Measurability in $C(2^\kappa)$ and Kunen cardinals

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- $\mathrm{Ba}(X,w) = \mathrm{Bo}(X)$ for $X = \ell^1(\omega_1)$. [Fremlin]

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for every **metric space** M with $\mathrm{weight}(M) = \kappa$.

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Question

Is there a **non-metrizable** K such that $\mathrm{Ba}(C_p(K)) = \mathrm{Bo}(C(K))$?

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- $\ell^1(\kappa) \hookrightarrow C(2^\kappa)$.
- $\ell^1(\omega_1) \hookrightarrow Y$ for every non-separable subspace $Y \hookrightarrow C(2^{\omega_1})$. [Hagler]

Main result

Theorem [Avilés-Plebanek-R.]

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Corollary

If G is a **compact group** and $\mathrm{weight}(G)$ is a Kunen cardinal, then

$$\mathrm{Ba}(C_p(G)) = \mathrm{Bo}(C(G)).$$

The case $\kappa = \omega_1$

Proposition

Let K be a compact space satisfying:

($\star$) *For each $n \in \mathbb{N}$ and each closed set $F \subseteq K^n$ there is a decreasing sequence (F_m) of closed separable subsets of K^n s.t. $F = \bigcap F_m$.*

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- **Consequence:**

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