The Birkhoff integral and the property of Bourgain

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Plan of the talk

- Fréchet's characterization of the Lebesgue integral and Birkhoff's definition.
- The Bourgain property of a family of real-valued functions.
- Characterization of Birkhoff integrability for a function $f : \Omega \longrightarrow X$ by means of the family

$$Z_f = \{x^* \circ f : \ x^* \in X^*, \ \|x^*\| \leq 1\}.$$

- A new characterization of Banach spaces not containing ℓ^1.

Fréchet (1915)

Let $f : \Omega \longrightarrow \mathbb{R}$ be a function.

- Given a countable partition $\Gamma = (A_n)$ of Ω in Σ, we say that f is *summable* with respect to Γ if $f(A_n)$ is bounded whenever $\mu(A_n) > 0$ and the series

$$J_*(f,\Gamma) = \sum_n \mu(A_n) \inf f(A_n), \quad J^*(f,\Gamma) = \sum_n \mu(A_n) \sup f(A_n),$$

  are absolutely convergent.
- The intersection

$$\bigcap \{[J_*(f,\Gamma), J^*(f,\Gamma)] : \ f \text{ is summable with respect to } \Gamma\}$$

  is a single point x if and only if f is Lebesgue integrable and $x = \int_\Omega f \, d\mu$.

Birkhoff (1935)

Let $f : \Omega \longrightarrow X$ be a function.

- Given a countable partition $\Gamma = (A_n)$ of Ω in Σ, we say that f is **summable** with respect to Γ if $f(A_n)$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f,\Gamma) = \left\{ \sum_n \mu(A_n) f(t_n) : \; t_n \in A_n \right\}$$

  is made up of *unconditionally* convergent series.
- We say that f is **Birkhoff integrable** if for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ for which f is summable and $\operatorname{diam}(J(f,\Gamma)) < \varepsilon$. In this case, the **Birkhoff integral** of f is the only point in the intersection

$$\bigcap \{ \overline{\mathrm{co}(J(f,\Gamma))} : \; f \text{ is summable with respect to } \Gamma \}.$$

Definition

A family $\mathscr{H} \subset \mathbb{R}^{\Omega}$ has the **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \ldots, A_n \subset A$, $A_i \in \Sigma$ with $\mu(A_i) > 0$, such that for every $h \in \mathscr{H}$

$$\min_{1 \leq i \leq n} \operatorname{diam}(h(A_i)) < \varepsilon.$$

In this case:

- $\mathscr{H}$ is made up of measurable functions.
- For each $h \in \overline{\mathscr{H}}^{\mathfrak{T}_p}$ there is a *sequence* (h_n) in $\mathscr{H}$ converging to h almost everywhere (Bourgain).

Lemma 1

Let $\mathscr{H} \subset \mathbb{R}^\Omega$ be a family of functions. TFAE:

(1) $\mathscr{H}$ has the Bourgain property;

(2) for every $\varepsilon > 0$ and every $\delta > 0$ there is a finite partition Γ of Ω in Σ such that for every $h \in \mathscr{H}$

$$\mu\Big(\bigcup\{A \in \Gamma : \operatorname{diam}(h(A)) > \varepsilon\}\Big) < \delta.$$

Moreover, if $\mathscr{H}$ is *uniformly bounded*, we can add

(3) for every $\varepsilon > 0$ there is a finite partition Γ of Ω in Σ such that for every $h \in \mathscr{H}$

$$\sum_{A \in \Gamma} \mu(A) \operatorname{diam}(h(A)) < \varepsilon.$$

Theorem 2

Let $f : \Omega \to X$ be a *bounded* function. TFAE:

(1) f is Birkhoff integrable;

(2) the family $Z_f = \{x^* \circ f : \ x^* \in B_{X^*}\}$ has the Bourgain property;

(3) there is a norming set $B \subset B_{X^*}$ such that the family $Z_{f,B} = \{x^* \circ f : \ x^* \in B\}$ has the Bourgain property.

Lemma 3

Let $B_1, \dots, B_n$ be subsets of X such that for every $x^* \in B_{X^*}$

$$\min_{1 \le i \le n} \operatorname{diam}(x^*(B_i)) < 1.$$

Then there is $1 \le j \le n$ such that B_j is bounded.

Lemma 4

Let $f : \Omega \longrightarrow X$ be a function such that $Z_f = \{x^* \circ f : \ x^* \in B_{X^*}\}$ has the Bourgain property. Then there is a countable partition (A_n) of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.

Theorem 5

Let $f : \Omega \longrightarrow X$ be a function. TFAE:

(1) f is Birkhoff integrable;

(2) the family $Z_f = \{x^* \circ f : x^* \in B_{X^*}\}$ is a uniformly integrable subset of $\mathscr{L}^1(\mu)$ with the Bourgain property.

Corollary 6

Let $f : \Omega \longrightarrow X$ be a Birkhoff integrable function. Then $\{\int_A f \, d\mu : A \in \Sigma\}$ is *norm* relatively compact.

Definition

A Banach space X has the **weak Radon-Nikodým property (WRNP)** if for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X$ of σ-finite variation, there is a Pettis integrable function $f : \Omega \longrightarrow X$ such that

$$\nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \Sigma.$$

Theorem 7

Let X be a Banach space. TFAE:

(1) X^* has the WRNP;

(2) X does not contain an isomorphic copy of ℓ^1;

(3) for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation, there is a *Birkhoff* integrable function $f : \Omega \longrightarrow X^*$ such that

$$\nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \Sigma.$$

Lemma 8

Let $\mathscr{H} \subset \mathbb{R}^{\Omega}$ be a uniformly bounded family. TFAE:

(1) $\mathscr{H}$ has the Bourgain property;

(2) for every $a < b$ in $\mathbb{R}$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \dots, A_n \subset A$, $A_i \in \Sigma$ with $\mu(A_i) > 0$, such that for every $h \in \mathscr{H}$ there is $1 \le i \le n$ such that

$$\text{either } \inf h(A_i) \ge a \quad \text{or } \sup h(A_i) \le b.$$

Lemma 9

Let $\mathfrak{T}$ be a topology on Ω with $\mathfrak{T} \subset \Sigma$ for which μ is hereditarily supported. Let $\mathscr{H} \subset \mathbb{R}^{\Omega}$ be a uniformly bounded family of continuous functions that does not contain ℓ^1-sequences (for the supremum norm). Then $\mathscr{H}$ has the Bourgain property.

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