Norming sets and integration with respect to vector measures

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Our General Problem

(Ω, Σ) measurable space, X Banach space with dual X^*,
$\nu : \Sigma \to X$ countably additive (c.a.) vector measure

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2. **strongly ν-integrable** if it is weakly ν-integrable and for each $A \in \Sigma$ there is a vector $\int_A f\, d\nu \in X$ such that

$$\boxed{x^*\left(\int_A f\, d\nu\right) = \int_A f\, d(x^*\nu)} \quad \text{for all } x^* \in X^*.$$

Here $x^*\nu : \Sigma \to \mathbb{R}$ denotes the composition of ν and x^*.

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Question

What happens if we replace X^* by a smaller set of functionals in the previous definition ?????

Weak Convergence Tests

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Grothendieck (1952)

Let (f_n) be a bounded sequence in $C(K)$ and $f \in C(K)$.
If $f_n(t) \to f(t)$ for all $t \in K$, then $f_n \to f$ weakly.

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Let (x_n) be a bounded sequence in X and $x \in X$. If $x^*(x_n) \to x^*(x)$ for all $x^* \in \mathrm{ext}(B_{X^*})$, then $x_n \to x$ weakly.

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Pfitzner (2008)

Let $B \subset B_{X^*}$ be a James boundary. Let $H \subset X$ be a bounded set.
If H is $\sigma(X, B)$-compact, then H is weakly compact.

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- Each of these properties characterizes w^*-thickness.

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When does the equality $\mathscr{L}^1_w(\nu) = \mathscr{L}^1_N(\nu)$ hold ?????

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2. If $\mathscr{L}^1_w(\mu) = \mathscr{L}^1_N(\mu)$ for every c.a. X-valued measure μ, then N is w^*-thick.

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If $X \not\supset c_0$, then every James boundary $B \subset B_{X^*}$ is w^*-thick.

Corollary

Suppose $X \not\supset c_0$ and let $B \subset B_{X^*}$ be a James boundary. Then

$$\boxed{\mathscr{L}^1(\nu) = \mathscr{L}^1_B(\nu)}.$$

Strong Integrability I

$N \subset X^*$ norming

That is, there is $\lambda \geq 1$ such that

$$\|x\| \leq \lambda \sup\{x^*(x) : x^* \in \mathrm{span}(N) \cap B_{X^*}\} \quad \text{for all } x \in X.$$

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Definition

Let $f \in \mathscr{L}^1_N(\nu)$. We say that $f \in \mathscr{L}^1_{N,s}(\nu)$ if for each $A \in \Sigma$ there is a vector $\int_A^N f \, d\nu \in X$ such that

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$$x^*\mu \text{ countably additive for all } x^* \in N \implies \mu \text{ countably additive.}$$

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Summarizing

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Summarizing

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- There are examples making clear that in the previous chain all combinations of "$\subsetneq$" and "$=$" are possible.

Some References

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