ON VECTOR MEASURES WITH VALUES IN ℓ_∞

S. OKADA, J. RODRÍGUEZ, AND E.A. SÁNCHEZ-PÉREZ

Abstract. We study some aspects of countably additive vector measures with values in ℓ_∞ and the Banach lattices of real-valued functions that are integrable with respect to such a vector measure. On the one hand, we prove that if $W \subseteq \ell_\infty^*$ is a total set not containing sets equivalent to the canonical basis of $\ell_1(\mathfrak{c})$, then there is a non-countably additive ℓ_∞-valued map ν defined on a σ-algebra such that the composition $x^* \circ \nu$ is countably additive for every $x^* \in W$. On the other hand, we show that a Banach lattice E is separable whenever it admits a countable, positively norming set and both E and E^* are order continuous. As a consequence, if ν is a countably additive vector measure defined on a σ-algebra and taking values in a separable Banach space, then the space $L_1(\nu)$ is separable whenever $L_1(\nu)^*$ is order continuous.

1. Introduction

Given a σ-algebra Σ and a Banach space X, we denote by $\mathrm{ca}(\Sigma, X)$ the set of all countably additive X-valued measures defined on Σ; when X is the real field, this set is simply denoted by $\mathrm{ca}(\Sigma)$. The topological dual of X is denoted by X^*. The Orlicz-Pettis theorem implies that a map $\nu : \Sigma \to X$ belongs to $\mathrm{ca}(\Sigma, X)$ if (and only if) the composition $x^* \circ \nu$ belongs to $\mathrm{ca}(\Sigma)$ for every $x^* \in X^*$ (see, e.g., [10, p. 22, Corollary 4]). It is natural to wonder whether testing on a "big" subset, instead of all X^*, is enough for countable additivity. For instance, one might consider a *total* subset of X^*, that is, a set $W \subseteq X^*$ satisfying $\bigcap_{x^* \in W} \ker x^* = \{0\}$. In general, this does not work. A typical example is given by the map $\nu : \mathcal{P}(\mathbb{N}) \to \ell_\infty$ defined on the power set of $\mathbb{N}$ by $\nu(A) := \chi_A$ (the characteristic function of A) for all $A \subseteq \mathbb{N}$. Indeed, ν is not countably additive, while the composition $\pi_n \circ \nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is countably additive for every $n \in \mathbb{N}$, where $\pi_n \in \ell_\infty^*$ is the nth-coordinate functional given by $\pi_n(x) := x(n)$ for all $x \in \ell_\infty$.

Thanks to the injectivity of ℓ_∞, such an example can be carried over to any Banach space containing subspaces isomorphic to ℓ_∞. Actually, the existence of such subspaces is the only obstacle, as the following result of Diestel and Faires [9] (cf. [10, p. 23, Corollary 7]) shows:

2020 *Mathematics Subject Classification.* Primary 46E30, 46G10.

Key words and phrases. Vector measure, space of integrable functions, Banach lattice, positively norming set.

Theorem 1.1 (Diestel-Faires). *Let X be a Banach space.*

(i) Suppose that X contains a subspace isomorphic to ℓ_∞. Then there exist a total set $W \subseteq X^*$ and a map $\nu : \mathcal{P}(\mathbb{N}) \to X$ such that $x^* \circ \nu \in \mathrm{ca}(\mathcal{P}(\mathbb{N}))$ for every $x^* \in W$ and $\nu \notin \mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$.

(ii) Suppose that X does not contain subspaces isomorphic to ℓ_∞. Let $W \subseteq X^*$ be a total set. If $\nu : \Sigma \to X$ is a map defined on a σ-algebra Σ such that $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for every $x^* \in W$, then $\nu \in \mathrm{ca}(\Sigma, X)$.

The following concept (going back to [32, Appendice II]) arises naturally:

Definition 1.2. Let X be a Banach space. A set $W \subseteq X^*$ is said to have the *Orlicz-Thomas (OT) property* if, for every map $\nu : \Sigma \to X$ defined on a σ-algebra Σ, we have $\nu \in \mathrm{ca}(\Sigma, X)$ whenever $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for all $x^* \in W$.

Any set having the OT property is necessarily total (Proposition 3.1). With this language, the Diestel-Faires Theorem 1.1 says that every total subset of the dual of a Banach space X has the OT property if and only if X contains no subspace isomorphic to ℓ_∞. Without additional assumptions on X, one can check that any norm-dense subset of X^* has the OT property as explicitly given in [22, Lemma 3.1]; alternatively, we can apply the Vitali-Hahn-Saks-Nikodým theorem (see, e.g., [10, p. 24, Corollary 10]) together with the Orlicz-Pettis theorem. Given any total set $W \subseteq X^*$, a classical result of Dieudonné and Grothendieck (see, e.g., [10, p. 16, Corollary 3]) states that an X-valued map defined on a σ-algebra is bounded and finitely additive if (and only if) $x^* \circ \nu$ is bounded and finitely additive for every $x^* \in W$. This result and the Rainwater-Simons theorem (see, e.g., [12, Theorem 3.134]) ensure that any James boundary of X has the OT property, [13, Proposition 2.9] (see also [22, Remark 3.2(i)]).

The previous discussion makes clear that the particular case of ℓ_∞ is the most interesting one as the OT property is concerned. In this paper we study the OT property in ℓ_∞ and some aspects of the Banach lattices of real-valued functions which are integrable with respect to countably additive ℓ_∞-valued measures. The paper is organized as follows.

In Section 2 we introduce some terminology and preliminary facts.

In Section 3 we focus on the OT property. We begin with some basic results in arbitrary Banach spaces, including an application to the factorization of vector measures and their integration operators (Proposition 3.7). The core of this section is devoted to studying the OT property in ℓ_∞. Our main result here (Theorem 3.15) states that any subset of ℓ_∞^* having the

OT property contains a copy of the usual basis of $\ell_1(\mathfrak{c})$, where $\mathfrak{c}$ stands for the cardinality of the continuum (i.e., $\mathfrak{c} = |\mathbb{R}|$). Some consequences of this result are given. Let us mention that the existence of copies of the usual basis of $\ell_1(\mathfrak{c})$ inside James boundaries (which always have the OT property) has been studied thoroughly (see [16, 17] and the references therein).

In Section 4 we study some structural properties of the Banach lattice $L_1(\nu)$ of real-valued functions which are integrable with respect to a given $\nu \in \mathrm{ca}(\Sigma, X)$, where Σ is a σ-algebra and X is a Banach space. These spaces play an important role in Banach lattices as well as in operator theory (see Subsection 2.3 and the references therein). It is known that any order continuous Banach lattice having a weak order unit is lattice-isometric to a space of the form $L_1(\nu)$. Suppose that ν has separable range (as every $\nu \in \mathrm{ca}(\Sigma, \ell_\infty)$ is known to have). Does it follow that $L_1(\nu)$ is separable? The answer is in the negative, in general, even if ν is a finite measure (see Subsection 2.4). However, the answer is in the affirmative with an additional assumption that $L_1(\nu)^*$ is order continuous, as asserted in Theorem 4.3. We shall instead prove a more general fact: Theorem 4.6, which asserts that if E is a Banach lattice admitting a countable, positively norming set and both E and E^* are order continuous, then E is separable. In particular, if μ is a finite measure for which $L_1(\mu)$ is non-separable and $1 < p < \infty$, then $L_p(\mu)$ is not isomorphic to $L_1(\nu)$ for any ν with separable range (Example 4.16).

2. Terminology and preliminaries

Given a set S, we denote by $\mathcal{P}(S)$ its power set, that is, the set of all subsets of S. The cardinality of S is denoted by $|S|$. The *density character* of a topological space $(T, \mathfrak{T})$, denoted by $\mathrm{dens}(T, \mathfrak{T})$ or simply $\mathrm{dens}(T)$, is the minimal cardinality of a $\mathfrak{T}$-dense subset of T.

2.1. **Banach spaces.** All Banach spaces considered in this paper are real. An *operator* is a continuous linear map between Banach spaces. Let X be a Banach space. The norm of X is denoted by $\|\cdot\|_X$ or $\|\cdot\|$. The closed unit ball of X is $B_X := \{x \in X : \|x\| \leq 1\}$. The weak (resp., weak*) topology on X (resp., X^*) is denoted by w (resp., w^*). By a *subspace* of X we mean a norm-closed linear subspace. In almost all cases we will deal with norm-closed linear subspaces, so we prefer to use such an abridged terminology; when norm-closedness is not assumed, we use the term *linear subspace* unless otherwise stated. By a *projection* from X onto a subspace $Y \subseteq X$ we mean an operator $P : X \to X$ such that $P(X) = Y$ and P is the identity when restricted to Y. The convex hull and linear span of a set $D \subseteq X$ are denoted by $\mathrm{co}(D)$ and $\mathrm{span}(D)$, respectively, and their closures are denoted by $\overline{\mathrm{co}}(D)$ and $\overline{\mathrm{span}}(D)$. A set $B \subseteq B_{X^*}$ is said to be *norming* if there is a constant $c > 0$ such that $\|x\| \leq c \sup_{x^* \in B} |x^*(x)|$ for every $x \in X$. A set $B \subseteq B_{X^*}$ is said to be a *James boundary* of X if for every $x \in X$ there is $x^* \in B$ such that $\|x\| = x^*(x)$. If X is a Banach lattice, then its positive cone is $X^+ = \{x \in X : x \geq 0\}$. Given a compact Hausdorff topological space K, we denote by $C(K)$ the Banach space of all real-valued continuous functions on K, equipped with the supremum norm.

2.2. **Banach function spaces.** Let (Ω, Σ, μ) be a finite measure space. To define Banach function spaces, we consider linear subspaces, not necessarily norm-closed, of $L_1(\mu)$. A Banach space E is said to be a *Banach function space* (or a *Köthe function space*) over (Ω, Σ, μ) if the following conditions hold:

(i) E is a linear subspace of $L_1(\mu)$;
(ii) if $f \in L_0(\mu)$ and $|f| \leq |g|$ μ-a.e. for some $g \in E$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$;
(iii) the characteristic function χ_A of each $A \in \Sigma$ belongs to E.

In this case, E is a Banach lattice when endowed with the μ-a.e. order and the inclusion map $E \to L_1(\mu)$ is an operator. The *Köthe dual* of E is

$$E' := \{g \in L_1(\mu) : \, fg \in L_1(\mu) \text{ for all } f \in E\}.$$

Any $g \in E'$ gives raise to a functional $\varphi_g \in E^*$ defined by $\varphi_g(f) := \int_\Omega fg \, d\mu$ for all $f \in E$. It is known that E is order continuous if and only if $E^* = \{\varphi_g : g \in E'\}$ (see, e.g., [21, p. 29]).

