Completeness in the Mackey topology

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▶▶ Let $Y \subset X^*$ be a **w*-dense** subspace.

Definition

The **Mackey topology** $\mu(X, Y)$ is the topology on X of uniform convergence on elements of the family

$$\mathscr{K}(Y) := \{K \subset Y : \ K \text{ is } \textbf{absolutely convex} \text{ and } \textbf{w}^*\textbf{-compact}\}.$$

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Theorem (Grothendieck)

$(X, \mu(X, Y))$ is **complete** $\iff$ for every linear functional $f : Y \to \mathbb{R}$ we have:
If $f|_K$ *is* **w***-**continuous** $\forall K \in \mathscr{K}(Y)$, *then* f *is* **w***-**continuous**
(i.e. $\exists x \in X$ *s.t.* $f(y) = \langle x, y\rangle \ \forall y \in Y$*).*

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Question (Arendt, Kunze)

Is $(X, \mu(X, Y))$ is **complete** for every **norming** and $\|\cdot\|$**-closed** $Y \subset X^*$ **???**

Negative answer by Bonet and Cascales

Example

Let $X = \ell_1([0,1])$ and $Y = C([0,1]) \subset X^* = \ell_\infty([0,1])$.
Then Y is norming and $\|\cdot\|$-closed, but $(X, \mu(X,Y))$ is **not** complete.

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Sketch of proof:

- Define $f : Y \to \mathbb{R}$ by $f(y) := \int_0^1 y(t)\,dt$.

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More generally:

Theorem

$X \supset \ell_1(\mathfrak{c}) \implies \exists Y \subset X^*$ norming and $\|\cdot\|$-closed subspace such that $(X, \mu(X,Y))$ is **not** complete.

Further advances by Guirao, Montesinos and Zizler

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Suppose Y is $\|\cdot\|$-**closed** and (B_{X^*}, w^*) is **Fréchet-Urysohn**. Then:

$$(X, \mu(X,Y)) \text{ is complete} \iff Y \text{ is norming} \iff (Y, w^*) \text{ is Mazur}.$$

Definition

We say that X is **universally Mackey complete** [resp. **universally Mazur**] iff

$$(X, \mu(X, Y)) \text{ is complete} \quad [\text{resp. } (Y, w^*) \text{ is Mazur}]$$

for every norming and $\|\cdot\|$-closed subspace $Y \subset X^*$.

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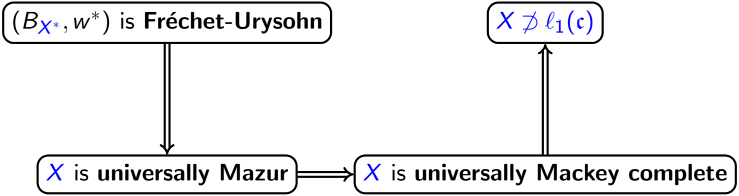

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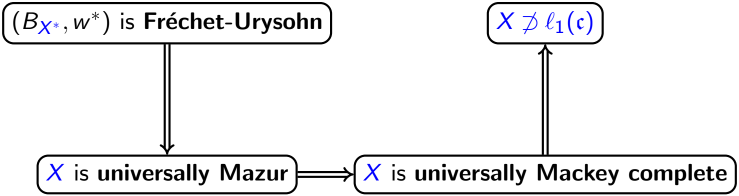

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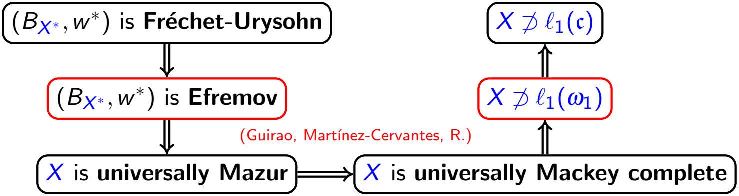

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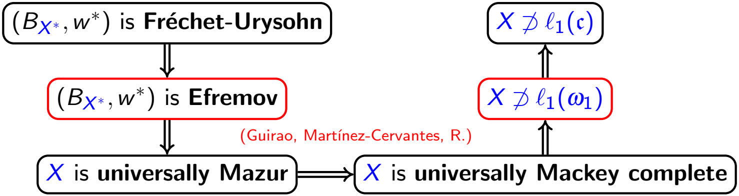

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$$\overline{C}^{w^*} = \{x^* \in X^* : \exists (x_n^*) \subset C \text{ s.t. } x_n^* \xrightarrow{w^*} x^*\}$$

for every convex set $C \subset B_{X^*}$.

