Uniqueness of measure extensions in Banach spaces

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Abstract

Let X be a Banach space, $B \subset B_{X^*}$ a norming set and $\sigma(X,B)$ the topology on X of pointwise convergence on B. We study the following question:

Given two (non negative, countably additive and finite) measures μ_1 and μ_2 on $\mathrm{Baire}(X,w)$,

$$\mu_1|_{\mathrm{Baire}(X,\sigma(X,B))} = \mu_2|_{\mathrm{Baire}(X,\sigma(X,B))} \quad \Longrightarrow \quad \mu_1 = \mu_2 \ ?$$

While this is not true in general, the answer is affirmative provided that both μ_1 and μ_2 are convexly τ-additive (and this condition holds automatically whenever X has the Pettis Integral Property).

For a Banach space Y not containing isomorphic copies of ℓ^1, we show that Y^* has the Pettis Integral Property if and only if every measure on $\mathrm{Baire}(Y^*,w^*)$ admits a unique extension to $\mathrm{Baire}(Y^*,w)$.

We also discuss the coincidence of the two σ-algebras involved in such results.

1 Introduction

All the measures considered in this poster are non negative, countably additive and finite.

From now on X is a real Banach space with topological dual X^*. We write w (resp. w^*) to denote the weak topology on X (resp. the weak* topology on X^*). As usual, we denote by B_X the closed unit ball of X.

According to a result of G. A. Edgar [3], the Baire σ-algebra of a locally convex space endowed with its weak topology is exactly the σ-algebra generated by the elements of the topological dual. Thus a function $f : \Omega \to X$, defined on a probability space $(\Omega, \Sigma, \lambda)$, is scalarly measurable if and only if it is Σ-$\mathrm{Baire}(X,w)$-measurable. Assume further that f is scalarly bounded, i.e. the family of compositions $\{x^* \circ f : x^* \in B_{X^*}\}$ is a bounded subset of $\mathcal{L}^\infty(\lambda)$. It was shown by M. Talagrand [14] that f is Pettis integrable if and only if the image measure λf^{-1} on $\mathrm{Baire}(X,w)$ is convexly τ-additive, in the following sense.

Definition 1 *A measure μ on $\mathrm{Baire}(X,w)$ is convexly τ-additive if for each decreasing net (C_α) of convex closed elements of $\mathrm{Baire}(X,w)$ with $\bigcap_\alpha C_\alpha = \emptyset$, we have $\lim_\alpha \mu(C_\alpha) = 0$.*

Recall that X has the Pettis Integral Property (PIP) if every scalarly bounded X-valued function is Pettis integrable; see for instance [9] and the references therein. As a consequence of Talagrand's result, one can deduce that

X has the PIP $\Longleftrightarrow$ every measure on $\mathrm{Baire}(X,w)$ is convexly τ-smooth.

In view of the characterization above, it is clear that every Banach space with the so-called property (C) of Corson [1] (e.g. a weakly compactly generated space or, more generally, a weakly Lindelöf one) has the PIP.

Now let $B \subset B_{X^*}$ be a norming set, i.e. $\|x\| = \sup\{|x^*(x)| : x^* \in B\}$ for every $x \in X$. The topology $\sigma(X,B)$ on X is the coarsest one for which each element of B is continuous. In view of the aforementioned theorem of G. A. Edgar, $\mathrm{Baire}(X,\sigma(X,B))$ is just the σ-algebra on X generated by B. Clearly, we have $\mathrm{Baire}(X,\sigma(X,B)) \subset \mathrm{Baire}(X,w)$.

The purpose of this work is to discuss the following general question, which turns out to be closely related to the Pettis integral theory:

Given two measures μ_1 and μ_2 on $\mathrm{Baire}(X,w)$ which coincide on $\mathrm{Baire}(X,\sigma(X,B))$, does it follow that $\mu_1 = \mu_2$?

2 Uniqueness of convexly τ-additive extensions

It is convenient to introduce a definition.