2.3. L_1 **of a vector measure.** For detailed information on the L_1 space of a vector measure we refer the reader to [25, Chapter 3] and the references given there. Here we just mention the basics needed in this paper. Let (Ω, Σ) be a measurable space, let X be a Banach space and let $\nu \in \mathrm{ca}(\Sigma, X)$. A set $A \in \Sigma$ is said to be *ν-null* if $\nu(B) = 0$ for every $B \in \Sigma$ with $B \subseteq A$. The family of all ν-null sets is denoted by $\mathcal{N}(\nu)$. By a *Rybakov control measure* of ν we mean a finite measure of the form $\mu = |x^* \circ \nu|$ (the variation of $x^* \circ \nu$) for some $x^* \in X^*$ such that $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ (see, e.g., [10, p. 268, Theorem 2] for a proof of the existence of Rybakov control measures). A Σ-measurable function $f : \Omega \to \mathbb{R}$ is called *ν-integrable* if $f \in L_1(|x^* \circ \nu|)$ for all $x^* \in X^*$ and, for each $A \in \Sigma$, there is $\int_A f \, d\nu \in X$ such that

$$x^*\left(\int_A f \, d\nu\right) = \int_A f \, d(x^* \circ \nu) \quad \text{for all } x^* \in X^*.$$

By identifying functions which coincide ν-a.e., the set $L_1(\nu)$ of all (equivalence classes of) ν-integrable functions is a Banach space with the norm

$$\|f\|_{L_1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f| \, d|x^* \circ \nu|.$$

The *integration operator* is the (norm one) operator $I_\nu : L_1(\nu) \to X$ defined by

$$I_\nu(f) := \int_\Omega f \, d\nu \quad \text{for all } f \in L_1(\nu).$$

If μ is any Rybakov control measure of ν, then $L_1(\nu)$ is a Banach function space over (Ω, Σ, μ). As a Banach lattice, $L_1(\nu)$ is order continuous and has a weak order unit (the function χ_Ω). Conversely, any order continuous Banach lattice having a weak order unit is lattice-isometric to the L_1 space of a countably additive vector measure defined on a σ-algebra. Indeed, on the one hand, such a Banach lattice is lattice-isometric to a Banach function space E over some finite measure space (Ω, Σ, μ) (see, e.g., [21, Theorem 1.b.14]). On the other hand, thanks to the order continuity of E, the map $\nu : \Sigma \to E$ given by $\nu(A) := \chi_A$ for all $A \in \Sigma$ is countably additive and one has $E = L_1(\nu)$ (see [5, Theorem 8], [11, Proposition 2.4(vi)]).

2.4. **The usual measure on $\{0,1\}^I$.** Let I be a non-empty set. For each $i \in I$ we denote by $\pi_i : \{0,1\}^I \to \{0,1\}$ the ith-coordinate projection. Let Λ_I be the σ-algebra on $\{0,1\}^I$ generated by all the sets of the form $\bigcap_{i \in F} \pi_i^{-1}(w(i))$, where $F \subseteq I$ is finite and $w \in \{0,1\}^F$. The usual product probability measure on $\{0,1\}^I$, denoted by λ_I, is defined on Λ_I and satisfies $\lambda_I(\pi_i^{-1}(\{0\})) = \lambda_I(\pi_i^{-1}(\{1\})) = \frac{1}{2}$ for all $i \in I$. For simplicity, we just call λ_I the *usual measure on* $\{0,1\}^I$. We have $\operatorname{dens}(L_1(\lambda_I)) = |I|$ if I is infinite. In particular, $L_1(\lambda_I)$ is not separable whenever I is uncountable. We refer the reader to [15, §254] for more information on infinite product measures and the usual measure on $\{0,1\}^I$.

2.5. **Measure algebras.** Let (Ω, Σ, μ) be a probability space. We consider the equivalence relation on Σ defined by $A \sim B$ if and only if $\mu(A \triangle B) = 0$. The set of equivalence classes, denoted by $\Sigma/\mathcal{N}(\mu)$, becomes a measure algebra when equipped with the usual Boolean algebra operations and the functional defined by $\mu^\bullet(A^\bullet) := \mu(A)$ for all $A \in \Sigma$, where $A^\bullet \in \Sigma/\mathcal{N}(\mu)$ denotes the equivalence class of A. Given another probability space $(\Omega_0, \Sigma_0, \mu_0)$, the measure algebras of μ and μ_0 are said to be *isomorphic* if there is a Boolean algebra isomorphism

$$\theta : \Sigma/\mathcal{N}(\mu) \to \Sigma_0/\mathcal{N}(\mu_0)$$

such that $\mu_0^\bullet \circ \theta = \mu^\bullet$. In this case, there is a lattice isometry $\Phi : L_1(\mu) \to L_1(\mu_0)$ such that $\int_\Omega f\,d\mu = \int_{\Omega_0} \Phi(f)\,d\mu_0$ and $\Phi(f\chi_B) = \Phi(f)\chi_C$ whenever $f \in L_1(\mu)$ and $\theta(B^\bullet) = C^\bullet$. For more information on measure algebras, see [14].

3. The Orlicz-Thomas property

3.1. **The OT property in arbitrary Banach spaces.** Throughout this subsection X is a Banach space. We begin with an observation:

Proposition 3.1. *If $W \subseteq X^*$ has the OT property, then W is total.*

Proof. If W is not total, then there is $x \in X \setminus \{0\}$ in such a way that $x^*(x) = 0$ for all $x^* \in W$. Let $\xi : \mathcal{P}(\mathbb{N}) \to [0,1]$ be a finitely additive measure which is not countably additive. Then there is a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathbb{N})$ such that the sequence $(\xi(\bigcup_{m>n} A_m))_{n\in\mathbb{N}}$ does not converge to 0. Define $\nu : \mathcal{P}(\mathbb{N}) \to X$ by $\nu(A) := \xi(A)x$ for all $A \subseteq \mathbb{N}$. For each $x^* \in W$ we have $(x^* \circ \nu)(A) = 0$ for all $A \subseteq \mathbb{N}$, hence $x^* \circ \nu \in \mathrm{ca}(\mathcal{P}(\mathbb{N}))$. Since $\|\nu(\bigcup_{m>n} A_m)\| = \xi(\bigcup_{m>n} A_m)\|x\|$ for every $n \in \mathbb{N}$, we have $\nu \notin \mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$. $\square$

The following proposition is straightforward.

Proposition 3.2. *Let $W \subseteq X^*$. The following statements are equivalent:*

(i) W has the OT property.
(ii) $\mathrm{co}(W)$ has the OT property.
(iii) $\mathrm{span}(W)$ has the OT property.

As usual, ω_1 denotes the first uncountable ordinal. Given a set $D \subseteq X^*$, we denote by $S_1(D) \subseteq X^*$ the set of all limits of w^*-convergent sequences contained in D. For any ordinal $\alpha \leq \omega_1$, we define $S_\alpha(D)$ by transfinite induction as follows:

- $S_0(D) := D$,
- $S_\alpha(D) := S_1(S_\beta(D))$ if $\alpha = \beta + 1$ for some ordinal $\beta < \omega_1$,
- $S_\alpha(D) := \bigcup_{\beta<\alpha} S_\beta(D)$ if α is a limit ordinal.

Then $S_{\omega_1}(D)$ is the smallest w^*-sequentially closed subset of X^* containing D. In general, we have

$$D \subseteq \overline{D}^{\|\cdot\|} \subseteq S_1(D) \subseteq S_{\omega_1}(D) \subseteq \overline{D}^{w^*}.$$

Proposition 3.3. *Let $W \subseteq X^*$. The following statements are equivalent:*

(i) W has the OT property.
(ii) $\overline{W}^{\|\cdot\|}$ has the OT property.

(iii) $S_1(W)$ *has the OT property.*
(iv) $S_{\omega_1}(W)$ *has the OT property.*
(v) $\overline{W}^w$ *has the OT property.*

Proof. The implications (i)⇒(ii)⇒(iii)⇒(iv) and (i)⇒(v) are obvious.

(iv)⇒(i): Suppose that $S_{\omega_1}(W)$ has the OT property. Let $\nu : \Sigma \to X$ be a map defined on a σ-algebra Σ such that $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for all $x^* \in W$. Bearing in mind the Vitali-Hahn-Saks-Nikodým theorem (see, e.g., [10, p. 24, Corollary 10]), a standard transfinite induction argument shows that, for each ordinal $\alpha \leq \omega_1$, we have $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for all $x^* \in S_\alpha(W)$. In particular, this holds for $\alpha = \omega_1$ and so $\nu \in \mathrm{ca}(\Sigma, X)$.

(v)⇒(i): Since $\overline{W}^w \subseteq \overline{\mathrm{co}(W)}^w = \overline{\mathrm{co}(W)}^{\|\cdot\|}$, the set $\overline{\mathrm{co}(W)}^{\|\cdot\|}$ has the OT property. By the equivalence (i)⇔(ii) applied to $\mathrm{co}(W)$, this set has the OT property and so does W (by Proposition 3.2). □

The following proposition gives an operator-theoretic reformulation of the OT property. Given a σ-algebra Σ, the set $\mathrm{ca}(\Sigma)$ is a Banach space when equipped with the variation norm. It is known that a set $H \subseteq \mathrm{ca}(\Sigma)$ is relatively weakly compact if and only if it is bounded and there is a non-negative $\mu_0 \in \mathrm{ca}(\Sigma)$ such that H is *uniformly* μ_0*-continuous*, i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{\mu \in H} |\mu(A)| \leq \varepsilon$ for every $A \in \Sigma$ satisfying $\mu_0(A) \leq \delta$ (see, e.g., [7, p. 92, Theorem 13]).

Proposition 3.4. *Let* $W \subseteq X^*$ *be a subspace and let* $\nu : \Sigma \to X$ *be a map defined on a* σ*-algebra* Σ *such that* $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ *for all* $x^* \in W$*. Then the map*

$$T : W \to \mathrm{ca}(\Sigma), \quad T(x^*) := x^* \circ \nu,$$

is an operator and the following statements hold:

(i) If $\nu \in \mathrm{ca}(\Sigma, X)$*, then* T *is weakly compact.*
(ii) If T *is weakly compact and* $B_W \subseteq B_{X^*}$ *is norming, then* $\nu \in \mathrm{ca}(\Sigma, X)$*.*

Proof. A routine application of the Closed Graph Theorem ensures that T is an operator.

(i) The fact that ν is countably additive ensures the existence of a non-negative $\mu_0 \in \mathrm{ca}(\Sigma)$ such that $\{x^* \circ \nu : x^* \in B_{X^*}\} \supseteq T(B_W)$ is uniformly μ_0-continuous (see, e.g., [10, p. 14, Corollary 6]).

(ii) Since B_W is absolutely convex and norming, we have

$$\overline{B_W}^{w^*} \supseteq cB_{X^*} \tag{3.1}$$

for some $c > 0$, by the Hahn-Banach separation theorem. Fix a non-negative $\mu_0 \in \mathrm{ca}(\Sigma)$ such that $T(B_W)$ is uniformly μ_0-continuous. Observe that ν is finitely additive because W is total. To prove that ν is countably additive it suffices to check that it is μ_0-continuous. Fix $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|x^*(\nu(A))| \leq c\varepsilon \quad \text{for every } A \in \Sigma \text{ with } \mu_0(A) \leq \delta \text{ and for every } x^* \in B_W.$$

Clearly, the previous inequality is also valid for all $x^* \in \overline{B_W}^{w^*}$ and then (3.1) implies that

$$\|\nu(A)\| = \sup_{x^* \in B_{X^*}} |x^*(\nu(A))| \leq \varepsilon \quad \text{for every } A \in \Sigma \text{ with } \mu_0(A) \leq \delta.$$

Therefore, ν is μ_0-continuous and so it is countably additive. □

Proposition 3.5. *Let $T : X \to Y$ be an operator between Banach spaces and let $\nu : \Sigma \to X$ be a map defined on a σ-algebra Σ. The following statements hold:*

(i) $T \circ \nu \in \mathrm{ca}(\Sigma, Y)$ if and only if $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for all $x^* \in T^*(B_{Y^*})$.
(ii) Suppose that $T^*(B_{Y^*})$ has the OT property. Then $\nu \in \mathrm{ca}(\Sigma, X)$ if and only if $T \circ \nu \in \mathrm{ca}(\Sigma, Y)$.
(iii) If T^{**} is injective, then $T^*(B_{Y^*})$ has the OT property.

