Measurable selectors and set-valued Pettis integral in non-separable Banach spaces

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Universidad Politécnica de Valencia

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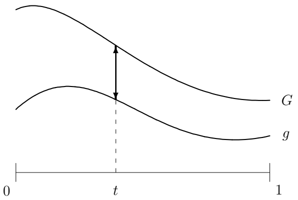
Example: a multi-function $F : [0,1] \to cwk(\mathbb{R})$ can be written as

$$F(t) = [g(t), G(t)]$$

for some real-valued functions $g \leq G$.



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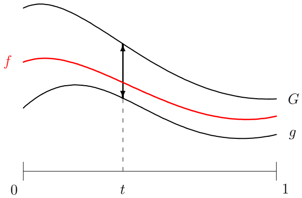

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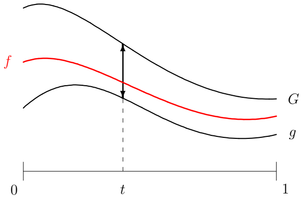

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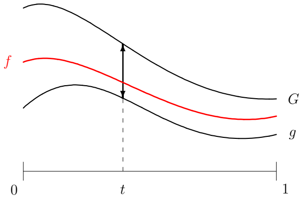

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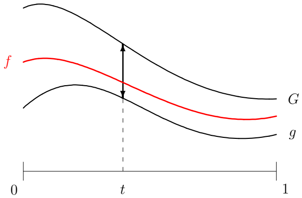

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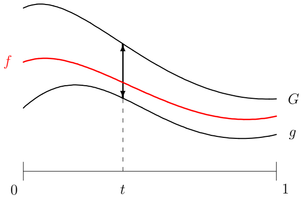

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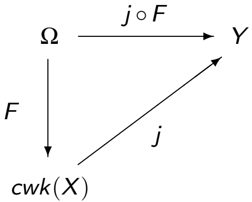

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Moreover, j is an **isometry** from $cwk(X)$ (equipped with the Hausdorff distance) into $\ell_\infty(B_{X^*})$.

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Theorem (Kuratowski and Ryll-Nardzewski, 1965)

Let M be a separable complete metric space and $F:\Omega\to 2^M$ an Effros measurable multi-function having closed non-empty values. Then F admits **Borel measurable selectors**.

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Suppose X is separable.
Then every scalarly measurable multi-function $F : \Omega \to cwk(X)$ admits **strongly measurable selectors**.

A characterization of set-valued Pettis integrability

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In this case:

$$\int_A F \, d\mu = \left\{ \int_A f \, d\mu : \; f \text{ Pettis integrable selector of } F \right\}$$

for all $A \in \Sigma$.

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The price

To find **scalarly measurable selectors** for scalarly measurable multi-functions without the separability assumption.

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Set-valued Pettis integral and selectors

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Remark

The "closure" can be removed if X^* is w^*-separable.

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▶ We **do not know** an example of a scalarly measurable multi-function without scalarly measurable selectors !!

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