Open problems in measure and integration in Banach spaces

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Integration, Vector Measures and Related Topics IV
Dedicated to Joe Diestel
La Manga del Mar Menor, March 3rd, 2011

1. L^p spaces of vector measures
2. Ranges of Pettis integrals
3. McShane vs Pettis integrals
4. σ-algebras on Banach spaces

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Fix $1 \leq p < \infty$. The space $L^p(\nu)$ of all functions $f : \Omega \to \mathbb{R}$ for which $|f|^p$ is ν-integrable is a **Banach lattice** with the $\|\nu\|$-a.e. order and the norm

$$\|f\|_{L^p(\nu)} = \sup_{x^* \in B_{X^*}} \left(\int_\Omega |f|^p \, d|x^*\nu|\right)^{\frac{1}{p}}.$$

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Known facts on $\sigma(L^p(\nu), \Gamma)$

1. $\sigma(L^1(\nu), \Gamma) =$ weak topology on bdd sets if $L^1(\nu) \not\supset \ell^1$. [Curbera, Okada]

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4. $\sigma(L^p(\nu), \Gamma)$ = weak topology on bdd sets if $p > 1$. [Ferrando-R., Galaz]

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3. YES if X is a Banach lattice and ν is positive. [Ferrando-R.]

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- To characterize when Γ a James boundary of $L^p(\nu)$, for $1 \leq p < \infty$.

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For a Pettis integrable function $f : \Omega \to X$, TFAE:

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For a Pettis integrable function $f : \Omega \to X$, TFAE:

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2. There is a sequence of **simple** functions $f_n : \Omega \to X$ such that

$$\lim_{n \to \infty} \sup_{x^* \in B_{X^*}} \int_\Omega |x^* f_n - x^* f| \, d\mu = 0.$$

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Known facts

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Is $\nu_f(\Sigma)$ norm relatively compact if μ is a quasi-Radon probability **??**

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Being quasi-Radon means that there is a topology $\mathfrak{T} \subset \Sigma$ on Ω such that:

- μ is outer regular with respect to $\mathfrak{T}$;
- $\mu(\bigcup \mathscr{G}) = \sup_{G \in \mathscr{G}} \mu(G)$ for every upwards directed family $\mathscr{G} \subset \mathfrak{T}$.

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A Banach space has the **Gelfand-Phillips property** if every limited subset is norm relatively compact.

Question: $X \not\supset c_0 \implies X$ has the Gelfand-Phillips property **??**

Let X be a Banach space.

Definition

A function $f : [0,1] \to X$ is **McShane integrable** if there is $x \in X$ such that:

for each $\varepsilon > 0$ there is a function $\delta : [0,1] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{n} \lambda(A_i) f(t_i) - x \right\| < \varepsilon$$

for every finite partition $A_1, \ldots, A_n$ of $[0,1]$ into intervals and every choice of points $t_1, \ldots, t_n \in [0,1]$ satisfying $A_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$.

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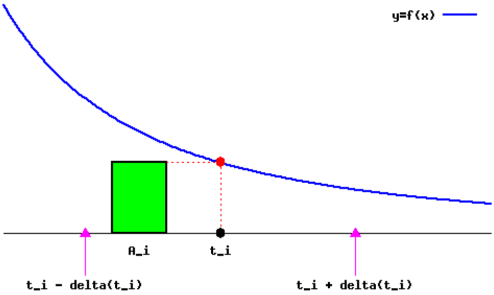

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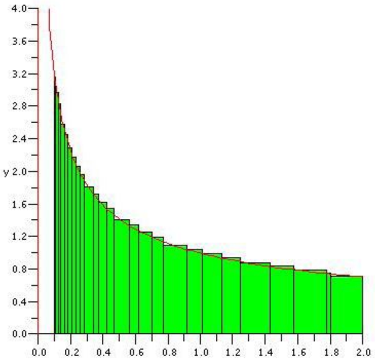

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- separable [Gordon]
- superreflexive (e.g. Hilbert) [Di Piazza-Preiss]

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Recall: a Banach space Y is **Hilbert generated** if there is a linear continuous map $T : \ell^2(\Gamma) \to Y$ having dense range.

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For more info, please attend Grzegorz Plebanek's lecture at 12:00.

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THANKS FOR YOUR ATTENTION !!