Proof. (i) follows at once from the Orlicz-Pettis theorem. (ii) is a consequence of (i). To prove (iii), note that the injectivity of T^{**} is equivalent (via the Hahn-Banach separation theorem) to the norm denseness of $T^*(Y^*)$ in X^*. From Proposition 3.3 it follows that $T^*(Y^*)$ has the OT property. Since $\mathrm{span}(T^*(B_{Y^*})) = T^*(Y^*)$, we can apply Proposition 3.2 to conclude that $T^*(B_{Y^*})$ has the OT property. □

Typical examples of non-isomorphic embeddings having injective biadjoints are the operators associated to the Davis-Figiel-Johnson-Pełczyński factorization (see, e.g., [2, Theorem 5.37]). The following simple example shows that the conclusion of Proposition 3.5(ii) can fail for arbitrary injective operators.

Example 3.6. Let $\nu : \mathcal{P}(\mathbb{N}) \to \ell_\infty$ be the finitely additive measure defined by $\nu(A) := \chi_A$ for all $A \subseteq \mathbb{N}$ and let $T : \ell_\infty \to \ell_1$ be the injective operator defined by $T((x_n)_{n\in\mathbb{N}}) := (2^{-n}x_n)_{n\in\mathbb{N}}$ for all $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$. Observe that $T \circ \nu$ is finitely additive and

$$\|(T \circ \nu)(A)\| = \sum_{n\in A} 2^{-n} \quad \text{for all } A \subseteq \mathbb{N},$$

so that $T \circ \nu \in \mathrm{ca}(\mathcal{P}(\mathbb{N}), \ell_1)$. However, $\nu \notin \mathrm{ca}(\mathcal{P}(\mathbb{N}), \ell_\infty)$.

Let $T : X \to Y$ be an injective operator between Banach spaces, let (Ω, Σ) be a measurable space and suppose that the integration operator I_ν of $\nu \in \mathrm{ca}(\Sigma, Y)$ factors as

$$\begin{tikzcd} L_1(\nu) \arrow[rr, "I_\nu"] \arrow[dr, "S"'] & & Y \\ & X \arrow[ur, "T"'] & \end{tikzcd}$$

for some operator $S : L_1(\nu) \to X$. Define $\tilde{\nu}(A) := S(\chi_A)$ for all $A \in \Sigma$. In [22, Theorem 3.7] it was proved that $\tilde{\nu} \in \mathrm{ca}(\Sigma, X)$ satisfies:

(a) $\nu = T \circ \tilde{\nu}$ and $\mathcal{N}(\nu) = \mathcal{N}(\tilde{\nu})$;
(b) $L_1(\nu) = L_1(\tilde{\nu})$ (with equivalent norms); and
(c) $S = I_{\tilde{\nu}}$.

This result improves [26, Lemma 3.1] in which (b) was obtained via the Diestel-Faires Theorem 1.1 under the additional assumption that X does not contain subspaces isomorphic to ℓ_∞. In a similar spirit, we have:

Proposition 3.7. *Let $T : X \to Y$ be an operator between Banach spaces such that $T^*(B_{Y^*})$ has the OT property. Let (Ω, Σ) be a measurable space and let $\nu \in \mathrm{ca}(\Sigma, Y)$ satisfy $I_\nu(L_1(\nu)) \subseteq T(X)$. Then there is $\tilde{\nu} \in \mathrm{ca}(\Sigma, X)$ such that:*

(a) $\nu = T \circ \tilde{\nu}$ and $\mathcal{N}(\nu) = \mathcal{N}(\tilde{\nu})$;
(b) $L_1(\nu) = L_1(\tilde{\nu})$ (with equivalent norms); and
(c) $I_\nu = T \circ I_{\tilde{\nu}}$.

Proof. Observe that $T^*(B_{Y^*})$ is total (by Proposition 3.1) and so T is injective. For each $A \in \Sigma$ we have $\nu(A) = I_\nu(\chi_A) \in T(X)$, so there is a unique $\tilde{\nu}(A) \in X$ such that

$$T(\tilde{\nu}(A)) = \nu(A).$$

The so-defined map $\tilde{\nu} : \Sigma \to X$ satisfies $\nu = T \circ \tilde{\nu}$ and belongs to $\mathrm{ca}(\Sigma, X)$ because $T^*(B_{Y^*})$ has the OT property and $T^*(y^*) \circ \tilde{\nu} = y^* \circ T \circ \tilde{\nu} = y^* \circ \nu \in \mathrm{ca}(\Sigma)$ for all $y^* \in B_{Y^*}$.

By [25, Lemma 3.27] we have $\mathcal{N}(\nu) = \mathcal{N}(\tilde{\nu})$, the inclusion $L_1(\tilde{\nu}) \subseteq L_1(\nu)$ and the equality $I_\nu = T \circ I_{\tilde{\nu}}$ on $L_1(\tilde{\nu})$.

To prove the reverse inclusion $L_1(\nu) \subseteq L_1(\tilde{\nu})$, let $f \in L_1(\nu)$. The fact that $I_\nu(L_1(\nu)) \subseteq T(X)$ enables us to define a finitely additive set function $\eta : \Sigma \to X$ such that $T(\eta(A)) = I_\nu(f\chi_A) = \int_A f \, d\nu$ for every $A \in \Sigma$. Then Proposition 3.5(ii) ensures that η is countably additive because $T^*(B_{Y^*})$ has the OT property and the indefinite integral $A \mapsto \int_A f \, d\nu$ on Σ is countably additive. Given $n \in \mathbb{N}$, let $A_n := |f|^{-1}([0, n]) \in \Sigma$ and $f_n := f\chi_{A_n}$. Fix

$A \in \Sigma$. Each f_n is bounded and Σ-measurable, hence it is $\tilde{\nu}$-integrable and, moreover, it satisfies

$$\int_A f_n \, d\tilde{\nu} = \eta(A \cap A_n). \tag{3.2}$$

Indeed, let $(s_k^{(n)})_{k\in\mathbb{N}}$ be a sequence of Σ-simple functions which are uniformly convergent to f_n as $k \to \infty$. Since $\nu = T \circ \tilde{\nu}$, it follows that

$$\begin{aligned} T\left(\int_A f_n \, d\tilde{\nu}\right) &= T\left(\lim_{k\to\infty} \int_A s_k^{(n)} \, d\tilde{\nu}\right) = \lim_{k\to\infty} T\left(\int_A s_k^{(n)} \, d\tilde{\nu}\right) \\ &= \lim_{k\to\infty} \int_A s_k^{(n)} \, d\nu = \int_A f_n \, d\nu = \int_{A\cap A_n} f \, d\nu = T\left(\eta(A \cap A_n)\right). \end{aligned}$$

This verifies (3.2) as T is injective. Now (3.2) together with countable additivity of η imply that $\lim_{n\to\infty} \int_A f_n \, d\tilde{\nu} = \lim_{n\to\infty} \eta(A \cap A_n) = \eta(A)$ (as $(A_n)_{n\in\mathbb{N}}$ is increasing with union Ω). Since this holds for an arbitrarily fixed $A \in \Sigma$ and since $\lim_{n\to\infty} f_n = f$ pointwise on Ω, it follows from a result by Lewis (see, e.g., [25, Theorem 3.5]) that $f \in L^1(\tilde{\nu})$. Therefore we have proved $L_1(\nu) \subseteq L_1(\tilde{\nu})$ and hence (c) holds. The Closed Graph Theorem can be used to show that both inclusions $L_1(\nu) \subseteq L_1(\tilde{\nu})$ and $L_1(\tilde{\nu}) \subseteq L_1(\nu)$ are continuous, so that the norms of $L_1(\nu)$ and $L_1(\tilde{\nu})$ are equivalent. □

We finish this subsection with two results showing that the study of countable additivity of vector measures in arbitrary Banach spaces can be reduced somehow to the ℓ_∞-valued case.

Proposition 3.8. *Let $W \subseteq B_{X^*}$ and let $i_W : X \to \ell_\infty(W)$ be the operator defined by*

$$i_W(x) := \big(x^*(x)\big)_{x^*\in W} \quad \text{for all } x \in X.$$

Let $\nu : \Sigma \to X$ be a map defined on a σ-algebra Σ and define $\widehat{\nu}_W := i_W \circ \nu$. Let us consider the following statements:

- *(i) $\nu \in \mathrm{ca}(\Sigma, X)$.*
- *(ii) $\widehat{\nu}_W \in \mathrm{ca}(\Sigma, \ell_\infty(W))$.*
- *(iii) $\varphi \circ \widehat{\nu}_W \in \mathrm{ca}(\Sigma)$ for every $\varphi \in \ell_1(W) \subseteq \ell_\infty(W)^*$.*
- *(iv) $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for every $x^* \in W$.*

Then (i)⇒(ii)⇒(iii)⇔(iv). Moreover:

- *(a) If W is w^*-compact, then (ii)⇔(iii)⇔(iv).*
- *(b) If i_W^{**} is injective, then (i)⇔(ii).*

Proof. The implications (i)⇒(ii)⇒(iii) are immediate.

To prove (iii)⇔(iv), observe first that for each $x^* \in W$ we have $x^* \circ \nu = e_{x^*} \circ \widehat{\nu}_W$, where $e_{x^*} \in \ell_1(W)$ is the vector defined by $e_{x^*}(z^*) = 0$ for all

$z^* \in W \setminus \{x^*\}$ and $e_{x^*}(x^*) = 1$. Since $\ell_1(W) = \overline{\mathrm{span}(\{e_{x^*} : x^* \in W\})}^{\|\cdot\|} \subseteq \ell_\infty(W)^*$, the equivalence (iii)$\Leftrightarrow$(iv) is a consequence of the Vitali-Hahn-Saks-Nikodým theorem (see, e.g., [10, p. 24, Corollary 10]).

(a) If W is w^*-compact, then i_W takes values in the subspace $C(W) \subseteq \ell_\infty(W)$ of all w^*-continuous real-valued functions on W. The set

$$\Gamma := \{\pm e_{x^*}|_{C(W)} : \, x^* \in W\} \subseteq B_{C(W)^*}$$

is a James boundary of $C(W)$ and so it has the OT property, as we already mentioned in the Introduction. Finally, observe that (iv) is equivalent to saying that $\gamma \circ \widehat{\nu}_W \in \mathrm{ca}(\Sigma)$ for every $\gamma \in \Gamma$.

(b) This follows at once from Proposition 3.5(ii-iii). $\square$

Observe that if $W \subseteq B_{X^*}$ is norming, then the operator i_W of Proposition 3.8 is an isomorphic embedding.

