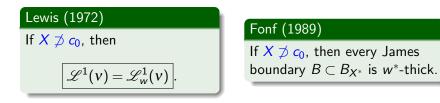
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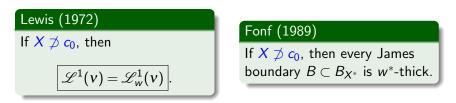
When $X \not\supseteq c_0$

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Corollary

Suppose $X \not\supseteq c_0$ and let $B \subset B_{X^*}$ be a James boundary. Then

$$\mathscr{L}^1(\mathbf{v}) = \mathscr{L}^1_B(\mathbf{v})$$
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$N \subset X^*$ norming

That is, there is $\lambda \geq 1$ such that

 $\|x\| \leq \lambda \sup\{x^*(x) : x^* \in \operatorname{span}(N) \cap B_{X^*}\}$ for all $x \in X$.

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Definition

Let $f \in \mathscr{L}^1_N(v)$. We say that $f \in \mathscr{L}^1_{N,s}(v)$ if for each $A \in \Sigma$ there is a vector $\int_A^N f \, dv \in X$ such that

$$x^*\left(\int_A^N f\,dv\right) = \int_A f\,d(x^*v) \quad \text{for all } x^* \in N.$$

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Theorem

Let $f \in \mathscr{L}^1_{N,s}(v)$. Then:

() The set function $\int_{(\cdot)}^{N} f \, dv$ is bounded and finitely additive.

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Theorem

Let $f \in \mathscr{L}^{1}_{N,s}(v)$. Then: **1** The set function $\int_{(\cdot)}^{N} f \, dv$ is bounded and finitely additive. **2** $f \in \mathscr{L}^{1}_{w}(v)$.

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That is, there is $\lambda \geq 1$ such that

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Theorem

Let f ∈ L¹_{N,s}(v). Then:
The set function ∫^N_(·) f dv is bounded and finitely additive.
f ∈ L¹_w(v).
f ∈ L¹(v) ⇔ ∫^N_(·) f dv is countably additive.

Question

When does the equality
$$\mathscr{L}^1(v) = \mathscr{L}^1_{N,s}(v)$$
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When does the equality $\mathscr{L}^1(v) = \mathscr{L}^1_{N,s}(v)$ hold ?????

► *N* has the **Orlicz property** (Thomas, 1970) if for every *X*-valued set function μ defined on a σ -algebra we have:

 $x^*\mu$ countably additive for all $x^* \in N \implies \mu$ countably additive.

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Examples of sets having the Orlicz property	
• Any norming set when $X \not\supseteq \ell_{\infty}$.	

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Examples of sets having the Orlicz property

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- Any norming set when $X \not\supseteq \ell_{\infty}$.
- James boundaries.

Final Remarks

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Theorem

Suppose $X = Y^*$ for a Banach space Y. Then

$$\mathscr{L}^{1}_{Y,s}(v) = \mathscr{L}^{1}_{w}(v) = \mathscr{L}^{1}_{Y}(v).$$

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Summarizing

For any norming set $N \subset X^*$ we have:

$$\mathscr{L}^1(v) \subset \mathscr{L}^1_{N,s}(v) \subset \mathscr{L}^1_w(v) \subset \mathscr{L}^1_N(v)$$

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Summarizing

For any norming set $N \subset X^*$ we have:

$$\mathscr{L}^1(v) \subset \mathscr{L}^1_{N,s}(v) \subset \mathscr{L}^1_w(v) \subset \mathscr{L}^1_N(v)$$

▶ There are examples making clear that in the previous chain all combinations of " \subseteq " and "=" are possible.

Some References

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