On the equivalence of McShane and Pettis integrability in non-separable Banach spaces

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Abstract

We show that McShane and Pettis integrability coincide for functions $f:[0,1]\to L^1(\mu)$, where μ is any finite measure. On the other hand, assuming the Continuum Hypothesis, we prove that there exist a weakly Lindelöf determined Banach space X, a scalarly null (hence Pettis integrable) function $h:[0,1]\to X$ and an absolutely summing operator $u:X\to Y$ (where Y is another Banach space) such that the composition $u\circ h:[0,1]\to Y$ is not Bochner integrable; in particular, h is not McShane integrable.

1 Introduction

The Riemann integral of a function $f:[0,1]\to\mathbb{R}$ is obtained as the limit of sums of the form $\sum_{i=1}^n (b_i-b_{i-1})f(t_i)$ when $\max_{1\le i\le n}(b_i-b_{i-1})\to 0$, where $0=b_0\le t_1\le b_1\le\cdots\le b_{n-1}\le t_n\le b_n=1$. Kurzweil and Henstock modified this limit process to obtain a notion of integral, usually called *Kurzweil-Henstock integral*, which extends Lebesgue's one, see e.g. [12]. Roughly, in this integration theory one requires that the integral is well approximated by means of the "Riemann sums" associated to all tagged partitions such that $b_i-b_{i-1}\le\delta(t_i)$ for all i, where δ is a certain positive function. The *McShane integral*, introduced in [14], is obtained from the Kurzweil-Henstock integral by dropping the restriction "$t_i\in[b_{i-1},b_i]$" and considering those tagged partitions such that $[b_{i-1},b_i]\subset[t_i-\delta(t_i),t_i+\delta(t_i)]$ for all i. Curiously, this variant allows to recover the Lebesgue integral:

Theorem (McShane). *A function $f:[0,1]\to\mathbb{R}$ is Lebesgue integrable if and only if there is $\alpha\in\mathbb{R}$ with the following property: for each $\varepsilon>0$ there is a function $\delta:[0,1]\to\mathbb{R}^+$ such that*

$$\Big|\sum_{i=1}^n (b_i-b_{i-1})f(t_i)-\alpha\Big|\le\varepsilon$$

for every partition $0=b_0<b_1<\cdots<b_n=1$ and every choice of points $t_1,\dots,t_n\in[0,1]$ such that $[b_{i-1},b_i]\subset[t_i-\delta(t_i),t_i+\delta(t_i)]$ for all i. In this case, $\alpha=\int_0^1 f(t)\,dt$.

Several methods of integration for functions taking values in a Banach space $(X,\|\cdot\|)$ have been studied over the years. Among these methods, those developed by Bochner [5] and Pettis [15, 19] have been the most popular ones. Recall that a function $f:[0,1]\to X$ is called:

- *Bochner integrable* if it is *strongly measurable* (i.e. there is a sequence of simple functions $f_n:[0,1]\to X$ converging to f a.e.) and $\int_0^1\|f(t)\|\,dt<\infty$.
- *Pettis integrable* if the composition $x^*\circ f$ is integrable for all $x^*\in X^*$ and for each measurable set $E\subset[0,1]$ there is $x_E\in X$ (the Pettis integral of f over E) such that $\int_E(x^*\circ f)(t)\,dt=x^*(x_E)$ for all $x^*\in X^*$.

McShane's alternative approach to Lebesgue's integration theory has also been extended to the case of vector-valued functions, see e.g. [9], [10] and [11].

Definition. *A function $f:[0,1]\to X$ is McShane integrable, with McShane integral $x\in X$, if for each $\varepsilon>0$ there is a function $\delta:[0,1]\to\mathbb{R}^+$ such that*

$$\Big\|\sum_{i=1}^n (b_i-b_{i-1})f(t_i)-x\Big\|\le\varepsilon$$

for every partition $0=b_0<b_1<\cdots<b_n=1$ and every choice of points $t_1,\dots,t_n\in[0,1]$ such that $[b_{i-1},b_i]\subset[t_i-\delta(t_i),t_i+\delta(t_i)]$ for all i.