Proposition 3.9. *Let $W \subseteq B_{X^*}$ be a norming set and let $\nu : \Sigma \to X$ be a map defined on a σ-algebra Σ such that $x^* \circ \nu \in \mathrm{ca}(\Sigma)$ for every $x^* \in W$. If $\nu \notin \mathrm{ca}(\Sigma, X)$, then there is a countable set $W_0 \subseteq W$ such that $\widehat{\nu}_{W_0} \notin \mathrm{ca}(\Sigma, \ell_\infty(W_0))$, and we have a commutative diagram*

$$\begin{tikzcd}
 & & \ell_\infty(W) \arrow[dd, "P_{W_0}"] \\
\Sigma \arrow[r, "\nu"] \arrow[urr, "\widehat{\nu}_W"] \arrow[drr, "\widehat{\nu}_{W_0}"'] & X \arrow[ur, "i_W"] \arrow[dr, "i_{W_0}"'] & \\
 & & \ell_\infty(W_0)
\end{tikzcd}$$

where P_{W_0} is the operator defined by $P_{W_0}(u) := u|_{W_0}$ for all $u \in \ell_\infty(W)$.

Proof. Observe that ν is finitely additive. Since ν is not countably additive, we can take a sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint elements of Σ such that the sequence $(\nu(\bigcup_{i>n} A_i))_{n\in\mathbb{N}}$ does not converge to 0. Write $B_n := \bigcup_{i>n} A_i$ for all $n \in \mathbb{N}$. Since W is norming, there is a constant $k > 0$ such that for each $n \in \mathbb{N}$ there is $x_n^* \in W$ such that $|x_n^*(\nu(B_n))| \geq k\|\nu(B_n)\|_X$. Let $W_0 := \{x_n^* : n \in \mathbb{N}\}$. Then

$$\|\widehat{\nu}_{W_0}(B_n)\|_{\ell_\infty(W_0)} \geq |x_n^*(\nu(B_n))| \geq k\|\nu(B_n)\|_X \quad \text{for all } n \in \mathbb{N},$$

hence the sequence $(\widehat{\nu}_{W_0}(B_n))_{n\in\mathbb{N}}$ does not converge to 0 in $\ell_\infty(W_0)$. It follows that $\widehat{\nu}_{W_0} \notin \mathrm{ca}(\Sigma, \ell_\infty(W_0))$. The last statement is immediate. $\square$

3.2. **The OT property in ℓ_∞.** As noted in the Introduction, the finitely additive map $\nu : \mathcal{P}(\mathbb{N}) \to \ell_\infty$ defined by $\nu(A) := \chi_A$ for $A \subseteq \mathbb{N}$ is not countably additive while $\pi_n \circ \nu \in \mathrm{ca}(\mathcal{P}(\mathbb{N}))$ for each coordinate functional $\pi_n \in \ell_\infty^*$. This example has been used to see that the set $\{\pi_n : n \in \mathbb{N}\} \subseteq \ell_\infty^*$ fails to have the OT property. To provide further examples of the same nature, we shall first determine the form of general ℓ_∞-valued countably additive measures in Proposition 3.12 below. The proof uses a couple of results which shall also be needed later. The first one goes back to Bartle, Dunford and Schwartz [4] (cf. [10, p. 14, Corollary 7]) while the second one is folklore (see, e.g., [12, Proposition 3.107]).

Theorem 3.10 (Bartle-Dunford-Schwartz). *Let Σ be a σ-algebra, let X be a Banach space and let $\nu \in \mathrm{ca}(\Sigma, X)$. Then the range of ν, that is, the set*

$$\mathcal{R}(\nu) := \{\nu(A) : A \in \Sigma\}$$

is relatively weakly compact in X.

Proposition 3.11. *Let X be a separable Banach space. Then any weakly compact subset of X^* is norm-separable.*

Proposition 3.12. *Let $\nu : \Sigma \to \ell_\infty$ be a map defined on a σ-algebra Σ such that $\{\pi_n \circ \nu : n \in \mathbb{N}\} \subseteq \mathrm{ca}(\Sigma)$. The following conditions are equivalent:*

(i) $\nu \in \mathrm{ca}(\Sigma, \ell_\infty)$.
(ii) $\{\nu(A) : A \in \Sigma\}$ *is relatively weakly compact in* ℓ_∞.
(iii) $\{\nu(A) : A \in \Sigma\}$ *is norm-separable in* ℓ_∞.
(iv) $\{\pi_n \circ \nu : n \in \mathbb{N}\}$ *is a uniformly countably additive subset of* $\mathrm{ca}(\Sigma)$.
(v) $\{\pi_n \circ \nu : n \in \mathbb{N}\}$ *is relatively weakly compact in* $\mathrm{ca}(\Sigma)$.
(vi) There exists a non-negative $\mu \in \mathrm{ca}(\Sigma)$ *such that* $\{\pi_n \circ \nu : n \in \mathbb{N}\}$ *is uniformly* μ*-continuous.*

Proof. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) follow from Theorem 3.10 and Proposition 3.11, respectively.

To prove (iii)$\Rightarrow$(i), note that (iii) implies that ν takes values in a separable subspace X of ℓ_∞. Since X contains no subspace isomorphic to ℓ_∞ and the set of restrictions $\{\pi_n|_X : n \in \mathbb{N}\} \subseteq X^*$ is total (for X), the Diestel-Faires Theorem 1.1(ii) implies that ν is countably additive.

Observe that

$$\sup_{n\in\mathbb{N}} |(\pi_n \circ \nu)(A)| = \|\nu(A)\|_{\ell_\infty} \quad \text{for every } A \in \Sigma. \tag{3.3}$$

The previous equality yields the equivalence (i)$\Leftrightarrow$(iv), because if $(A_i)_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint elements of Σ, then

$$\sup_{n\in\mathbb{N}}\left|(\pi_n\circ\nu)\left(\bigcup_{i>m}A_i\right)\right| = \left\|\nu\left(\bigcup_{i>m}A_i\right)\right\|_{\ell_\infty} \quad \text{for all } m\in\mathbb{N}.$$

Moreover, (3.3) and the Nikodým boundedness theorem (see, e.g., [10, p. 14, Theorem 1]) apply to conclude that $\{\pi_n\circ\nu : n\in\mathbb{N}\}$ is bounded in $\mathrm{ca}(\Sigma)$. So, the equivalences (iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi) follow from a well known characterization of relatively weakly compact subsets of $\mathrm{ca}(\Sigma)$ (see, e.g., [7, p. 92, Theorem 13]). $\square$

Motivated by condition (vi) in Proposition 3.12 above, we present the following:

Example 3.13. Let λ be the Lebesgue measure on the Lebesgue σ-algebra Σ of $[0,1]$. Take a norm-bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $L_1[0,1]$. Let $\nu:\Sigma\to\ell_\infty$ be the map defined by

$$\nu(A) := \left(\int_A f_n\,d\lambda\right)_{n\in\mathbb{N}} \quad \text{for all } A\in\Sigma.$$

According to the proof of Proposition 3.12 and the Dunford-Pettis theorem (see, e.g., [7, p. 93]), the map ν is countably additive if and only if $(f_n)_{n\in\mathbb{N}}$ is uniformly λ-integrable, that is,

$$\lim_{\lambda(A)\to 0}\sup_{n\in\mathbb{N}}\int_A |f_n|\,d\lambda = 0.$$

This holds when there is $g\in L_1[0,1]$ such that $|f_n|\le g$ for all $n\in\mathbb{N}$. For further criteria for uniform integrability, see [8], for example.

It is easy to check that the norm-bounded sequence $(n\chi_{[0,1/n)})_{n\in\mathbb{N}}$ in $L_1[0,1]$ is not uniformly λ-integrable. The same holds for the sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_n(t) := \int_0^t n\chi_{[1-1/n,1)}(s)/(1-s)ds$ for $t\in[0,1]$ and $n\in\mathbb{N}$, [23, p. 295]. More generally, [3, Example 1] provides a norm-bounded sequence in $L_1[0,1]$ whose restriction to any $A\in\Sigma\setminus\mathcal{N}(\lambda)$ is not uniformly λ-integrable.

Exploiting the fact that the Banach spaces ℓ_∞ and $L_\infty[0,1]$ are isomorphic (see, e.g., [1, Theorem 4.3.10]), let us see that $L_1[0,1]$ considered as a total subspace of $L_\infty[0,1]^*$ fails to have the OT property.

Example 3.14. Let λ and Σ be as in Example 3.13. Define $\nu:\Sigma\to L_\infty[0,1]$ by $\nu(A)=\chi_A$ for all $A\in\Sigma$. Then $\nu\notin\mathrm{ca}(\Sigma,L_\infty[0,1])$ and for each $\varphi\in L_1[0,1]$ we have $\varphi\circ\nu\in\mathrm{ca}(\Sigma)$, because $(\varphi\circ\nu)(A)=\int_A\varphi\,d\lambda$ for all $A\in\Sigma$. So, $L_1[0,1]\subseteq L_\infty[0,1]^*$ fails to have the OT property.

It turns out that the existence of a norm-separable subset of X^* having the OT property prevents the Banach space X from containing subspaces isomorphic to ℓ_∞. This is asserted in Corollary 3.22 below, whose proof requires our main result:

Theorem 3.15. *Let $W \subseteq \ell_\infty^*$ be a set not containing sets equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. Then W fails the OT property.*

Given a Banach space X and a non-empty set I, a set $\{x_i : i \in I\} \subseteq X$ is said to be *equivalent to the canonical basis of* $\ell_1(I)$ if it is bounded and there is a constant $c > 0$ such that

$$\left\| \sum_{i\in I} a_i x_i \right\| \geq c \sum_{i \in I} |a_i|$$

for every $(a_i)_{i\in I} \in \ell_1(I)$. In this case, $\overline{\mathrm{span}}(\{x_i : i \in I\})$ is isomorphic to $\ell_1(I)$.

The proof of Theorem 3.15 requires some known facts and uses the following result of Talagrand (see [31, Théorème 4]). Recall that the *cofinality* (denoted by $cf(\kappa)$) of a cardinal κ is the smallest cardinal κ' such that κ is the union of κ' many sets of cardinality $< \kappa$. Both $cf(\omega_1)$ and $cf(\mathfrak{c})$ are uncountable (see, e.g., [18, Corollary 5.12]).

Theorem 3.16 (Talagrand)**.** *Let I be a set such that $|I|$ has uncountable cofinality. Let X be a Banach space and let $D \subseteq X$ be a set such that $X = \overline{\mathrm{span}}(D)$. If X contains a subspace isomorphic to $\ell_1(I)$, then D contains a set which is equivalent to the canonical basis of $\ell_1(I)$.*

The following result is an application of [28, Lemma 1.1]:

Lemma 3.17. *Let X be the ℓ_1-sum of a family of Banach spaces $\{X_i : i \in I\}$ and, for each $i \in I$, let $\pi_i : X \to X_i$ be the canonical projection. Let $W \subseteq X$ be a subspace. If the set $\{i \in I : \pi_i(W) \neq \{0\}\}$ contains a set J such that $|J|$ has uncountable cofinality, then W contains a subspace isomorphic to $\ell_1(J)$.*

Proof. For each $i \in J$ we fix $x_i \in B_W$ and $x_i^* \in B_{X_i^*}$ with $x_i^*(\pi_i(x_i)) \neq 0$. Define an operator $T : X \to \ell_1(J)$ by

$$T(x) := \big(x_i^*(\pi_i(x))\big)_{i\in J} \quad \text{for all } x \in X.$$

For each $k \in \mathbb{N}$ we define $J_k := \{i \in J : |x_i^*(\pi_i(x_i))| > \frac{1}{k}\}$, so that $J = \bigcup_{k\in\mathbb{N}} J_k$. Since $|J|$ has uncountable cofinality, there is $k \in \mathbb{N}$ such that $|J_k| = |J|$.