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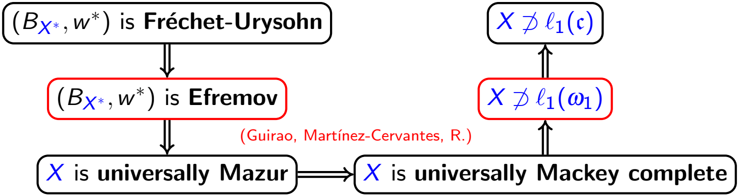

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Efremov $\neq$ Fréchet-Urysohn under CH.

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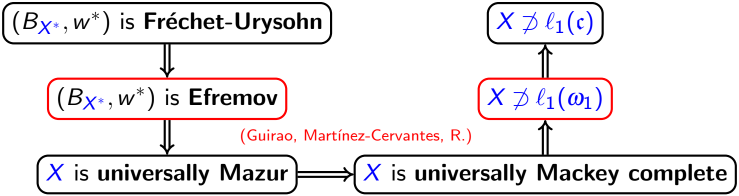

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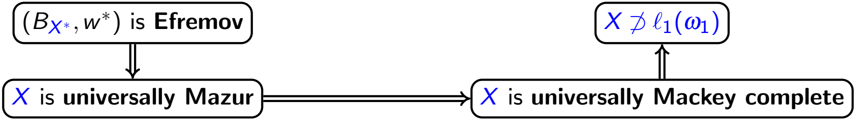
It is unknown what happens in ZFC or under other axioms.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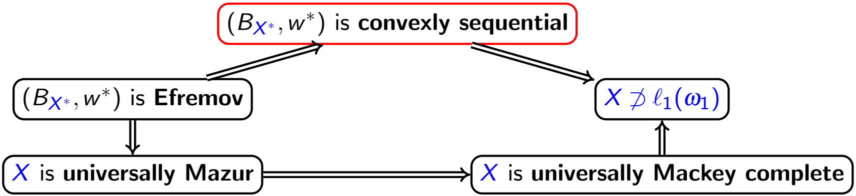
(B_{X^*}, w^*) is **Efremov** $\Downarrow$ X is **universally Mazur** $\Longrightarrow$ X is **universally Mackey complete** $\Rightarrow$ $X \not\supset \ell_1(\omega_1)$

Definition

We say that (B_{X^*}, w^*) is **(convexly) sequential** iff

for every **(convex) non**-w^*-closed set $C \subset B_{X^*}$

there exist $x^* \in X^* \setminus C$ and $(x_n^*) \subset C$ such that $x_n^* \xrightarrow{w^*} x^*$.



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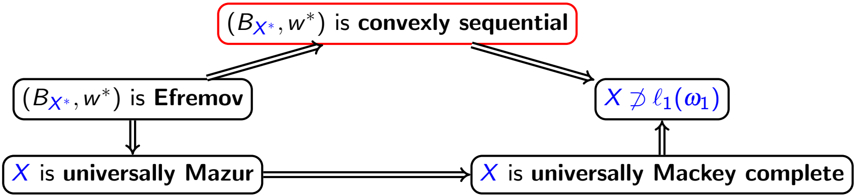

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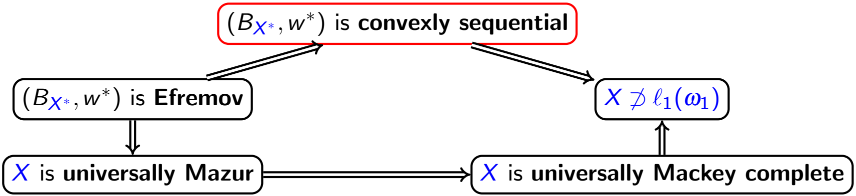
Theorem (Guirao, Martínez-Cervantes, R.)

Suppose (B_{X^*}, w^*) is convexly sequential. Then:

$$X \text{ is universally Mazur} \iff X \text{ is universally Mackey complete.}$$



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Theorem (Guirao, Martínez-Cervantes, R.)

Under CH, there is a Banach space X such that (B_{X^*}, w^*) is sequential and X is **not** universally Mackey complete.