It is known that for a function $f:[0,1]\to X$ the following implications hold:

Bochner integrable $\Rightarrow$ McShane integrable $\Rightarrow$ Pettis integrable

and the corresponding "integrals" coincide, while no reverse arrow is true in general, see [10] and [11]. However, McShane and Pettis integrability are always equivalent for functions taking values in **separable** Banach spaces, see [10] and [11]. The point here is that *a function $f:[0,1]\to X$ is strongly measurable if and only if it is scalarly measurable (i.e. $x^*\circ f$ is measurable for all $x^*\in X^*$) and there is a null set $E\subset[0,1]$ such that $f([0,1]\setminus E)$ is separable* (Pettis' measurability theorem).

Recently, Di Piazza and Preiss [3] discussed the equivalence of McShane and Pettis integrability for functions taking values in certain **non-separable** Banach spaces. They showed that such equivalence holds for:

- Banach spaces admitting an equivalent uniformly convex norm (for instance, $L^p(\mu)$ with $1<p<\infty$ and μ any finite measure).
- $c_0(\Gamma)$ (where Γ is any non-empty set).

An essential part of their proofs relies, on the one hand, on the existence of a suitable PRI (projectional resolution of the identity) in those spaces (which are WCG –weakly compactly generated–) and, on the other hand, on the reduction to the case of scalarly null functions. Recall that $f:[0,1]\to X$ is called *scalarly null* if for each $x^*\in X^*$ we have $x^*\circ f=0$ a.e. When X is WCG (or, more generally, weakly Lindelöf), for each scalarly measurable function $f:[0,1]\to X$ there is a strongly measurable one $g:[0,1]\to X$ such that $f-g$ is scalarly null, see [6].

Question (Di Piazza-Preiss). *Suppose X is WCG and let $f:[0,1]\to X$ be a Pettis integrable function. Is f McShane integrable?*

In view of the comments above, an affirmative answer to the following question (attributed to Musial in [3]) within the setting of WCG spaces would imply that the previous question has affirmative answer too.

Question (Musial). *Let $f:[0,1]\to X$ be a scalarly null function. Is f McShane integrable?*

In [3] Musial's question was answered in the negative, *under CH (the Continuum Hypothesis)*, by means of an example of a function taking values in $\ell^\infty([0,1])$ (which is not WCG).

2 McShane and Pettis integrability for $L^1(\mu)$-valued functions

Let μ be a finite, non-negative, countably additive measure defined on a σ-algebra. The Banach space $L^1(\mu)$ is always WCG, while it may be non-separable. Our main result answers affirmatively to the previous questions for $L^1(\mu)$-valued functions:

Theorem. *A function $f:[0,1]\to L^1(\mu)$ is McShane integrable if and only if it is Pettis integrable.*

We next present a brief **sketch of the proof** in order to give an idea of the techniques involved:

