The Bourgain property and convex hulls

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Definition (Birkhoff, 1935)

$f : \Omega \longrightarrow X$ is **Birkhoff integrable**, with integral $x \in X$, if for each $\varepsilon > 0$ there is a partition of Ω into countably many measurable sets $A_1, A_2, \ldots$ such that

$$\left\| \sum_n \mu(A_n) f(t_n) - x \right\| \leq \varepsilon$$

for every choice $t_n \in A_n$, $n \in \mathbb{N}$, the series involved being *unconditionally* convergent.

In general ...

$$\text{Bochner} \underset{\not\Leftarrow}{\Rightarrow} \text{Birkhoff} \underset{\not\Leftarrow}{\Rightarrow} \text{Pettis}$$

Given $f : \Omega \longrightarrow X$ and $B \subset B_{X^*}$ *norming*, we define

$$\boxed{Z_{f,B} = \{x^* \circ f : \ x^* \in B\} \subset \mathbb{R}^\Omega.}$$

Definition (Bourgain, ∼1980)

$\mathscr{H} \subset \mathbb{R}^\Omega$ has the **Bourgain property** if, for each $\varepsilon > 0$ and each measurable set $A \subset \Omega$ of positive measure, there are finitely many measurable sets $A_1, \dots, A_n \subset A$ of positive measure such that

$$\sup_{h \in \mathscr{H}} \min_{1 \le i \le n} \sup_{t,s \in A_i} |h(t) - h(s)| \le \varepsilon.$$

Theorem (Cascales & R., 2005)

$f : \Omega \longrightarrow X$ is Birkhoff integrable if and only if $Z_{f,B_{X^*}}$ is uniformly integrable and has the Bourgain property.

We can **replace** $Z_{f,B_{X^*}}$ **with** $Z_{f,B}$ in some cases:

- f is *bounded* and $B \subset B_{X^*}$ is *any* norming set.
- $X = Y^*$ and $B = B_Y$.

(Cascales & R., 2005)

Question 1

Let $f : \Omega \longrightarrow X$ and $B \subset B_{X^*}$ norming.

$Z_{f,B}$ is uniformly integrable and has the Bourgain property.

$\Downarrow$???

f is Birkhoff integrable.

Question 2

Let $\mathscr{H} \subset \mathbb{R}^{\Omega}$ be pointwise bounded.

$\mathscr{H}$ has the Bourgain property.

$\Downarrow$???

$\mathrm{co}(\mathscr{H})$ has the Bourgain property.

Remark

Question 1 has *affirmative* answer if and only if there is a partition of Ω into countably many measurable sets $A_1, A_2, \ldots$ such that

$$\text{the restriction } f|_{A_n} \text{ is bounded whenever } \mu(A_n) > 0.$$

This is the case if B is *convex*.

Remark

Question 2 has *affirmative* answer if and only if there is a partition of Ω into countably many measurable sets $A_1, A_2, \ldots$ such that

$$\mathscr{H}|_{A_n} = \{h|_{A_n} : \ h \in \mathscr{H}\}$$

is *uniformly* bounded whenever $\mu(A_n) > 0$.

Example (Fremlin)

There is a pointwise bounded $\mathscr{H} \subset \mathbb{R}^{[0,1]}$ such that:

- $\mathscr{H}$ has the Bourgain property.
- For each $h \in \mathscr{H}$ we have $h = 0$ a.e.
- $\mathrm{co}(\mathscr{H})$ does not have the Bourgain property.

Corollary

There are $f : [0,1] \longrightarrow c_0(\mathfrak{c})$ and $B \subset B_{c_0(\mathfrak{c})^*}$ norming such that:

- $Z_{f,B}$ has the Bourgain property.
- For each $x^* \in c_0(\mathfrak{c})^*$ we have $x^* \circ f = 0$ a.e.
- f is not Birkhoff integrable.

Theorem

Let $\mathscr{H} \subset \mathbb{R}^{\Omega}$ be pointwise bounded.
If $\mathscr{H}$ has the Bourgain property, then

$$\overline{\mathrm{aco}(\mathscr{H})}^{\mathfrak{T}_p(\Omega)} \text{ is made up of measurable functions.}$$

Corollary

Let $f : \Omega \longrightarrow X$ and $B \subset B_{X^*}$ norming.
If $Z_{f,B}$ has the Bourgain property, then f is *scalarly measurable*:

$$x^* \circ f \text{ is measurable for every } x^* \in X^*.$$

We consider the following cardinal number:

$$\kappa(\mu) = \min\{\#(\mathscr{E}) : \mathscr{E} \subset \Sigma,\ \mu(E) = 0\ \forall E \in \mathscr{E},\ \mu^*(\cup\mathscr{E}) > 0\}$$

provided that such families $\mathscr{E}$ exist (e.g. μ atomless).

Lemma

Suppose that $\mathrm{dens}(B_{X^*}, w^*) < \kappa(\mu)$.
If $f : \Omega \longrightarrow X$ is scalarly measurable, then the function

$$\|f\| : \Omega \longrightarrow \mathbb{R}, \quad t \mapsto \|f(t)\|,$$

is measurable.

Theorem

Suppose that $\mathrm{dens}(B_{X^*}, w^*) < \kappa(\mu)$.
Let $f : \Omega \longrightarrow X$ and $B \subset B_{X^*}$ norming such that

$Z_{f,B}$ is uniformly integrable and has the Bourgain property.

Then f is Birkhoff integrable.

Examples where $\mathrm{dens}(B_{X^*}, w^*) < \kappa(\mu)$:

- B_{X^*} is w^*-separable.
- (Under *Martin's Axiom*) $\mathrm{dens}(X) < \mathfrak{c}$ and $\mu \equiv$ Lebesgue measure on $[0,1]$.

A few references

- G. Birkhoff, *Integration of functions with values in a Banach space*, Trans. Amer. Math. Soc. **38** (1935), no. 2, 357–378.
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