Definition 2 *Let $\Sigma' \subset \Sigma$ be two σ-algebras on a set Ω and let $\mathcal{M}$ be a family of measures on Σ. We say that Σ' has the uniqueness property with respect to $\mathcal{M}$ if for every pair $\mu_1, \mu_2 \in \mathcal{M}$ we have*

$$\mu_1|_{\Sigma'} = \mu_2|_{\Sigma'} \quad \Longrightarrow \quad \mu_1 = \mu_2.$$

As a consequence of the results of R. G. Douglas [2] we can characterize the uniqueness property as follows.

Proposition 1 *Let $\Sigma' \subset \Sigma$ be two σ-algebras on a set Ω and let $\mathcal{M}$ be a family of measures on Σ such that:*

- *$\mu_1 + \mu_2 \in \mathcal{M}$ for every $\mu_1, \mu_2 \in \mathcal{M}$.*
- *If $\nu \in \mathcal{M}$ and μ is a measure on Σ such that $\mu \leq \nu$, then $\mu \in \mathcal{M}$.*

Then the following statements are equivalent:

(i) Σ' has the uniqueness property with respect to $\mathcal{M}$.

(ii) For every $\mu \in \mathcal{M}$ and every $E \in \Sigma$ there exists $E' \in \Sigma'$ such that $\mu(E \triangle E') = 0$.

Our approach to the uniqueness problem relies on G. A. Edgar's work [5] (going back to A. Ionescu-Tulcea [8]) about the continuity of the mapping

$$I : \mathcal{F} \to \mathbb{R}, \quad I(f) = \int_\Omega f \, d\lambda,$$

where $(\Omega, \Sigma, \lambda)$ is a probability space and $\mathcal{F} \subset \mathcal{L}^1(\lambda) \subset \mathbb{R}^\Omega$ is endowed with the pointwise convergence topology. For a thorough study on this subject we refer the reader to [15].

We can now state a general result on the uniqueness of convexly τ-additive extensions.

Theorem 2 *Let $B \subset B_{X^*}$ be a norming set. Then $\mathrm{Baire}(X,\sigma(X,B))$ has the uniqueness property with respect to the family of all convexly τ-additive measures on $\mathrm{Baire}(X,w)$.*

Corollary 3 *Suppose that X has the PIP and let $B \subset B_{X^*}$ be a norming set. Then $\mathrm{Baire}(X,\sigma(X,B))$ has the uniqueness property with respect to the family of all measures on $\mathrm{Baire}(X,w)$.*

3 Uniqueness in dual Banach spaces

In this section we deal with a particular case of the situation considered above: we assume that $X = Y^*$ for some Banach space Y and we take $B = B_Y$, so that $\sigma(X,B)$ is just the weak* topology.

Definition 3 *We say that Y^* has property (U) if $\mathrm{Baire}(Y^*,w^*)$ has the uniqueness property with respect to the family of all measures on $\mathrm{Baire}(Y^*,w)$.*

From Corollary 3 it follows that

Y^* has the PIP $\Longrightarrow$ Y^* has property (U).

We do not know whether the reverse implication holds in general, but this is always the case provided that Y contains no subspace isomorphic to ℓ^1 (we write $Y \not\supset \ell^1$ for short), as we point out below.

When $Y \not\supset \ell^1$, a result of R. Haydon [7] ensures that the identity mapping $I : B_{Y^*} \to Y^*$ is Pettis integrable with respect to the completion of each Radon measure on $\mathrm{Borel}(B_{Y^*}, w^*)$. This fact is a basic tool to prove the following

Theorem 4 *Suppose that $Y \not\supset \ell^1$. Then every measure on $\mathrm{Baire}(Y^*,w^*)$ can be extended in a unique way to a convexly τ-additive measure on $\mathrm{Baire}(Y^*,w)$.*

Corollary 5 *Suppose that $Y \not\supset \ell^1$. The following statements are equivalent:*

(i) Y^* has the PIP.

(ii) Every measure on $\mathrm{Baire}(Y^*,w^*)$ can be extended in a unique way to a measure on $\mathrm{Baire}(Y^*,w)$.