- Let $\mathcal{P}$ be the class of all Banach spaces X for which every scalarly null function $f:[0,1]\to X$ is McShane integrable. Since $L^1(\mu)$ is WCG, we only have to check that this space belongs to $\mathcal{P}$.
- Given an infinite cardinal κ, we denote by Σ_κ the product σ-algebra on $\{0,1\}^\kappa$ and λ_κ stands for the usual product probability on Σ_κ. Recall that $\mathrm{dens}(L^1(\lambda_\kappa))$, the density character of $L^1(\lambda_\kappa)$, is exactly κ.
- As a consequence of Maharam's theorem, $L^1(\mu)$ is isometrically isomorphic to an ℓ^1-sum
$$\Big(\bigoplus_{i\in I}X_i\Big)_1$$
where I is countable and each X_i is either $\ell^1(\Gamma_i)$ (with Γ_i countable) or $L^1(\lambda_{\kappa_i})$ (with κ_i an infinite cardinal).
- Since the class $\mathcal{P}$ is closed under countable ℓ^1-sums and contains all separable Banach spaces, it only remains to prove that $L^1(\lambda_\kappa)$ belongs to $\mathcal{P}$ whenever κ is an uncountable cardinal.
- Let $i:L^2(\lambda_\kappa)\to L^1(\lambda_\kappa)$ be the "inclusion" operator. Using that Hilbert spaces belong to $\mathcal{P}$, one can prove that every $L^1(\lambda_\kappa)$-valued scalarly null function whose range is contained in $i(L^2(\lambda_\kappa))$ is McShane integrable.
- Roughly, the proof finishes by "approximating" $L^1(\lambda_\kappa)$-valued scalarly null functions by $i(L^2(\lambda_\kappa))$-valued ones. This is the most technical part of the proof and requires the following lemmas.
- A collection $\{P_\alpha\}_{\omega\le\alpha\le\kappa}$ of bounded linear projections on a Banach space X with $\mathrm{dens}(X)=\kappa$ is called a SPRI (separable projectional resolution of the identity) if $P_\omega\equiv 0$, P_κ is the identity on X and:
  - For each $\omega\le\alpha<\kappa$, the subspace $(P_{\alpha+1}-P_\alpha)(X)$ is separable.
  - $P_\alpha\circ P_\beta=P_\beta\circ P_\alpha=P_\beta$ whenever $\omega\le\beta\le\alpha\le\kappa$.
  - $x\in\overline{\mathrm{span}}\{(P_{\alpha+1}-P_\alpha)(x):\omega\le\alpha<\kappa\}$ for every $x\in X$.
- **Lemma 1.** *$L^1(\lambda_\kappa)$ admits a SPRI $\{P_\alpha\}_{\omega\le\alpha\le\kappa}$ such that, for each $\omega\le\alpha<\kappa$, the subspace $(P_{\alpha+1}-P_\alpha)(L^1(\lambda_\kappa))$ has a Schauder basis made up of Σ_κ-simple functions.*
- **Lemma 2.** *Let $\{P_\alpha\}_{\omega\le\alpha\le\kappa}$ be a SPRI on a WCG Banach space X with $\mathrm{dens}(X)=\kappa$. Then, for each $x^*\in X^*$, the set of all $\omega\le\alpha<\kappa$ for which $x^*|_{(P_{\alpha+1}-P_\alpha)(X)}\not\equiv 0$ is countable.*
- **Lemma 3.** *The pointwise limit of a sequence of scalarly null McShane integrable functions is again scalarly null and McShane integrable.*

  This lemma is a consequence of "Vitali's convergence theorem" for the McShane integral, see [9] and [10].

Remark. Our theorem can be seen as an strengthening of the equivalence of McShane and Pettis integrability in Hilbert spaces, because $\ell^2(\kappa)$ is isomorphic to a closed subspace of $L^1(\lambda_\kappa)$ for any infinite cardinal κ.

3 A scalarly null function which is not McShane integrable

A Banach space X is *weakly Lindelöf determined (WLD)* if (B_{X^*},w^*) is a *Corson compactum*, i.e. it embeds into

$$\Sigma(\Gamma)=\{s\in[-1,1]^\Gamma:\ s(\gamma)=0 \text{ for all but countably many } \gamma\in\Gamma\}$$

endowed with the product topology, for some set Γ. The class of WLD spaces is strictly bigger than that of WCG spaces and is made up of weakly Lindelöf spaces. Every non-separable WLD space admits a PRI as well as a SPRI. In addition, the conclusion of Lemma 2 (and its analogue for a PRI) is still valid within this class of spaces. For a detailed account on WLD spaces we refer the reader to [7] and [8].

In view of the above, it is also natural to think about the questions of Di Piazza-Preiss and Musial inside this class of Banach spaces. It turns out that we cannot expect a general result on the coincidence of McShane and Pettis integrability in WLD spaces. Indeed, we have the following:

Example. *Under CH, there exist a WLD Banach space X and a scalarly null function $f:[0,1]\to X$ which is not McShane integrable.*

In this example, the key to distinguish Pettis integrability from McShane integrability has to do with the behavior of the composition of a vector-valued function with an absolutely summing operator, as we next explain.