For each $i \in J$, let $\phi_i \in \ell_1(J)^*$ be the ith-coordinate functional. Since J_k is contained in the set

$$J' := \left\{ i \in J : \, \big|\phi_i(T(x))\big| > \frac{1}{k} \text{ for some } x \in B_W \right\},$$

we have $|J'| = |J|$. We can now apply [28, Lemma 1.1] to conclude that W contains a subspace isomorphic to $\ell_1(J)$. □

Another ingredient for proving Theorem 3.15 is Theorem 3.18 below. It can be proved as [19, Proposition 4] (which can also be found in [1, Theorem 2.5.4]), bearing in mind that there is an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ with $|\mathcal{A}| = \mathfrak{c}$ (see, e.g., the proof of [1, Lemma 2.5.3]).

Theorem 3.18 (Kalton)**.** *Let $T : \ell_\infty \to \ell_\infty(I)$ be an operator, where I is a non-empty set with $|I| < \mathfrak{c}$. If T vanishes on c_0, then there is an infinite set $A \subseteq \mathbb{N}$ such that T vanishes on the subspace*

$$Z_A := \big\{(x_n)_{n\in\mathbb{N}} \in \ell_\infty : \, x_n = 0 \textit{ for all } n \in \mathbb{N} \setminus A\big\}.$$

Let K be a compact Hausdorff topological space. By Riesz's representation theorem, the dual $C(K)^*$ is the Banach space of all real-valued regular Borel measures on K with the variation norm. The subset of $C(K)^*$ consisting of all regular Borel probability measures on K is denoted by $P(K)$. Given $\mu \in P(K)$, any $\xi \in C(K)^*$ can be written (in a unique way) as $\xi = f\,d\mu + \xi'$ for some $f \in L_1(\mu)$ and some $\xi' \in C(K)^*$ which is singular with respect to μ; here $f\,d\mu \in C(K)^*$ is given by $(f\,d\mu)(B) = \int_B f\,d\mu$ for every Borel set $B \subseteq K$ and, as usual, we write $f = \frac{d\xi}{d\mu}$.

The Banach spaces ℓ_∞ and $C(\beta\mathbb{N})$ are isometrically isomorphic, where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$ with the discrete topology. Recall that $\beta\mathbb{N}$ is the set of all ultrafilters on $\mathbb{N}$, which is a compact Hausdorff topological space such that the family $\widehat{A} := \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$, for $A \subseteq \mathbb{N}$, forms a basis of clopen sets. Each $n \in \mathbb{N}$ is identified with the ultrafilter $\{A \subseteq \mathbb{N} : n \in A\} \in \beta\mathbb{N}$.

We are now ready to prove the main result of this section:

Proof of Theorem 3.15. Let $R : C(\beta\mathbb{N}) \to \ell_\infty$ be the isometric isomorphism satisfying $R(\chi_{\widehat{A}}) = \chi_A$ for all $A \subseteq \mathbb{N}$. Let $\mu_0 \in P(\beta\mathbb{N})$ be the regular Borel probability measure on $\beta\mathbb{N}$ satisfying $\mu_0(\{n\}) = 2^{-n}$ for all $n \in \mathbb{N}$. Observe that for each $f \in L_1(\mu_0)$ the series of real numbers $\sum_{n\in\mathbb{N}} f(n)2^{-n}$ is absolutely convergent and we have

$$\int_{\widehat{B}} f\,d\mu_0 = \sum_{n\in B} f(n)2^{-n} \quad \text{for every } B \subseteq \mathbb{N}. \tag{3.4}$$

Zorn's lemma ensures the existence of a set $\Delta \subseteq P(\beta\mathbb{N})$ containing μ_0 and consisting of mutually singular elements of $P(\beta\mathbb{N})$ such that Δ is maximal (with respect to the inclusion) among all subsets of $P(\beta\mathbb{N})$ satisfying those properties. Then for any $\xi \in C(\beta\mathbb{N})^*$ we have

$$\xi = \sum_{\mu \in \Delta} \frac{d\xi}{d\mu} \, d\mu, \tag{3.5}$$

the series being absolutely convergent in $C(\beta\mathbb{N})^*$, and the space $C(\beta\mathbb{N})^*$ is isometrically isomorphic to the ℓ_1-sum Z of the family of Banach spaces $\{L_1(\mu) : \mu \in \Delta\}$ via the operator $S : C(\beta\mathbb{N})^* \to Z$ defined by

$$S(\xi) := \left(\frac{d\xi}{d\mu}\right)_{\mu \in \Delta} \quad \text{for all } \xi \in C(\beta\mathbb{N})^*$$

(see, e.g., the proof of [1, Proposition 4.3.8(iii)]).

By Theorem 3.16 we can assume without loss of generality that W is a subspace of ℓ_∞^*. The conclusion is obvious if $W = \{0\}$, so we assume that $W \neq \{0\}$. For each $\mu \in \Delta$, let $\pi_\mu : Z \to L_1(\mu)$ be the canonical projection. Since W does not contain subspaces isomorphic to $\ell_1(\mathfrak{c})$, the same holds for the subspace $(S \circ R^*)(W) \subseteq Z$ and so the set

$$\Delta_0 := \big\{\mu \in \Delta :\ \pi_\mu\big((S \circ R^*)(W)\big) \neq \{0\}\big\}$$

is non-empty and has cardinality $|\Delta_0| < \mathfrak{c}$ (by Lemma 3.17).

Let $T : \ell_\infty \to \ell_\infty(\Delta_0)$ be the operator defined by

$$T(x)(\mu) := \begin{cases} \int_{\beta\mathbb{N}} R^{-1}(x) \, d\mu & \text{if } \mu \neq \mu_0 \\ 0 & \text{if } \mu = \mu_0 \end{cases}$$

for every $\mu \in \Delta_0$ and for every $x \in \ell_\infty$. Every $\mu \in \Delta \setminus \{\mu_0\}$ is singular with respect to μ_0 and hence, $\mu(\widehat{A}) = 0$ for every finite set $A \subseteq \mathbb{N}$. Bearing in mind that $c_0 = \overline{\text{span}}(\{\chi_A : A \subseteq \mathbb{N} \text{ finite}\}) \subseteq \ell_\infty$, we deduce that $T(x) = 0$ for every $x \in c_0$. The fact that $|\Delta_0| < \mathfrak{c}$ allows us to apply Theorem 3.18 to get an infinite set $A \subseteq \mathbb{N}$ such that T vanishes on

$$Z_A := \big\{(x_n)_{n \in \mathbb{N}} \in \ell_\infty :\ x_n = 0 \text{ for all } n \in \mathbb{N} \setminus A\big\},$$

that is,

$$\int_{\beta\mathbb{N}} R^{-1}(x) \, d\mu = 0 \quad \text{for every } x \in Z_A \text{ and for every } \mu \in \Delta_0 \setminus \{\mu_0\}. \tag{3.6}$$

Define a finitely additive map $\nu : \mathcal{P}(A) \to \ell_\infty$ by $\nu(B) := \chi_B$ for all $B \subseteq A$. Note that ν is not countably additive, because A is infinite and $\|\nu(\{n\})\| = 1$ for every $n \in A$. To finish the proof we will show that W fails the OT property by checking that $\varphi \circ \nu \in \mathrm{ca}(\mathcal{P}(A))$ for arbitrarily $\varphi \in W$. Observe first that

$$R^*(\varphi) \overset{(3.5)}{=} \sum_{\mu\in\Delta} \frac{dR^*(\varphi)}{d\mu}\, d\mu = \sum_{\mu\in\Delta_0} \frac{dR^*(\varphi)}{d\mu}\, d\mu,$$

the series being absolutely convergent in $C(\beta\mathbb{N})^*$. Moreover, for each $B \subseteq A$ we have $\chi_B \in Z_A$ and so (3.6) yields $\mu(\widehat{B}) = 0$ for all $\mu \in \Delta_0 \setminus \{\mu_0\}$. Hence,

$$(\varphi\circ\nu)(B) = R^*(\varphi)(\chi_{\widehat{B}}) = \sum_{\mu\in\Delta_0} \int_{\widehat{B}} \frac{dR^*(\varphi)}{d\mu}\, d\mu = \begin{cases} 0 & \text{if } \mu_0 \notin \Delta_0 \\ \int_{\widehat{B}} \frac{dR^*(\varphi)}{d\mu_0}\, d\mu_0 & \text{if } \mu_0 \in \Delta_0. \end{cases}$$

Therefore, if $\mu_0 \notin \Delta_0$, then $\varphi\circ\nu$ is identically null and so countably additive. If $\mu_0 \in \Delta_0$, then

$$(\varphi \circ \nu)(B) = \int_{\widehat{B}} \frac{dR^*(\varphi)}{d\mu_0}\, d\mu_0 \overset{(3.4)}{=} \sum_{n\in B} \frac{dR^*(\varphi)}{d\mu_0}(n)2^{-n}$$

for every $B \subseteq A$, where the series $\sum_{n\in A} \frac{dR^*(\varphi)}{d\mu_0}(n)2^{-n}$ is absolutely convergent, hence $\varphi \circ \nu$ is countably additive. The proof is finished. $\square$

The converse of Theorem 3.15 fails to hold, in general. An example follows:

Example 3.19. Let $2\mathbb{N} - 1$ (resp., $2\mathbb{N}$) be the set of all odd (resp., even) natural numbers. With the notation of Theorem 3.18, let $W \subseteq \ell_\infty^*$ be the norm-closure of $(Z_{2\mathbb{N}-1})^\perp + \ell_1$. Then:

(i) W is total and contains a subspace isometric to $\ell_1(2^{\mathfrak{c}})$; and
(ii) W fails the OT property.

Indeed, W is total because it contains ℓ_1. Let $P : \ell_\infty \to Z_{2\mathbb{N}}$ be the canonical projection and define $\Phi : Z_{2\mathbb{N}}^* \to (Z_{2\mathbb{N}-1})^\perp$ by $\Phi(\xi) := \xi \circ P$ for all $\xi \in Z_{2\mathbb{N}}^*$. Then Φ is an isometric embedding and therefore W contains a subspace isometric to $Z_{2\mathbb{N}}^*$. Since $Z_{2\mathbb{N}}$ is isometric to $C(\beta\mathbb{N})$, its dual contains a subspace isometric to $\ell_1(|\beta\mathbb{N}|)$, and the same holds for W. Now, bear in mind that $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$ (see, e.g., [33, 19.13(d)]) to get (i). In order to show that W fails the OT property, define $\nu : \mathcal{P}(\mathbb{N}) \to \ell_\infty$ by $\nu(A) := \chi_{A\cap(2\mathbb{N}-1)} = Q(\chi_A)$ for all $A \subseteq \mathbb{N}$, where $Q : \ell_\infty \to Z_{2\mathbb{N}-1}$ is the canonical projection. Clearly, ν is not countably additive. However, we claim that $\varphi \circ \nu \in \mathrm{ca}(\mathcal{P}(\mathbb{N}))$ for every $\varphi \in W$. Indeed, by the Vitali-Hahn-Saks-Nikodým theorem (see, e.g., [10, p. 24, Corollary 10]), it suffices to check it whenever $\varphi \in (Z_{2\mathbb{N}-1})^\perp \cup \{e_n : n \in \mathbb{N}\}$, where $\{e_n : n \in \mathbb{N}\}$ is the canonical basis of ℓ_1. On the one hand, we have $\varphi \circ \nu = 0$ (hence it is countably additive) whenever $\varphi \in (Z_{2\mathbb{N}-1})^\perp$. On the other hand, for each $n \in \mathbb{N}$ the composition $e_n \circ \nu$ is countably additive, because $(e_n \circ \nu)(A) = \chi_{A\cap(2\mathbb{N}-1)}(n)$ for all $A \subseteq \mathbb{N}$. This establishes the claim and hence, W fails the OT property.

