Birkhoff integral and the property of Bourgain

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Abstract

We study the Birkhoff integral of functions $f : \Omega \longrightarrow X$ defined on a complete probability space (Ω, Σ, μ) with values in a Banach space X. We prove that if f is bounded then its Birkhoff integrability is equivalent to the fact that the set of compositions of f with elements of the dual unit ball $Z_f = \{\langle x^*, f\rangle : x^* \in B_{X^*}\}$ has Bourgain property. We characterize the weak Radon-Nikodým property in dual Banach spaces via Birkhoff integrable Radon-Nikodým derivatives. A non necessarily bounded function f is shown to be Birkhoff integrable if, and only if, Z_f is uniformly integrable, has Bourgain property and f is bounded when restricted to the members of positive measure of some measurable countable partition of Ω. As a consequence it turns out that the range of the indefinite integral of a Birkhoff integrable function is relatively norm compact. Some other applications are given.

1 Introduction

From now on (Ω, Σ, μ) is a complete probability space and $(X, \|\cdot\|)$ is a real Banach space. The starting point of our investigation goes back to the paper by Garrett Birkhoff [1], dated in 1935, in which he studied the integration of functions $f : \Omega \longrightarrow X$. Birkhoff's idea was to extend, to the setting of Banach-valued functions, *"Fréchet's elegant interpretation of the Lebesgue integral"*, see [2]. Fréchet's views inspired Birkhoff to give the following definition.

Definition 1 *Let $f : \Omega \longrightarrow X$ be a function. If Γ is a partition of Ω into countably many sets (A_n) of Σ, the function f is called* **summable** *with respect to Γ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums*

$$J(f,\Gamma) = \Big\{ \sum_n f(t_n)\mu(A_n) : t_n \in A_n \Big\}$$

is made up of unconditionally convergent series. The function f is said **Birkhoff integrable** *if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of Ω in Σ for which f is summable and $\|\cdot\|$-diam$(J(f,\Gamma)) < \varepsilon$. In this case, the* **Birkhoff integral** *$(B)\int_\Omega f\, d\mu$ of f is the only point in the intersection*

$$\bigcap \{\overline{\mathrm{co}(J(f,\Gamma))} : f \text{ is summable with respect to } \Gamma\}.$$

It has been known for long that

$$\begin{array}{ccccc} f \textbf{ Bochner} \text{ integrable} & \Longrightarrow & f \textbf{ Birkhoff} \text{ integrable} & \Longrightarrow & f \textbf{ Pettis} \text{ integrable}, \\ & & \Uparrow & & \\ & & f \textbf{ Riemann} \text{ integrable} & & \end{array}$$

and none of the reverse implications hold in general, see [1, 10, 11]. If f is Birkhoff integrable then $(B)\int_\Omega f\, d\mu =$ (Pettis) $\int_\Omega f\, d\mu$ and both integrals are, from now onwards, simply written as $\int_\Omega f\, d\mu$. When the range space X is separable, Birkhoff and Pettis integrability are the same, [10].

2 Birkhoff integral for bounded functions

Theorem 1 *Let $f : \Omega \longrightarrow X$ be a bounded function. The following statements are equivalent:*

(i) f is Birkhoff integrable;

(ii) for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that for each $t_k, t'_k \in A_k$, $k \in \mathbb{N}$, we have

$$\Big| \sum_{k=1}^m \langle x^*, f\rangle(t_k)\mu(A_k) - \sum_{k=1}^m \langle x^*, f\rangle(t'_k)\mu(A_k) \Big| < \varepsilon$$

for every $m \in \mathbb{N}$ and every $x^* \in B_{X^*}$;

(iii) $Z_f = \{\langle x^*, f\rangle : x^* \in B_{X^*}\}$ has Bourgain property;

(iv) there is a norming set $B \subset B_{X^*}$ such that $Z_{f,B} = \{\langle x^*, f\rangle : x^* \in B\}$ has Bourgain property.

Definition 2 ([5, 8, 12]) *We say that a family $\mathcal{F} \subset \mathbb{R}^\Omega$ has* **Bourgain property** *if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \dots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathcal{F}$*

$$\inf_{1 \le i \le n} |\cdot|\text{-diam}(f(B_i)) < \varepsilon.$$

If $\mathcal{F} \subset \mathbb{R}^\Omega$ has Bourgain property, then $\mathcal{F}$ is stable, see [13, 9-5-4]. While speaking about a bounded function $f : \Omega \longrightarrow X$, its Bochner integrability is equivalent to strong measurability; a deep result by Talagrand, [13, Theorem 6-1-2], establishes that f is Pettis integrable when $Z_f = \{\langle x^*, f\rangle : x^* \in B_{X^*}\}$ is stable. Along this line, our Theorem 1 says somehow that

Bourgain property is to Birkhoff integrability what strong measurability is to Bochner integrability.

Riddle and Saab proved in [12, Theorem 13] that any bounded function $f : \Omega \longrightarrow X^*$ is Pettis integrable whenever $\{\langle f, x\rangle : x \in B_X\}$ has Bourgain property. The particular case of Theorem 1 that is isolated below improves Riddle and Saab's result.