Recall that an operator $u:X\to Y$ between Banach spaces is *absolutely summing* if it takes unconditionally convergent series to absolutely convergent ones. As one may expect, absolutely summing operators also "improve" the integrability properties of vector-valued functions. This topic has been studied by several authors over the years, see e.g. [2], [4], [13] and [17]. Given an X-valued *Pettis integrable* function f, the Y-valued composition $u\circ f$ is *Bochner integrable* in many cases (but not always), for instance:

- When f is McShane integrable, see [13, 17].
- When X is a subspace of a weakly Lindelöf $C(K)$ space, see [17].

The latter is the case if X is WLD and (B_{X^*},w^*) has *property (M)* (i.e. every Radon probability on it has separable support). It is known that for a Banach space X the following implications hold:

WCG $\Rightarrow$ WLD and (B_{X^*},w^*) has property (M) $\Rightarrow$ WLD

$\Updownarrow$

$C(B_{X^*})$ WLD

and no reverse arrow is true in general. In fact, the statement "every Corson compactum has property (M)" is undecidable in ZFC (true under Martin's Axiom and the negation of CH, false under CH). See [1], [7] and [16].

The results in [17] left open the question of whether $u\circ f$ is Bochner integrable provided that X is WLD or f is scalarly null or the indefinite Pettis integral of f has norm relatively compact range.

It turns out that this is not true in general, since **the composition of the function f given in our example with certain absolutely summing operator is not Bochner integrable**. In particular, this property implies that f is not McShane integrable. Our example is based on the WLD Banach space whose dual unit ball fails property (M) constructed, under CH, by Plebanek and Kalenda [16].

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References

[1] S. Argyros, S. Mercourakis, and S. Negrepontis, *Functional-analytic properties of Corson-compact spaces*, Studia Math. **89** (1988), no. 3, 197–229.

[2] A. Belanger and P. N. Dowling, *Two remarks on absolutely summing operators*, Math. Nachr. **136** (1988), 229–232.

[3] L. Di Piazza and D. Preiss, *When do McShane and Pettis integrals coincide?*, Illinois J. Math. **47** (2003), no. 4, 1177–1187.

[4] J. Diestel, *An elementary characterization of absolutely summing operators*, Math. Ann. **196** (1972), 101–105.

[5] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, No. 15, AMS, Providence, R.I., 1977.

[6] G. A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), no. 4, 663–677.

[7] M. Fabian, *Gâteaux differentiability of convex functions and topology. Weak Asplund spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1997.

[8] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 8, Springer-Verlag, New York, 2001.

[9] D. H. Fremlin, *The generalized McShane integral*, Illinois J. Math. **39** (1995), no. 1, 39–67.

[10] D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Illinois J. Math. **38** (1994), no. 1, 127–147.

[11] R. A. Gordon, *The McShane integral of Banach-valued functions*, Illinois J. Math. **34** (1990), no. 3, 557–567.

[12] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics, No. 4, AMS, Providence, R.I., 1994.

[13] V. Marraffa, *A characterization of absolutely summing operators by means of McShane integrable functions*, J. Math. Anal. Appl. **293** (2004), no. 1, 71–78.

[14] E. J. McShane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. Amer. Math. Soc., No. 88, AMS, Providence, R.I., 1969.

[15] K. Musial, *Pettis integral*, Handbook of measure theory, Vol. I, II, North-Holland, Amsterdam, 2002, pp. 531–586.

[16] G. Plebanek, *Convex Corson compacta and Radon measures*, Fund. Math. **175** (2002), no. 2, 143–154.

[17] J. Rodríguez, *Absolutely summing operators and integration of vector-valued functions*, J. Math. Anal. Appl. **316** (2006), no. 2, 579–600.

[18] J. Rodríguez, *On the equivalence of McShane and Pettis integrability in non-separable Banach spaces*, preprint.

[19] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **51** (1984), no. 307.