Corollary 3.20. *Let X be a Banach space such that there is a subset of X^* having the OT property but not containing sets equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. Then X does not contain subspaces isomorphic to ℓ_∞.*

Proof. Let $W \subseteq X^*$ be a set having the OT property such that W does not contain sets equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. Given any subspace $Y \subseteq X$, the set $W|_Y := \{x^*|_Y : x^* \in W\} \subseteq Y^*$ has the OT property and does not contain sets equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. By Theorem 3.15, Y cannot be isomorphic to ℓ_∞. $\square$

Remark 3.21. The converse of the previous corollary is not true in general, as witnessed by the space $X = c_0(\mathfrak{c})$. Indeed, $c_0(\mathfrak{c})$ contains no subspace isomorphic to ℓ_∞ (because any separable subspace of $c_0(\mathfrak{c})$ is isomorphic to a subspace of c_0, while any separable Banach space is isomorphic to a subspace of ℓ_∞). The space $c_0(\mathfrak{c})$ is weakly compactly generated, so its dual $c_0(\mathfrak{c})^* = \ell_1(\mathfrak{c})$ satisfies $\mathrm{dens}(\ell_1(\mathfrak{c}), w^*) = \mathrm{dens}(c_0(\mathfrak{c}), \|\cdot\|) = \mathfrak{c}$ (see, e.g., [12, Theorem 13.3]). Fix a total set $W \subseteq \ell_1(\mathfrak{c})$. Then W has the OT property by the Diestel-Faires Theorem 1.1(ii). Since the subspace $W_0 := \overline{\mathrm{span}(W)}^{\|\cdot\|}$ is w^*-dense in $\ell_1(\mathfrak{c})$, we have $\mathrm{dens}(W_0, \|\cdot\|) = \mathfrak{c}$. Now, a classical result of Köthe (see, e.g., [29, p. 29]) ensures that W_0 contains a subspace isomorphic to $\ell_1(\mathfrak{c})$. Finally, Theorem 3.16 applies to conclude that W contains a set equivalent to the canonical basis of $\ell_1(\mathfrak{c})$.

Corollary 3.22. *Let X be a Banach space. The following statements are equivalent:*

(i) X does not contain subspaces isomorphic to ℓ_∞ and X^* is w^*-separable.
(ii) There is a countable subset of X^* having the OT property.
(iii) There is a norm-separable subset of X^* having the OT property.

Proof. (i)$\Rightarrow$(ii) follows from the Diestel-Faires Theorem 1.1(ii), bearing in mind that the w^*-separability of X^* is equivalent to the existence of a countable total subset of X^*. The implication (ii)$\Rightarrow$(i) is a consequence of Corollary 3.20 and Proposition 3.1. The equivalence (ii)$\Leftrightarrow$(iii) follows from Proposition 3.3. $\square$

4. L_1 SPACES OF VECTOR MEASURES WITH SEPARABLE RANGE

Let Σ be a σ-algebra, let X be a Banach space and let $\nu \in \mathrm{ca}(\Sigma, X)$. By Theorem 3.10 and Proposition 3.11, the set

$$\mathcal{R}(\nu) = \{\nu(A) : A \in \Sigma\} \subseteq X$$

(the range of ν) is separable when $X = \ell_\infty$ and so, in this case, ν can be seen as an element of $\mathrm{ca}(\Sigma, Y)$ for some separable subspace $Y \subseteq \ell_\infty$. Conversely, if X is separable, then it is isometric to a subspace of ℓ_∞ (see, e.g., [1, Remark 1.4.2(b)]) and so ν can be seen as an element of $\mathrm{ca}(\Sigma, \ell_\infty)$. As a consequence, we get:

Proposition 4.1. *Let E be a Banach function space over a finite measure space (Ω, Σ, μ). The following statements are equivalent:*

(i) There is $\nu_0 \in \mathrm{ca}(\Sigma, \ell_\infty)$ such that E is lattice-isomorphic to $L_1(\nu_0)$.

(ii) There exist a separable Banach space X and $\nu_1 \in \mathrm{ca}(\Sigma, X)$ such that E is lattice-isomorphic to $L_1(\nu_1)$.

Moreover, both statements hold if E is order continuous and separable.

Proof. The equivalence (i)$\Leftrightarrow$(ii) has already been explained above. For the "moreover" part, note that the order continuity of E implies that the map

$$\nu : \Sigma \to E, \quad \nu(E) := \chi_E \text{ for all } E \in \Sigma,$$

is countably additive and $E = L_1(\nu)$ (see Subsection 2.3). $\square$

In general, the separability of $\mathcal{R}(\nu)$ does not imply that the space $L_1(\nu)$ is separable, as witnessed by the space $L_1(\lambda_I)$ of the usual measure on $\{0,1\}^I$ for any uncountable set I (see Subsection 2.4).

The following example provides a vector measure ν such that $\mathcal{R}(\nu)$ is separable, $\overline{\mathrm{span}}(\mathcal{R}(\nu))$ is infinite-dimensional and $L_1(\nu)$ is neither separable nor lattice-isomorphic to any AL-space.

Recall that a Banach lattice E is said to be an *AL-space* whenever its norm is 1-additive, that is, $\|x + y\|_E = \|x\|_E + \|y\|_E$ for all $x, y \in E^+$ with $x \wedge y = 0$. It is known that a Banach lattice is an AL-space if and only if it is lattice-isometric to the usual space $L_1(\mu)$ of some non-negative measure μ (possibly infinite); see, e.g., [2, Theorem 4.27]. We refer the reader to [6, Proposition 2] which determines exactly when the L_1-spaces of vector measures are lattice-isomorphic to an AL-space.

Example 4.2. Let G be any non-metrizable compact abelian group (e.g., the product $\{0,1\}^I$ or $\mathbb{T}^I$ for any uncountable set I). By μ we denote the Haar probability measure on the Borel σ-algebra $\mathcal{B}(G)$. Fix $g \in L_1(\mu)$ and define $\nu : \mathcal{B}(G) \to L_\infty(\mu)$ by $\nu(A) := \chi_A * g$ (the convolution) for all $A \in \mathcal{B}(G)$. Then:

(i) $\nu \in \mathrm{ca}(\mathcal{B}(G), L_\infty(\mu))$;
(ii) $\mathcal{R}(\nu)$ is relatively norm-compact (hence separable);
(iii) $L_1(\nu)$ is not separable; and
(iv) if, in addition, $g \notin L_\infty(\mu)$, then $\overline{\mathrm{span}}(\mathcal{R}(\nu))$ is infinite-dimensional and $L_1(\nu)$ is not lattice-isomorphic to any AL-space.

Indeed, (i) and (iii) follow from parts (I) and (II)(iv) of [24, Theorem 1] (where ν is denoted by $m_g^{(\infty)}$). Statement (ii) was noticed in [24, Remark 2]. For statement (iv), suppose that $g \notin L_\infty(\mu)$. Then ν has infinite variation, [24, Theorem 2], and so $L_1(\nu)$ is not lattice-isomorphic to any AL-space, [6, Proposition 2]. To prove that $\overline{\mathrm{span}}(\mathcal{R}(\nu))$ is infinite-dimensional, assume by way of contradiction that $\overline{\mathrm{span}}(\mathcal{R}(\nu))$ is finite-dimensional. Then the integration operator $I_\nu : L_1(\nu) \to L_\infty(\mu)$ is absolutely 1-summing, because its range is contained in $\overline{\mathrm{span}}(\mathcal{R}(\nu))$. Therefore, the restriction of I_ν to $L_\infty(\mu)$, which coincides with the convolution operator $C_g^{(\infty)} : L_\infty(\mu) \to L_\infty(\mu)$ (see part (II)(ii) of [24, Theorem 1]), is absolutely 1-summing as well. This contradicts the fact that $g \notin L_\infty(\mu)$, [24, Theorem 2].

This section is devoted to proving the following:

Theorem 4.3. *Let Σ be a σ-algebra, let X be a separable Banach space and let $\nu \in \mathrm{ca}(\Sigma, X)$. If $L_1(\nu)^*$ is order continuous, then $L_1(\nu)$ is separable.*

We will obtain Theorem 4.3 as a consequence of a more general approach dealing with the concept of positively norming set introduced in [30]:

Definition 4.4. Let E be a Banach lattice. A set $B \subseteq B_{E^*} \cap (E^*)^+$ is said to be *positively norming* if there is a constant $c > 0$ such that

$$\|x\|_E \leq c \sup_{\varphi \in B} \varphi(|x|) \quad \text{for every } x \in E.$$

Lemma 4.5. *Let (Ω, Σ) be a measurable space, let X be a separable Banach space and let $\nu \in \mathrm{ca}(\Sigma, X)$. Then $L_1(\nu)$ admits a countable, positively norming set.*

Proof. Let μ be a Rybakov control measure of ν, so that $L_1(\nu)$ is a Banach function space over (Ω, Σ, μ). Since X is separable, B_{X^*} is w^*-separable and so we can take a w^*-dense sequence $(x_n^*)_{n\in\mathbb{N}}$ in B_{X^*}. For each $n \in \mathbb{N}$, the measure $|x_n^* \circ \nu|$ is μ-continuous and we consider its Radon-Nikodým derivative $g_n := \frac{d|x_n^*\circ\nu|}{d\mu} \in (L_1(\nu))'$ and the associated functional $\varphi_{g_n} \in$

$B_{L_1(\nu)^*} \cap (L_1(\nu)^*)^+$ (see Subsections 2.2 and 2.3). Since $(x_n^*)_{n\in\mathbb{N}}$ is w^*-dense in B_{X^*}, for each $f \in L_1(\nu)$ we can apply [27, Lemma 2.2] to get

$$\|f\|_{L_1(\nu)} = \sup_{n\in\mathbb{N}} \int_\Omega |f|\, d|x_n^* \circ \nu| = \sup_{n\in\mathbb{N}} \varphi_{g_n}(|f|),$$

that is, $\{\varphi_{g_n} : n \in \mathbb{N}\}$ is positively norming. $\square$

It is now clear that Theorem 4.3 will be an immediate consequence of the following:

Theorem 4.6. *Let E be a Banach lattice such that both E and E^* are order continuous. If E admits a countable, positively norming set, then E is separable.*

The proof of Theorem 4.6 requires some previous work. Proposition 4.10 below presents a special case when E is a Banach function space and will be used to prove Theorem 4.6.

Lemma 4.7. *Let E be a Banach function space over a finite measure space (Ω, Σ, μ).*

(i) If E is separable, then $L_1(\mu)$ is separable.
(ii) If $L_1(\mu)$ is separable and E is order continuous, then E is separable.