(iii) Y^* has property (U).

Recall that a topological space is realcompact if it is homeomorphic to a closed subset of $\mathbb{R}^I$ for some set I. G. A. Edgar [4] showed that every Banach space with the PIP is realcompact for its weak topology. We can now obtain the same conclusion for any dual Banach space with property (U).

Proposition 6 *If Y^* has property (U), then (Y^*,w) is realcompact.*

The converse of Proposition 6 does not hold in general. Indeed, the space ℓ_∞ is weakly realcompact [1], whereas it fails property (U), as D. H. Fremlin pointed out to us. His example involves the so-called Talagrand's measure [13] already used in [6] when proving that ℓ_∞ does not enjoy the PIP.

4 Coincidence of $\mathrm{Baire}(X,\sigma(X,B))$ and $\mathrm{Baire}(X,w)$

Of course, our results on the uniqueness of measure extensions are not of interest when the norming set $B \subset B_{X^*}$ satisfies

$$\mathrm{Baire}(X,\sigma(X,B)) = \mathrm{Baire}(X,w).$$

For instance, this equality holds true whenever (B_{X^*}, w^*) is angelic (e.g. for a weakly compactly generated X).

In this section we discuss the coincidence of both σ-algebras in two particular cases: (i) Dual Banach spaces and the weak* topology. (ii) Banach spaces of real-valued continuous functions on a compact Hausdorff topological space and the pointwise convergence topology.

The Odell-Rosenthal theorem [11] states that a separable Banach space Y is w^*-sequentially dense in Y^{**} if and only if $Y \not\supset \ell^1$. For non necessarily separable spaces we have the following characterization due to M. Raja.

Proposition 7 *For a Banach space Y the following statements are equivalent:*

(i) Y is weak*-sequentially dense in Y^{**}.

(ii) $\mathrm{Baire}(Y^*,w^*) = \mathrm{Baire}(Y^*,w)$.

(iii) $Y \not\supset \ell^1$ and for every $y^{**} \in Y^{**}$ there exists a countable set $D \subset Y$ such that $y^{**} \in \overline{D}^{w^*}$.

For instance, $\ell^1(\omega_1)$ is an example of dual Banach space with the PIP for which $\mathrm{Baire}(\ell^1(\omega_1),w^*) \neq \mathrm{Baire}(\ell^1(\omega_1),w)$.

On the other hand, R. Pol's [12] dual characterization of property (C) allows us to deduce the following

Corollary 8 *Let Y be a Banach space such that Y^* has property (C). Then $\mathrm{Baire}(Y^*,w^*) = \mathrm{Baire}(Y^*,w)$.*

Given a compact Hausdorff topological space K, we write $C(K)$ to denote the Banach space of all real-valued continuous functions on K endowed with the supremum norm. Notice that the set $B = \{\delta_t : t \in K\} \subset B_{C(K)^*}$ of "point masses" is norming and that $\sigma(C(K),B)$ is just the pointwise convergence topology $\mathfrak{T}_p(K)$.

Proposition 9 *Let K be a compact Hausdorff topological space and ν a Radon measure on K. If the functional*

$$\iota(\nu) : C(K) \to \mathbb{R}, \quad h \mapsto \int_K h \, d\nu,$$

is $\mathrm{Baire}(C(K),\mathfrak{T}_p(K))$-measurable, then there exists a closed separable set $F \subset K$ such that $\nu(K \setminus F) = 0$.

Corollary 10 *Let K be a compact Hausdorff topological space such that $\mathrm{Baire}(C(K),\mathfrak{T}_p(K)) = \mathrm{Baire}(C(K),w)$. Then for each Radon measure ν on K there exists a closed separable set $F \subset K$ such that $\nu(K \setminus F) = 0$.*

Under the continuum hypothesis, there exists a compact Hausdorff topological space K (the Kunen-Haydon-Talagrand space, cf. [10]) such that $C(K)$ has the PIP and $\mathrm{Baire}(C(K),\mathfrak{T}_p(K)) \neq \mathrm{Baire}(C(K),w)$.

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