Corollary 2 *Let $f : \Omega \longrightarrow X^*$ be a bounded function. Then f is Birkhoff integrable if, and only if, $\{\langle f, x\rangle : x \in B_X\} \subset \mathbb{R}^\Omega$ has Bourgain property.*

3 The weak Radon-Nikodým property in dual Banach spaces via Birkhoff integrable derivatives

Theorem 3 *Let X be a Banach space. The following statements are equivalent:*

(i) X^* has the weak Radon-Nikodým property;

(ii) X does not contain a copy of ℓ^1;

(iii) for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation there is a Birkhoff integrable function $f : \Omega \longrightarrow X^*$ such that

$$\nu(E) = \int_E f\, d\mu$$

for every $E \in \Sigma$;

(iv) for every complete probability space (Ω, Σ, μ) and every bounded operator $T : L^1(\mu) \longrightarrow X^*$ there is a bounded Birkhoff integrable function $f : \Omega \longrightarrow X^*$ such that

$$\langle x^{**}, T(g)\rangle = \int_\Omega g \langle x^{**}, f\rangle\, d\mu$$

for every $x^{**} \in X^{**}$ and every $g \in L^1(\mu)$.

Recall that a dual Banach space X^* is said to have the **weak Radon-Nikodým property** (WRNP, for short), [15, Definition 5.8], if for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation there is a Pettis integrable function $f : \Omega \longrightarrow X^*$ such that $\nu(E) = \int_E f\, d\mu$ for every $E \in \Sigma$. Efforts of several mathematicians led to the well-known characterization of Banach spaces X without copies of ℓ^1 as those for which X^* has the WRNP, see [8, 13, 15] and the references therein. Our Theorem 3 states that

for dual spaces X^* the presence of WRNP entitle us to change Pettis integrable Radon-Nikodým derivatives to Birkhoff integrable Radon-Nikodým derivatives.

Theorem 4 *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) f is Birkhoff integrable;

(ii) Z_f is uniformly integrable, Z_f has Bourgain property and there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.

Fremlin proved in [3] that for every Birkhoff integrable function $f : \Omega \longrightarrow X$ the set Z_f is stable. Since Bourgain property is more restrictive than stability, [13, 9-5-4], Fremlin's result is a weaker form of Theorem 4.

Corollary 5 *If $f : \Omega \longrightarrow X$ is Birkhoff integrable, then the range of the indefinite integral $\{\int_A f\, d\mu : A \in \Sigma\}$ is relatively norm compact.*

A consequence of Theorem 3 is that if X^* has the WRNP, then every Pettis integrable function $f : \Omega \longrightarrow X^*$ is scalarly equivalent to a Birkhoff integrable function.

Theorem 6 *Let $f : \Omega \longrightarrow X^*$ be a function such that Z_f is uniformly integrable and has Bourgain property. Then there is a Birkhoff integrable function $g : \Omega \longrightarrow X^*$ which is scalarly equivalent to f.*

4 Relationship with other recently studied integrals of Banach-valued functions

We end up this poster by paying a visit to the **Riemann-Lebesgue integrals** recently introduced in [6, 7]. Given a function $f : \Omega \longrightarrow X$, a countable partition $\Gamma = (A_n)$ of Ω in Σ and a *choice* $T = (t_n)$ in Γ (i.e. $t_n \in A_n$ for every n), we consider the formal series

$$S(f, \Gamma, T) := \sum_n f(t_n)\mu(A_n).$$

As usual, we say that another partition Γ' of Ω, into countable elements of Σ, is finer than Γ when each element of Γ' is contained in some element of Γ.

Proposition 7 below shows that for a function $f : \Omega \longrightarrow X$ its Birkhoff integral (upon its existence) can be realized as the limit refining partitions of the net $\{S(f, \Gamma, T)\}_\Gamma$. We mention that functions $f : \Omega \longrightarrow X$ satisfying (iii) in Proposition 7 are called **unconditionally Riemann-Lebesgue integrable** functions in [7].

Proposition 7 *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) f is Birkhoff integrable;

(ii) there exists $x \in X$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that f is summable with respect to Γ and $\|S(f, \Gamma, T) - x\| < \varepsilon$ for every choice T in Γ;

(iii) there exists $y \in X$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that f is summable with respect to each countable partition Γ' finer than Γ and $\|S(f, \Gamma', T') - y\| < \varepsilon$ for every choice T' in Γ'.

In this case, $x = y = \int_\Omega f\, d\mu$.

It turns out that

unconditional Riemann-Lebesgue integrability = Birkhoff integrability.

Functions $f : \Omega \to X$ for which Z_f is stable and such that $\|f\|$ has a μ-integrable majorant have caught the attention of several authors over the years, see [4, 9, 13, 14] amongst others. These functions are called by Fremlin **Talagrand integrable functions** [4] and they are characterized by Talagrand as those functions satisfying *the law of large numbers*, see [14]. As the last application of our techniques here we characterize those functions f for which Z_f has Bourgain property and $\|f\|$ has a μ-integrable majorant.

Proposition 8 *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) f is Riemann-Lebesgue integrable;

(ii) Z_f has Bourgain property and there is $g \in \mathcal{L}^1(\mu)$ such that $\|f\| \le g$ μ-almost everywhere.

Recall that a function $f : \Omega \longrightarrow X$ is said to be **Riemann-Lebesgue integrable**, [6, 7], if there exists $x \in X$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that for every countable partition Γ' finer than Γ and every choice T' in Γ', the series $S(f, \Gamma', T')$ is absolutely convergent and $\|S(f, \Gamma', T') - x\| < \varepsilon$. Every Riemann-Lebesgue integrable function is Birkhoff integrable after Proposition 7.

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