Proof. (i) If E is separable, then so is $S := \operatorname{span}(\{\chi_A : A \in \Sigma\}) \subseteq E$. Since the inclusion map $\iota : E \to L_1(\mu)$ is an operator, the set $\iota(S)$ is separable. Since $\iota(S)$ is dense in $L_1(\mu)$, we conclude that $L_1(\mu)$ is separable.

(ii) Since E is order continuous, S is dense in E (see, e.g., [25, Remark 2.6]). So, it suffices to check that $\{\chi_A : A \in \Sigma\}$ is separable as a subset of E. Now, since $L_1(\mu)$ is separable, there is a sequence $(A_n)_{n\in\mathbb{N}}$ in Σ such that

$$\inf_{n\in\mathbb{N}} \mu(A_n \triangle A) = 0 \quad \text{for every } A \in \Sigma.$$

Therefore, the order continuity of E implies that the set $\{\chi_{A_n} : n \in \mathbb{N}\}$ is dense in $\{\chi_A : A \in \Sigma\}$ as subsets of E (see, e.g., [25, Lemma 2.37(ii)]). $\square$

Given a finite measure space (Ω, Σ, μ) and $A \in \Sigma \setminus \mathcal{N}(\mu)$, we define

$$\Sigma_A := \{B \in \Sigma : \, B \subseteq A\} \quad \text{and} \quad \mu_A(B) := \mu(A)^{-1}\mu(B) \quad \text{for every } B \in \Sigma_A,$$

so that μ_A is a probability measure on the measurable space (A, Σ_A).

Lemma 4.8. *Let (Ω, Σ, μ) be a finite measure space such that $L_1(\mu)$ is not separable and let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $L_1(\mu)$. Then there exist $A \in$*

$\Sigma \setminus \mathcal{N}(\mu)$ and a sequence $(A_m)_{m\in\mathbb{N}}$ of pairwise disjoint sets of $\Sigma_A \setminus \mathcal{N}(\mu)$ such that

$$\int_A f g_n \, d\mu_A = \left(\int_A f \, d\mu_A\right)\left(\int_A g_n \, d\mu_A\right)$$

for every $f \in \operatorname{span}(\{\chi_{A_m} : m \in \mathbb{N}\})$ and for every $n \in \mathbb{N}$.

Proof. Since $L_1(\mu)$ is not separable, Maharam's theorem (see, e.g., [14, Section 3] or [20, §14]) ensures the existence of $A \in \Sigma \setminus \mathcal{N}(\mu)$ such that the measure algebra of μ_A is isomorphic to the measure algebra of the usual measure λ_I on $\{0,1\}^I$, for some uncountable set I. Therefore, there is a lattice isometry $\Phi : L_1(\mu_A) \to L_1(\lambda_I)$ satisfying

$$\int_A v \, d\mu_A = \int_{\{0,1\}^I} \Phi(v) \, d\lambda_I \quad \text{for every } v \in L_1(\mu_A) \tag{4.1}$$

and

$$\Phi(uv) = \Phi(u)\Phi(v) \quad \text{whenever } u \in \operatorname{span}(\{\chi_B : B \in \Sigma_A\}) \text{ and } v \in L_1(\mu_A) \tag{4.2}$$

(see Subsection 2.5).

We denote by

$$\rho_{J'J} : \{0,1\}^{J'} \to \{0,1\}^J$$

the canonical projection for any $J \subseteq J' \subseteq I$.

For each $n \in \mathbb{N}$ we have $\Phi(g_n|_A) \in L_1(\lambda_I)$ and so there exist a countable set $I_n \subseteq I$ and $h_n \in L_1(\lambda_{I_n})$ such that $\Phi(g_n|_A) = h_n \circ \rho_{II_n}$ (see, e.g., [15, 254Q]). Then the set $I' := \bigcup_{n\in\mathbb{N}} I_n$ is countable and for each $n \in \mathbb{N}$ we have

$$\Phi(g_n|_A) = \widetilde{h}_n \circ \rho_{II'}, \tag{4.3}$$

where $\widetilde{h}_n := h_n \circ \rho_{I'I_n} \in L_1(\lambda_{I'})$. Note that the set $J := I \setminus I'$ is uncountable and, in particular, infinite. So, we can find a sequence $(B_m)_{m\in\mathbb{N}}$ of pairwise disjoint elements of $\Lambda_J \setminus \mathcal{N}(\lambda_J)$. Define

$$C_m := B_m \times \{0,1\}^{I'} \in \Lambda_I \quad \text{for all } m \in \mathbb{N},$$

so that the C_m's are pairwise disjoint with $\lambda_I(C_m) = \lambda_J(B_m)$.

Claim. For every $h \in \operatorname{span}(\{\chi_{C_m} : m \in \mathbb{N}\})$ and for every $n \in \mathbb{N}$ we have

$$\int_{\{0,1\}^I} h \, \Phi(g_n|_A) \, d\lambda_I = \left(\int_{\{0,1\}^I} h \, d\lambda_I\right)\left(\int_{\{0,1\}^I} \Phi(g_n|_A) \, d\lambda_I\right). \tag{4.4}$$

Indeed, note that $h = \widetilde{h} \circ \rho_{IJ}$ for some $\widetilde{h} \in \operatorname{span}(\{\chi_{B_m} : m \in \mathbb{N}\})$ and then Fubini's theorem yields

$$\begin{aligned}(4.5) \quad \int_{\{0,1\}^I} h\,\Phi(g_n|_A)\,d\lambda_I &\overset{(4.3)}{=} \int_{\{0,1\}^I} (\widetilde{h} \circ \rho_{IJ})\,(\widetilde{h}_n \circ \rho_{II'})\,d\lambda_I \\ &= \left(\int_{\{0,1\}^J} \widetilde{h}\,d\lambda_J\right)\left(\int_{\{0,1\}^{I'}} \widetilde{h}_n\,d\lambda_{I'}\right).\end{aligned}$$

Since the function $\rho_{II'}$ is Λ_I-to-$\Lambda_{I'}$ measurable and $\lambda_{I'}(A) = \lambda_I(\rho_{II'}^{-1}(A))$ for every $A \in \Lambda_{I'}$ (see, e.g., [15, 254O]), we have

$$(4.6) \qquad \int_{\{0,1\}^I} \Phi(g_n|_A)\,d\lambda_I \overset{(4.3)}{=} \int_{\{0,1\}^I} \widetilde{h}_n \circ \rho_{II'}\,d\lambda_I = \int_{\{0,1\}^{I'}} \widetilde{h}_n\,d\lambda_{I'}.$$

Similarly, or by direct computation, we also have

$$(4.7) \qquad \int_{\{0,1\}^I} h\,d\lambda_I = \int_{\{0,1\}^J} \widetilde{h}\,d\lambda_J.$$

By putting together (4.5), (4.6) and (4.7) we get (4.4), as claimed.

Finally, let $(A_m)_{m\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of Σ_A such that $\Phi(\chi_{A_m}|_A) = \chi_{C_m}$ for all $m \in \mathbb{N}$. Then $\mu(A_m) = \mu(A)\lambda_I(C_m) > 0$ for all $m \in \mathbb{N}$. Given any $f \in \operatorname{span}(\{\chi_{A_m} : m \in \mathbb{N}\})$, we have $\Phi(f|_A) \in \operatorname{span}(\{\chi_{C_m} : m \in \mathbb{N}\})$ and for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_A f g_n\,d\mu_A &\overset{(4.1)}{=} \int_{\{0,1\}^I} \Phi(f|_A\, g_n|_A)\,d\lambda_I \overset{(4.2)}{=} \int_{\{0,1\}^I} \Phi(f|_A)\Phi(g_n|_A)\,d\lambda_I \\ &\overset{(4.4)}{=} \left(\int_{\{0,1\}^I} \Phi(f|_A)\,d\lambda_I\right)\left(\int_{\{0,1\}^I} \Phi(g_n|_A)\,d\lambda_I\right) \overset{(4.1)}{=} \left(\int_A f\,d\mu_A\right)\left(\int_A g_n\,d\mu_A\right).\end{aligned}$$

The proof is finished. $\square$

We shall make use of the following well known characterization of Banach lattices with order continuous dual (see, e.g., [2, Theorem 4.69]):

Theorem 4.9. *Let E be a Banach lattice. The following statements are equivalent:*

- *(i) E^* is order continuous.*
- *(ii) There is no disjoint sequence in E^+ which is equivalent to the canonical basis of ℓ_1.*
- *(iii) E does not contain sublattices which are lattice isomorphic to ℓ_1.*
- *(iv) E^* does not contain subspaces isomorphic to ℓ_∞.*

Proposition 4.10. *Let E be a Banach function space over a finite measure space (Ω, Σ, μ) such that both E and E^* are order continuous. If E admits a countable, positively norming set, then E is separable.*

Proof. Fix a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $B_{E^*}\cap(E^*)^+$ and a constant $c>0$ such that

$$\|f\|_E \le c\sup_{n\in\mathbb{N}} \phi_n(|f|) \quad \text{for every } f\in E. \tag{4.8}$$

Since E is order continuous, we can identify E^* with E', hence for each $n\in\mathbb{N}$ we have $\phi_n=\varphi_{g_n}$ for some $g_n\in E'$ (see Subsection 2.2). Then (4.8) reads as

$$\|f\|_E \le c\sup_{n\in\mathbb{N}} \int_\Omega |f|g_n\,d\mu \quad \text{for every } f\in E. \tag{4.9}$$

Suppose, by contradiction, that E is not separable. By Lemma 4.7(ii), the space $L_1(\mu)$ is not separable. Then we can apply Lemma 4.8 to $(g_n)_{n\in\mathbb{N}}$ (as a sequence in $L_1(\mu)$) to find $A\in\Sigma\setminus\mathcal{N}(\mu)$ and a sequence $(A_m)_{m\in\mathbb{N}}$ of pairwise disjoint sets of $\Sigma_A\setminus\mathcal{N}(\mu)$ such that

$$\int_A fg_n\,d\mu_A = \left(\int_A f\,d\mu_A\right)\left(\int_A g_n\,d\mu_A\right) \tag{4.10}$$

for every $f\in \operatorname{span}(\{\chi_{A_m}: m\in\mathbb{N}\})$ and for every $n\in\mathbb{N}$.

It follows that for each $f\in \operatorname{span}(\{\chi_{A_m}: m\in\mathbb{N}\})$ we have

$$\begin{aligned}
(c\mu(A))^{-1}\|f\|_E &\overset{(4.9)}{\le} \sup_{n\in\mathbb{N}}\int_A |f|g_n\,d\mu_A \\
&\overset{(4.10)}{=} \sup_{n\in\mathbb{N}}\left(\int_A |f|\,d\mu_A\right)\left(\int_A g_n\,d\mu_A\right) \\
&\le \mu(A)^{-2}\,\|\chi_A\|_E\,\|f\|_{L_1(\mu)} \\
&\le \mu(A)^{-2}\|\chi_A\|_E\,\|\iota\|\,\|f\|_E,
\end{aligned}$$

where $\iota: E\to L_1(\mu)$ is the inclusion map.

Therefore, there exist constants $\alpha,\beta>0$ such that

$$\alpha\|f\|_{L_1(\mu)} \le \|f\|_E \le \beta\|f\|_{L_1(\mu)} \quad \text{for every } f\in \operatorname{span}(\{\chi_{A_m}: m\in\mathbb{N}\}). \tag{4.11}$$

Define $f_m:=\mu(A_m)^{-1}\chi_{A_m}\in E$ for all $m\in\mathbb{N}$. We can use (4.11) to prove that the disjoint sequence $(f_m)_{m\in\mathbb{N}}$ in E^+ is equivalent to the canonical basis of ℓ_1. This contradicts the order continuity of E^* (see Theorem 4.9). $\square$

Recall that an *unconditional Schauder decomposition* of a Banach space X is a family $\{X_i : i\in I\}$ of subspaces of X such that each $x\in X$ can be written in a unique way as $x=\sum_{i\in I}x_i$, where $x_i\in X_i$ for all $i\in I$, the series being unconditionally convergent. In this case, for each $i\in I$ one has a projection P_i from X onto X_i in such a way that $x=\sum_{i\in I}P_i(x)$ for all $x\in X$.

Lemma 4.11. *Let X be a Banach space and let $\{X_i : i \in I\}$ be an unconditional Schauder decomposition of X. For each $i \in I$, let P_i be the associated projection from X onto X_i. If X^* contains no subspace isomorphic to ℓ_∞, then for every $x^* \in X^*$ the set $\{i \in I : x^* \circ P_i \neq 0\}$ is countable.*

Proof. For each $J \subseteq I$, let Q_J be the projection from X onto $\overline{\mathrm{span}}(\bigcup_{i\in J} X_i)$ defined by $Q_J(x) := \sum_{i\in J} P_i(x)$ for all $x \in X$.

Fix $x^* \in X^*$ and define $\nu : \mathcal{P}(I) \to X^*$ by

$$\nu(J) := x^* \circ Q_J \quad \text{for all } J \subseteq I.$$

We identify X as a subspace of X^{**} in the canonical way. Note that $x \circ \nu \in \mathrm{ca}(\mathcal{P}(I))$ for every $x \in X$, because the series of real numbers $\sum_{i\in I} x^*(P_i(x))$ is absolutely convergent and

$$(x \circ \nu)(J) = x^*(Q_J(x)) = \sum_{i\in J} x^*(P_i(x)) \quad \text{for all } J \subseteq I.$$

Since X^* contains no subspace isomorphic to ℓ_∞ and X is a total subset of X^{**}, we can apply the Diestel-Faires Theorem 1.1(ii) to conclude that $\nu \in \mathrm{ca}(\mathcal{P}(I), X^*)$.

We claim that for every $\varepsilon > 0$ the set $I_\varepsilon := \{i \in I : \|x^* \circ P_i\| \geq \varepsilon\}$ is finite. Indeed, if not, then there is a sequence $(i_n)_{n\in\mathbb{N}}$ of distinct elements of I_ε. However, the countable additivity of ν implies that the series $\sum_{n\in\mathbb{N}} \nu(\{i_n\}) = \sum_{n\in\mathbb{N}} x^* \circ P_{i_n}$ is unconditionally convergent in X^*, which is impossible because $\|x^* \circ P_{i_n}\| \geq \varepsilon$ for all $n \in \mathbb{N}$. Therefore, the set $\{i \in I : x^* \circ P_i \neq 0\} = \bigcup_{n\in\mathbb{N}} I_{1/n}$ is countable. □

Lemma 4.12. *Let E be a Banach lattice admitting a countable, positively norming set. Then E admits a strictly positive functional, that is, there is $\varphi \in (E^*)^+$ such that $\varphi(x) > 0$ whenever $x \in E^+ \setminus \{0\}$.*

Proof. Take a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $B_{E^*} \cap (E^*)^+$ and a constant $c > 0$ such that $\|x\|_E \leq c \sup_{n\in\mathbb{N}} \varphi_n(|x|)$ for every $x \in E$. Now, it is clear that the functional $\varphi := \sum_{n\in\mathbb{N}} 2^{-n}\varphi_n$ satisfies the required property. □

We have gathered all the tools needed to prove the main result of this section:

Proof of Theorem 4.6. Since E is order continuous, it admits an unconditional Schauder decomposition $\{E_i : i \in I\}$ consisting of pairwise disjoint bands, each having a weak order unit (see, e.g., [21, Proposition 1.a.9]). For each $i \in I$, let P_i be the associated projection from E onto E_i.

Fix $i \in I$. Since E_i is order continuous and has a weak order unit, it is lattice-isometric to a Banach function space over a finite measure space (see, e.g., [21, Proposition 1.b.14]). Since E^* is order continuous, so is E_i^* (bear in mind the equivalence (i)$\Leftrightarrow$(iii) in Theorem 4.9). Moreover, E_i admits a countable, positively norming set (consider the restriction to E_i of a countable, positively norming set for E). Then, E_i is separable by Proposition 4.10.

Fix $\varphi \in (E^*)^+$ such that $\varphi(x) > 0$ whenever $x \in E^+ \setminus \{0\}$ (see Lemma 4.12). For each $i \in I$ we have $E_i \neq \{0\}$ and so $\varphi \circ P_i \neq 0$. Since E^* is order continuous, it contains no subspace isomorphic to ℓ_∞ (see Theorem 4.9) and then Lemma 4.11 implies that I is countable. From the separability of each E_i it follows that E is separable. $\square$

Example 4.13. The conclusion of Theorem 4.6 can fail if the order continuity of E is dropped. For instance, the non-separable Banach lattice ℓ_∞ admits a countable, positively norming set (the coordinate functionals form a norming set) and ℓ_∞^* is an AL-space, hence it is order continuous (see, e.g., [2, p. 194 and Theorem 4.23]).

Any reflexive Banach lattice is order continuous (see, e.g., [2, Theorem 4.9]), so we get:

Corollary 4.14. *Let E be a reflexive Banach lattice. If E admits a countable, positively norming set, then E is separable.*

The previous corollary and Lemma 4.5 yield:

Corollary 4.15. *Let Σ be a σ-algebra, let X be a separable Banach space and let $\nu \in \mathrm{ca}(\Sigma, X)$. If $L_1(\nu)$ is reflexive, then $L_1(\nu)$ is separable.*

Example 4.16. Let (Ω, Σ, μ) be a finite measure space such that $L_1(\mu)$ is not separable and let $1 < p < \infty$. Then

(i) $L_p(\mu)$ is a non-separable reflexive Banach function space over (Ω, Σ, μ).

(ii) $L_p(\mu) = L_1(\nu)$, where $\nu \in \mathrm{ca}(\Sigma, L_p(\mu))$ is defined by $\nu(A) := \chi_A$ for all $A \in \Sigma$.

(iii) If X is a *separable* Banach space, Σ_0 is a σ-algebra and $\nu_0 \in \mathrm{ca}(\Sigma_0, X)$, then $L_p(\mu)$ is not isomorphic to $L_1(\nu_0)$ even as Banach spaces. To see this, apply Corollary 4.15 and part (i) above.

Acknowledgements. We thank A. Avilés for valuable comments on Theorem 3.15. We also thank the referee for several suggestions that improved the exposition. The research of J. Rodríguez was partially supported by grants MTM2017-86182-P (funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe") and 21955/PI/22 (funded by *Fundación Séneca - ACyT Región de Murcia*). The research of E.A. Sánchez-Pérez was partially supported by grant PID2020-112759GB-I00 funded by MCIN/AEI/10.13039/501100011033.

References

[1] F. Albiac and N. J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.

[2] C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Reprint of the 1985 original, Springer, Dordrecht, 2006.

[3] J. M. Ball and F. Murat, *Remarks on Chacon's biting lemma*, Proc. Amer. Math. Soc. **107** (1989), no. 3, 655–663.

[4] R. G. Bartle, N. Dunford, and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.

[5] G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), no. 2, 317–330.

[6] G. P. Curbera, *When L^1 of a vector measure is an AL-space*, Pacific J. Math. **162** (1994), no. 2, 287–303.

[7] J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.

[8] J. Diestel, *Uniform integrability: an introduction*, Rend. Ist. Mat. Univ. Trieste **23** (1991), 41–80.

[9] J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc. **198** (1974), 253–271.

[10] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

[11] P. G. Dodds, B. de Pagter, and W. Ricker, *Reflexivity and order properties of scalar-type spectral operators in locally convex spaces*, Trans. Amer. Math. Soc. **293** (1986), no. 1, 355–380.

[12] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.

[13] A. Fernández, F. Mayoral, F. Naranjo, and J. Rodríguez, *Norming sets and integration with respect to vector measures*, Indag. Math. (N.S.) **19** (2008), no. 2, 203–215.

[14] D. H. Fremlin, *Measure algebras*, Handbook of Boolean algebras, Vol. 3, North-Holland, Amsterdam, 1989, pp. 877–980.

[15] D. H. Fremlin, *Measure theory. Vol. 2. Broad foundations*, Torres Fremlin, Colchester, 2003.

[16] G. Godefroy, *James boundaries and Martin's axiom*, Serdica Math. J. **42** (2016), no. 1, 59–64.

[17] A. S. Granero and J. M. Hernández, *On James boundaries in dual Banach spaces*, J. Funct. Anal. **263** (2012), no. 2, 429–447.

[18] T. Jech, *Set theory*, The third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[19] N. J. Kalton, *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

[20] H. E. Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften, Band 208, Springer-Verlag, New York, 1974.

[21] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II. Function spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, Berlin, 1979.

[22] O. Nygaard and J. Rodríguez, *Isometric factorization of vector measures and applications to spaces of integrable functions*, J. Math. Anal. Appl. **508** (2022), no. 1, Paper No. 125857, 16 p.

[23] S. Okada and W. J. Ricker, *Nonweak compactness of the integration map for vector measures*, J. Austral. Math. Soc. Ser. A **54** (1993), no. 3, 287–303.

[24] S. Okada and W. J. Ricker, *Ideal properties and integral extension of convolution operators on* $L^\infty(G)$, Note Mat. **31** (2011), no. 1, 149–172.

[25] S. Okada, W. J. Ricker, and E. A. Sánchez Pérez, *Optimal domain and integral extension of operators. Acting in function spaces*, Operator Theory: Advances and Applications, vol. 180, Birkhäuser Verlag, Basel, 2008.

[26] J. Rodríguez, *Factorization of vector measures and their integration operators*, Colloq. Math. **144** (2016), no. 1, 115–125.

[27] J. Rodríguez, *On non-separable* L^1*-spaces of a vector measure*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 4, 1039–1050.

[28] H. P. Rosenthal, *On injective Banach spaces and the spaces* $L^\infty(\mu)$ *for finite measure* μ, Acta Math. **124** (1970), 205–248.

[29] H. P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36.

[30] E. A. Sánchez-Pérez and P. Tradacete, *Positively norming sets in Banach function spaces*, Q. J. Math. **65** (2014), no. 3, 1049–1068.

[31] M. Talagrand, *Sur les espaces de Banach contenant $l^1(\tau)$*, Israel J. Math. **40** (1981), no. 3-4, 324–330 (1982).

[32] G. E. F. Thomas, *L'intégration par rapport à une mesure de Radon vectorielle*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. 2, 55–191.

[33] S. Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

112 Marcorni Crescent, Kambah, ACT 2902, Australia
Email address: sus.okada@outlook.com

Dpto. de Ingeniería y Tecnología de Computadores, Facultad de Informática, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
Current address: Dpto. de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales de Albacete, Universidad de Castilla-La Mancha, 02071 Albacete, Spain
Email address: joserr@um.es

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
Email address: easancpe@mat.upv.es